DYNAMICS

ME 34010

HOMEWORK SOLUTIONS

Mahmoud M. Safadi[1], M.B. Rubin[2]

[1]safadi@technion.ac.il, [2]mbrubin@technion.ac.il

Faculty of Mechanical Engineering

Technion – Israel Institute of Technology

Spring Semester 2016

Last revision: December 2017

Problem Set 1 Solutions

Problem 1

The vertical slotted guide shown in Fig. 1.1 moves horizontally at a speed 20 [mm/s].

This causes the pin P to move in the fixed parabolic slot whose shape in given by

$$y = \frac{x^2}{b} \quad , \quad b = 160 \text{ [mm]} \quad .$$

1. Find the velocity and acceleration of P.
2. Find the velocity and acceleration of the P for the position $x = 60$ [mm].

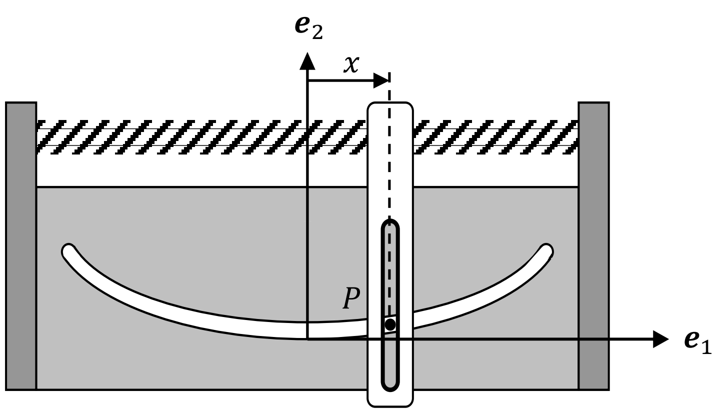

Figure 1.1

Solution:

The position of the particle is given by

$$\boldsymbol{x} = x\boldsymbol{e}_1 + \frac{x^2}{b}\boldsymbol{e}_2 \quad .$$

Thus, the velocity of the particle becomes

$$\boldsymbol{v} = \dot{\boldsymbol{x}} = \dot{x}\left(\boldsymbol{e}_1 + \frac{2x}{b}\boldsymbol{e}_2\right) = 20\boldsymbol{e}_1 + \frac{x}{4}\boldsymbol{e}_2 \text{ [mm/s]} \quad .$$

Moreover, the acceleration of the particle takes the form

$$\boldsymbol{a} = \dot{\boldsymbol{v}} = \frac{2\dot{x}^2}{b}\boldsymbol{e}_2 = 5\boldsymbol{e}_2 \text{ [mm/s}^2\text{]} \quad .$$

Next, at the instant when $x = 60$ [mm], the velocity and acceleration of the particle are given, respectively, by

$$\boldsymbol{v} = 20\boldsymbol{e}_1 + 15\boldsymbol{e}_2 \text{ [mm/s]} \quad , \quad \boldsymbol{a} = 5\boldsymbol{e}_2 \text{ [mm/s}^2\text{]} \quad .$$

Problem 2

The absolute acceleration vector of a particle, expressed in Cartesian coordinates with basis vectors $\boldsymbol{e}_i$, is given by

$$\boldsymbol{a}(t) = (4t - 3)\boldsymbol{e}_1 + t^2\boldsymbol{e}_2 \text{ [m/s}^2\text{]} \quad .$$

The particle is initially $(t = 0)$ at rest at the origin.

1. Find the velocity of the particle as a function of time.
2. Find the position of the particle as a function of time.

Solution:

The velocity of the particle is given by

$$\boldsymbol{v}(t) = \int_0^t a(\tau)d\tau = \left[(2\tau^2 - 3\tau)\boldsymbol{e}_1 + \frac{\tau^3}{3}\boldsymbol{e}_2\right]_0^t = (2t^2 - 3t)\boldsymbol{e}_1 + \frac{t^3}{3}\boldsymbol{e}_2 \quad \text{[m/s].}$$

Furthermore, the position of the particle takes the form

$$\boldsymbol{x}(t) = \int_0^t v(\tau)d\tau = \left[\left(\frac{2\tau^3}{3} - \frac{3\tau^2}{2}\right)\boldsymbol{e}_1 + \frac{\tau^3}{3}\boldsymbol{e}_2\right]_0^t = \left(\frac{2t^3}{3} - \frac{3t^2}{2}\right)\boldsymbol{e}_1 + \frac{t^3}{3}\boldsymbol{e}_2 \quad \text{[m].}$$

Problem 3

A particle passes through the points $A: (1,1,1)$ [m] and $B: (-1,4,7)$ [m] during its motion along a straight line. Let $\boldsymbol{e}_{B/A}$ denote the unit vector pointing from A to B, and $s(t)$ the distance traveled by the particle from the point A. The position vector of the particle is given by

$$\boldsymbol{x}(s) = \boldsymbol{x}_A + s\boldsymbol{e}_{B/A} = x_i(s)\boldsymbol{e}_i \text{ [m]} \quad , \quad i = \{1, 2, 3\} \quad ,$$

where the repeated index i implies a summation over i (Einstein summation convention).

1. Find the components $x_i(s)$ of $\boldsymbol{x}(s)$.
2. Let C denote the closest point to the origin along the straight line. Find the coordinates of this point.
3. Find the distance between the point C and the origin.
4. Find the distance between the points C and B.

Solution:

The unit vector $\boldsymbol{e}_{B/A}$ can be expressed as

$$\boldsymbol{e}_{B/A} = \frac{\boldsymbol{x}_{B/A}}{|\boldsymbol{x}_{B/A}|} = \frac{-2\boldsymbol{e}_1 + 3\boldsymbol{e}_2 + 6\boldsymbol{e}_3}{\sqrt{49}} = \frac{1}{7}(-2\boldsymbol{e}_1 + 3\boldsymbol{e}_2 + 6\boldsymbol{e}_3) \quad .$$

Thus, the position vector $\boldsymbol{x}(s)$ is given by

$$\boldsymbol{x}(s) = \left(1 - \frac{2s}{7}\right)\boldsymbol{e}_1 + \left(1 + \frac{3s}{7}\right)\boldsymbol{e}_2 + \left(1 + \frac{6s}{7}\right)\boldsymbol{e}_3 \text{ [m]} \quad .$$

Now, the direction $\boldsymbol{e}_{C/O}$ of $\boldsymbol{x}_{C/O}$ is perpendicular to the direction $\boldsymbol{e}_{B/A}$ of $\boldsymbol{x}(s)$ provided that C is the closest point to the origin O along $\boldsymbol{x}(s)$. Denoting the coordinates of C by (x_1, x_2, x_3), it follows that

$$\boldsymbol{e}_{B/A} \cdot \boldsymbol{e}_{C/O} = \frac{1}{7}(-2\boldsymbol{e}_1 + 3\boldsymbol{e}_2 + 6\boldsymbol{e}_3) \cdot \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3) = 0 \quad \Rightarrow$$

$$-2x_1 + 3x_2 + 6x_3 = 0 \quad .$$

However, since C lies on $\boldsymbol{x}(s)$

$$x_1 = 1 - \frac{2s}{7} \quad , \quad x_2 = 1 + \frac{3s}{7} \quad , \quad x_3 = 1 + \frac{6s}{7} \quad .$$

Consequently,

$$-2\left(1 - \frac{2s}{7}\right) + 3\left(1 + \frac{3s}{7}\right) + 6\left(1 + \frac{6s}{7}\right) = 7 + \frac{49s}{7} = 0 \quad \Rightarrow \quad s = -1 \text{ [m]} \quad .$$

Moreover, the coordinates of C are given by

$$\boldsymbol{x}_C = \frac{1}{7}(9\boldsymbol{e}_1 + 4\boldsymbol{e}_2 + \boldsymbol{e}_3) \text{ [m]} \quad .$$

Next, the distances $\left|\boldsymbol{x}_{C/O}\right|$ and $\left|\boldsymbol{x}_{C/B}\right|$ are given, respectively, by

$$\left|\boldsymbol{x}_{C/O}\right| = |\boldsymbol{x}_C| = \sqrt{2} \text{ [m]} \quad , \quad \left|\boldsymbol{x}_{C/B}\right| = |\boldsymbol{x}_C - \boldsymbol{x}_B| = 8 \text{ [m]} \quad .$$

Problem 4

A moving object is influenced by the aerodynamic drag, which is proportional to the square of the object's speed, such that the acceleration of this object is given by

$$a = -c_1 - c_2 v^2 \text{ [m/s}^2\text{]} \quad ,$$

where c_1 $[\mathrm{m/s^2}]$ and c_2 $[1/\mathrm{m}]$ are constant parameters.

The object starts its motion from the origin with speed 80 $[\mathrm{km/h}]$. Furthermore, the speeds of the object after traveling the distances of $\{200, 400\}$ [m] are given, respectively, by $\{60, 36\}$ $[\mathrm{km/h}]$.

Find the total distance traveled until the object stops.

Solution:

Denoting the distance traveled by the object by x and using the chain rule of differentiation it follows that

$$\frac{dv}{dt} = \frac{dv}{dx}\dot{x} = \frac{dv}{dx}v = a(v) \;\; \Rightarrow \;\; \frac{v}{a(v)}dv = dx \;\; \Rightarrow \;\; x = x_0 + \int_{v_0}^{V} \frac{v}{a(v)}dv \quad .$$

Now,

$$\int_{v_0}^{V} \frac{v}{a(v)}dv = -\int_{v_0}^{V} \frac{v}{c_1 + c_2 v^2}dv = -\left[\frac{1}{2c_2}\ln(c_1 + c_2 v^2)\right]_{v_0}^{V} = -\frac{1}{2c_2}\ln\left(\frac{c_1 + c_2 V^2}{c_1 + c_2 v_0^2}\right) \quad ,$$

such that

$$x = x_0 - \frac{1}{2c_2}\ln\left(\frac{c_1 + c_2 V^2}{c_1 + c_2 v_0^2}\right) \quad .$$

Next, with $x_0 = 0$, $v_0 = 80$ $[\mathrm{km/h}]$, and

$$@x = 0.2 \text{ [km]: } V = 60 \text{ [km/h]} \quad , \quad @x = 0.4 \text{ [km]: } V = 36 \text{ [km/h]} \quad ,$$

it follows that

$$0.2 = \frac{1}{2c_2}\ln\left(\frac{c_1 + 3600c_2}{c_1 + 6400c_2}\right) \quad , \quad 0.4 = \frac{1}{2c_2}\ln\left(\frac{c_1 + 1296c_2}{c_1 + 6400c_2}\right) \quad .$$

Solving these two equation for $\{c_1, c_2\}$ yields

$$c_1 = 4585\ [\mathrm{km/h^2}] \quad , \quad c_2 = 0.4874\ [1/\mathrm{km}] \quad .$$

Hence, the total distance traveled until the object stops is given by substituting the values of $\{c_1, c_2\}$ together with $\{@x = 0.4\ [\mathrm{km}]:\ V = 0\}$ into x, such that

$$x = 0.5324\ [\mathrm{km}] = 532.4\ [\mathrm{m}] \quad .$$

Problem Set 2 Solutions

Problem 1

Figure 2.1 shows a block being hauled to the surface over a curved track by a cable wound around a 750 [mm] drum, which turns at the constant clockwise speed of 120 [rpm]. The shape of the track is designed so that $y = x^2/16$, where x and y are in meters.

1. Determine the acceleration of the block as a function of x.
2. Find the magnitude of the acceleration of the block as it reaches a level of 1 [m] below the top.

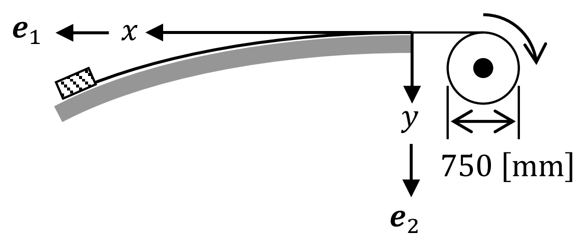

Figure 2.1

Solution:

The velocity of this block takes the form

$$\boldsymbol{v} = \dot{s}\boldsymbol{e}_t \quad ,$$

where the speed $\dot{s}$ of the block is given by

$$\dot{s} = \frac{\omega D}{2} \quad , \quad \omega = 120 \text{ [rpm]} = 120\left(\frac{2\pi}{60}\right) = 4\pi \text{ [rad/s]} \quad \Rightarrow$$

$$\dot{s} = 2\pi D \text{ [m/s]} \quad .$$

Moreover, the unit tangent vector can be expressed as

$$\boldsymbol{e}_t = \frac{d\boldsymbol{x}}{ds} = \frac{d\boldsymbol{x}}{dx}\frac{dx}{ds} \quad , \quad \boldsymbol{x} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 \quad , \quad \frac{dx}{ds} = -\frac{1}{\sqrt{1+(dy/dx)^2}} \quad , \quad y = \frac{x^2}{16} \quad \Rightarrow$$

$$\boldsymbol{e}_t = -\frac{1}{\sqrt{x^2+64}}(8\boldsymbol{e}_1 + x\boldsymbol{e}_2) \quad .$$

Notice that the minus sign in dx/ds must be included since each time s increases, x decreases (cf. Fig. 1.1). Therefore,

$$\boldsymbol{v} = \dot{s}\boldsymbol{e}_t = -\frac{2\pi D}{\sqrt{x^2+64}}(8\boldsymbol{e}_1 + x\boldsymbol{e}_2) \quad .$$

Next, the acceleration of the block is given by

$$\boldsymbol{a} = \dot{\boldsymbol{v}} = \frac{d\boldsymbol{v}}{dx}\dot{x} = \frac{16\pi D\dot{x}}{(x^2+64)^{3/2}}(x\boldsymbol{e}_1 - 8\boldsymbol{e}_2) \quad , \quad \dot{x} = -\frac{\dot{s}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = -\frac{2\pi D}{\sqrt{1+\frac{x^2}{64}}} \quad \Rightarrow$$

$$\boldsymbol{a} = \frac{256\pi^2 D^2}{(x^2+64)^2}(-x\boldsymbol{e}_1 + 8\boldsymbol{e}_2) \quad .$$

Hence, as the block reaches a level of 1 [m] below the top it follows that the magnitude of the acceleration of the block reduces to

$$y = 1\ [\mathrm{m}] \ \Rightarrow \ x = 4\ [\mathrm{m}] \quad ; \quad D = 0.75\ [m] \ \Rightarrow$$

$$|\boldsymbol{a}| = \frac{256\pi^2(0.75)^2}{80\sqrt{80}} \approx 1.986\ [\mathrm{m/s^2}] \quad .$$

Problem 2

The pin P shown in Fig. 2.2 is constrained to move in the slotted guides A and B which move at right angles to one another. At the instant represented, A has a velocity to the right of 0.2 $[\mathrm{m/s}]$ which is decreasing at the rate of 0.75 $[\mathrm{m/s}]$ each second. At the same time, B is moving down with a velocity of 0.15 $[\mathrm{m/s}]$ which is decreasing at the rate of 0.5 $[\mathrm{m/s}]$ each second.

1. For this instant, find the radius of curvature ρ of the path followed by P.
2. Is it possible to also determine the time rate of change of ρ.

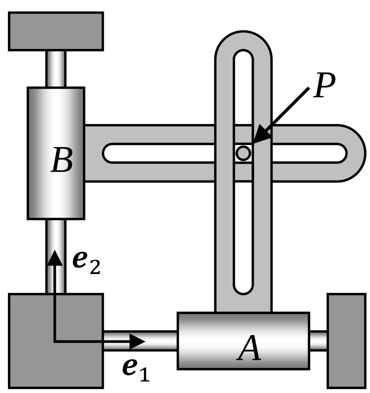

Figure 2.2

Solution:

The velocity and acceleration of the pin P are given, respectively, by

$$\boldsymbol{v} = 0.2\boldsymbol{e}_1 - 0.15\boldsymbol{e}_2 \ [\mathrm{m/s}] \ \ , \ \ \boldsymbol{a} = 0.75\boldsymbol{e}_1 - 0.5\boldsymbol{e}_2 \ [\mathrm{m/s^2}] \ \ .$$

Moreover, the unit tangent vector to the path followed by P takes the form

$$\boldsymbol{e}_t = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = 0.8\boldsymbol{e}_1 - 0.6\boldsymbol{e}_2 \ \ .$$

Therefore, the normal component of the acceleration of P becomes

$$a_n = |\boldsymbol{a} - (\boldsymbol{a} \cdot \boldsymbol{e}_t)\boldsymbol{e}_t| = 0.05 \ [\mathrm{m/s^2}] \quad .$$

Now, using the relation

$$a_n = \frac{\dot{s}^2}{\rho} = \frac{|\boldsymbol{v}|^2}{\rho} \quad ,$$

it follows that

$$\rho = \frac{|\boldsymbol{v}|^2}{a_n} = 1.25 \ [\mathrm{m}] \quad .$$

Next, recall that the radius of curvature can be expressed in terms of the speed $\dot{s}$ of P and the angular rate $\dot{\beta}$ of the radial line from P to the center of curvature in the form

$$\dot{s} = \rho\dot{\beta} \quad .$$

Hence,

$$\ddot{s} = \dot{\rho}\dot{\beta} + \rho\ddot{\beta} \quad \Rightarrow$$

$$\dot{\rho} = \frac{\ddot{s} - \rho\ddot{\beta}}{\dot{\beta}} \quad ; \quad \ddot{s} = |\boldsymbol{a} \cdot \boldsymbol{e}_t| \quad , \quad \dot{\beta} = \frac{\dot{s}}{\rho} \quad .$$

This shows that $\dot{\rho}$ cannot be determined until the angular acceleration $\ddot{\beta}$ of the radial line from P to the center of curvature is known.

Problem 3

A particle is constrained to move along a track characterized by the function $y = 2x^{3/2}$, where x and y are in meters. The distance $s(t)$ actually traveled by the particle as it moves along the track is given by $s(t) = 2t^3$, where t denotes the time in seconds.

Initially, at the time $t = 0$, $x = 0$.

At the instant when $t = 1$ [s]:

1. Find the radius of curvature of the particle path.
2. Find the magnitude of the acceleration of the particle.

Solution:

First, recall that

$$\dot{s} = \dot{x}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad .$$

Hence,

$$6t^2 = \frac{dx}{dt}\sqrt{1+9x} \;\Rightarrow\; \int_0^{x(1)} \sqrt{1+9x}\, dx = \int_0^1 6t^2 dt \;\Rightarrow\; \frac{2}{27}[1+9x(1)]^{3/2} = 2 \;\Rightarrow$$

$$x(1) = \frac{8}{9}\ [m] \quad .$$

Next, the position, velocity, and acceleration of the particle at $t = 1$[s] are given, respectively, by

$$\boldsymbol{x} = x\boldsymbol{e}_1 + 2x^{3/2}\boldsymbol{e}_2 \;\;,\;\; \boldsymbol{v} = \dot{x}\left(\boldsymbol{e}_1 + 3\sqrt{x}\boldsymbol{e}_2\right) \;\;,\;\; \boldsymbol{a} = \ddot{x}\boldsymbol{e}_1 + 3\left(\ddot{x}\sqrt{x} + \frac{\dot{x}^2}{2\sqrt{x}}\right)\boldsymbol{e}_2 \;\;,$$

where,

$$\dot{x} = \frac{\dot{s}}{\sqrt{1+9x}} = \frac{6t^2}{\sqrt{1+9x}} \;\;,\;\; \ddot{x} = \frac{12t}{(1+9x)^{1/2}} - \frac{27\dot{x}}{(1+9x)^2} \;\Rightarrow$$

$$\dot{x}(1) = 2 \text{ [m/s]} \quad , \quad \ddot{x}(1) = 2 \left[\text{m/s}^2\right] \quad .$$

Therefore, the values of $\boldsymbol{v}$ and $\boldsymbol{a}$ at $t = 1$ [s] reduce, respectively, to

$$\boldsymbol{v}(1) = 2\boldsymbol{e}_1 + 4\sqrt{2}\,\boldsymbol{e}_2 \text{ [m/s]} \quad , \quad \boldsymbol{a}(1) = 2\boldsymbol{e}_1 + \frac{17}{\sqrt{2}}\boldsymbol{e}_2 \left[\text{m/s}^2\right] \quad .$$

Next, the normal component of the acceleration of the particle at $t = 1$ [s] takes the form

$$a_n(1) = |\boldsymbol{a} - (\boldsymbol{a} \cdot \boldsymbol{e}_t)\boldsymbol{e}_t|_{t=1\text{ [s]}} \quad , \quad \boldsymbol{e}_t(1) = \frac{\boldsymbol{v}(1)}{|\boldsymbol{v}(1)|} = \frac{1}{3}\boldsymbol{e}_1 + \frac{2\sqrt{2}}{3}\boldsymbol{e}_2 \quad \Rightarrow$$

$$a_n(1) = \frac{3\sqrt{2}}{2} \left[\text{m/s}^2\right] \quad ,$$

Consequently, the radius of curvature of the particle path is given by

$$\rho(1) = \frac{|\boldsymbol{v}(1)|^2}{a_n(1)} = 12\sqrt{2} \text{ [m]} \approx 17 \text{ [m]} \quad .$$

Problem 4

A particle moves in the x-y plane at constant speed b along a track characterized by the function $y = y(x)$, where x and y are in meters. Also, let s denote the actual distance traveled by the particle along the track.

1. Assuming that $dx/ds > 0$, show that

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad .$$

2. Use the chain rule of differentiation to determine the velocity of the particle as a function of x.
3. Use the chain rule of differentiation to determine the acceleration of the particle as a function of x.
4. Show that the radius of curvature at any point along the particle path is given by

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \quad .$$

5. Determine the unit normal vector $\boldsymbol{e}_n$ at any point along the particle path as a function of x.

Solution:

Recall that the unit tangent vector $\boldsymbol{e}_t$ is defined by

$$\boldsymbol{e}_t = \frac{d\boldsymbol{x}}{ds} \quad ; \quad \boldsymbol{e}_t \cdot \boldsymbol{e}_t = 1 \quad , \quad \boldsymbol{x} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 \quad ,$$

so that

$$\frac{d\boldsymbol{x}}{ds} \cdot \frac{d\boldsymbol{x}}{ds} = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1 \quad \Rightarrow \quad (dx)^2 \left[1 + \left(\frac{dy}{dx}\right)^2\right] = (ds)^2 \quad \Rightarrow$$

$$\frac{ds}{dx} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} \quad .$$

Thus, the velocity of the particle is given by

$$\boldsymbol{v} = \dot{s}\boldsymbol{e}_t \quad , \quad \boldsymbol{e}_t = \frac{d\boldsymbol{x}}{ds} = \frac{d\boldsymbol{x}}{dx}\,\frac{dx}{ds} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{-1/2} \left(\boldsymbol{e}_1 + \frac{dy}{dx}\boldsymbol{e}_2\right) \;\Rightarrow$$

$$\boldsymbol{v} = b\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{-1/2} \left(\boldsymbol{e}_1 + \frac{dy}{dx}\boldsymbol{e}_2\right)$$

Next, recall that the derivative of $\boldsymbol{e}_t$ with respect to s is given by

$$\frac{d\boldsymbol{e}_t}{ds} = \frac{1}{\rho}\boldsymbol{e}_n \quad \Rightarrow \quad \rho = \frac{1}{\left|\frac{d\boldsymbol{e}_t}{ds}\right|} = \frac{1}{\left|\frac{d\boldsymbol{e}_t}{dx}\,\frac{dx}{ds}\right|} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}}{\left|\frac{d\boldsymbol{e}_t}{dx}\right|} \quad ,$$

where,

$$\frac{d\boldsymbol{e}_t}{dx} = -\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{-3/2} \frac{dy}{dx}\,\frac{d^2y}{dx^2}\left(\boldsymbol{e}_1 + \frac{dy}{dx}\boldsymbol{e}_2\right) + \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{-1/2} \frac{d^2y}{dx^2}\boldsymbol{e}_2$$

$$= \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{-3/2} \frac{d^2y}{dx^2}\left(-\frac{dy}{dx}\,\boldsymbol{e}_1 + \boldsymbol{e}_2\right) \quad ,$$

Consequently,

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{-3/2} \frac{d^2y}{dx^2}\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \quad .$$

Now, the acceleration of the particle takes the form

$$\boldsymbol{a} = \ddot{s}\boldsymbol{e}_t + \frac{\dot{s}^2}{\rho}\boldsymbol{e}_n = \frac{b^2\,\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}\;\boldsymbol{e}_n \quad ,$$

where,

$$\boldsymbol{e}_n = \rho \frac{d\boldsymbol{e}_t}{ds} = \rho \frac{d\boldsymbol{e}_t}{dx}\frac{dx}{ds} = \frac{1}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}}\left(-\frac{dy}{dx}\ \boldsymbol{e}_1 + \boldsymbol{e}_2\right) \quad ,$$

such that

$$\boldsymbol{a} = \frac{b^2 \frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^2}\left(-\frac{dy}{dx}\ \boldsymbol{e}_1 + \boldsymbol{e}_2\right) \quad .$$

Problem Set 3 Solutions

Problem 1

A particle moving along a curve in space has coordinates in millimeters which vary with time t in seconds according to

$$x = 60\cos(\omega t) \quad , \quad y = 40\sin(\omega t) \quad , \quad z = 30t^2 \quad ,$$

where $\omega = 2$ [rad/s].

1. Plot the path of the particle over the time interval $0 \leq t \leq 20$ [s].

At the instant when $t = 4$ [s]:

2. Determine the unit normal and unit tangent vectors of the particle path.
3. Find the velocity of the particle.
4. Find the acceleration of the particle.
5. Find the radius of curvature of the particle path.

Solution:

The path of the particle is shown in Fig. 3.1. Now, the position of the particle is given by

$$\boldsymbol{x} = 60\cos(\omega t)\,\boldsymbol{e}_1 + 40\sin(\omega t)\,\boldsymbol{e}_2 + 30t^2\boldsymbol{e}_3 \quad .$$

Hence, the unit tangent vector to the particle path can be expressed as

$$\boldsymbol{e}_t = \frac{d\boldsymbol{x}/dt}{|d\boldsymbol{x}/dt|} = \frac{-3\omega\sin(\omega t)\,\boldsymbol{e}_1 + 2\omega\cos(\omega t)\;\boldsymbol{e}_2 + 3t\boldsymbol{e}_3}{\sqrt{9t^2 + \omega^2[9 - 5\cos(\omega t)]}} \quad ,$$

such that

$$\boldsymbol{e}_t(4) = -0.443\boldsymbol{e}_1 - 0.0434\boldsymbol{e}_2 + 0.955\boldsymbol{e}_3 \quad .$$

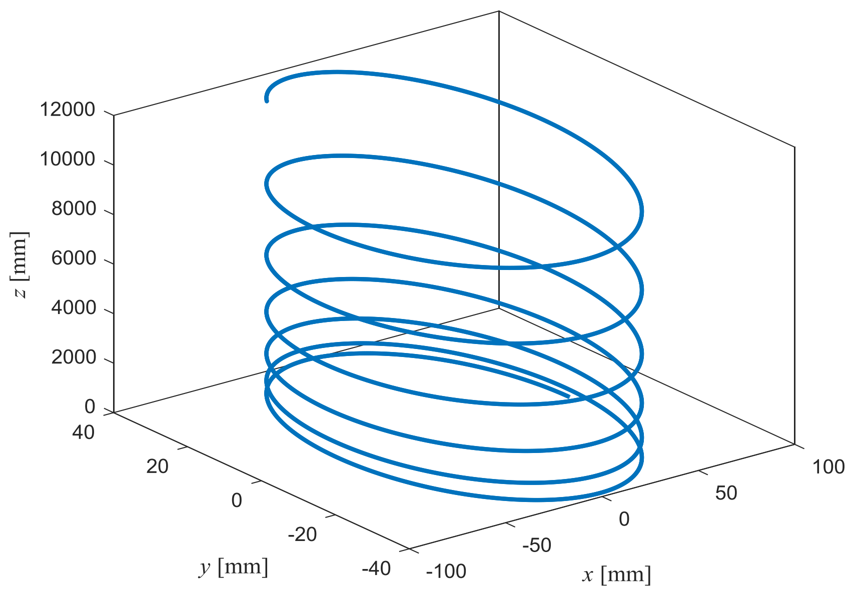

Fig. 3.1

Also, the unit normal to the particle path is given by

$$\boldsymbol{e}_n = \frac{d\boldsymbol{e}_t/dt}{|d\boldsymbol{e}_t/dt|} \quad .$$

However, it is more convenient to calculate $\boldsymbol{e}_n$ using the acceleration of the particle as will be shown next.

The velocity of the particle takes the form

$$\boldsymbol{v} = -60\omega \sin(\omega t)\, \boldsymbol{e}_1 + 40\omega \cos(\omega t)\, \boldsymbol{e}_2 + 60t\boldsymbol{e}_3 \quad ,$$

such that

$$\boldsymbol{v}(4) = -118.7\boldsymbol{e}_1 - 11.64\boldsymbol{e}_2 + 240\boldsymbol{e}_3 \; [\text{mm/s}] \quad .$$

Furthermore, the acceleration of the particle becomes

$$\boldsymbol{a} = -60\omega^2 \cos(\omega t)\, \boldsymbol{e}_1 - 40\omega^2 \sin(\omega t)\, \boldsymbol{e}_2 + 60\boldsymbol{e}_3 \quad ,$$

such that

$$\boldsymbol{a}(4) = 34.92\boldsymbol{e}_1 - 158.3\boldsymbol{e}_2 + 60\boldsymbol{e}_3 \; [\text{mm/s}^2] \quad .$$

Next, the normal component of the total acceleration at the time $t = 4$ [s] is given by

$$\boldsymbol{a}_n(4) = [\boldsymbol{a} - (\boldsymbol{a} \cdot \boldsymbol{e}_t)\boldsymbol{e}_t]_{t=4\ [\mathrm{s}]} = 95.91\boldsymbol{e}_1 - 156.3\boldsymbol{e}_2 + 19.58\boldsymbol{e}_3\ [\mathrm{mm/s^2}] \quad .$$

Thus, the unit normal vector to the particle path at the time $t = 4$ [s] reduces to

$$\boldsymbol{e}_n = \frac{\boldsymbol{a}_n(4)}{|\boldsymbol{a}_n(4)|} = 0.329\boldsymbol{e}_1 - 0.937\boldsymbol{e}_2 + 0.117\boldsymbol{e}_3 \quad .$$

Also, the radius of curvature of the particle path at the time $t = 4$ [s] takes the form

$$\rho = \frac{|\boldsymbol{v}(4)|^2}{a_n(4)} = 430.5\ [\mathrm{mm}]\ .$$

Problem 2

Figure 3.2 shows a particle moving along a track inside a vertical cylinder of radius 2 [m]. At the instant represented, the particle passes through the point A with an acceleration of $10\ [\mathrm{m/s^2}]$ at an angle of 30^{o} with respect to the horizontal plane, and it increases its speed along the track at the rate of $8\ [\mathrm{m/s}]$ each second.

For this instant:

1. Determine the velocity of the particle in terms of cylindrical-polar coordinates.
2. Find the angular speed $\dot{\theta}$ of the particle.
3. Find the angular acceleration $\ddot{\theta}$ of the particle.
4. Find the vertical component of the acceleration of the particle.

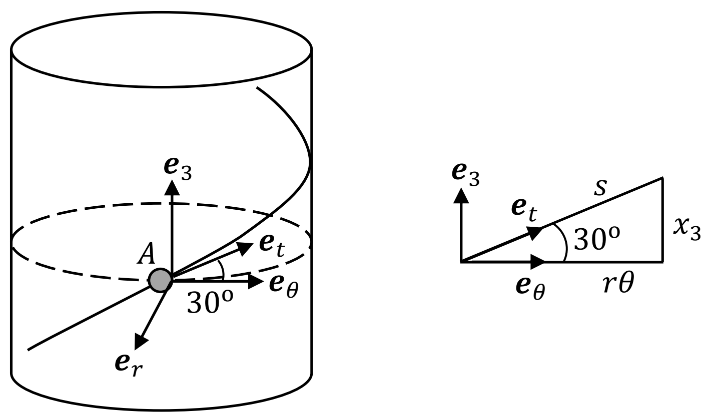

Figure 3.2

Solution:

The total $|\boldsymbol{a}|$, tangential $|\boldsymbol{a}_t|$ and normal $|\boldsymbol{a}_n|$ accelerations of the particle at the instant represented in Fig. 3.2 are given, respectively, by

$$|\boldsymbol{a}| = 10\,[\mathrm{m/s^2}] \ \ , \ \ |\boldsymbol{a}_t| = \ddot{s} = 8\,[\mathrm{m/s^2}] \ \ , \ \ |\boldsymbol{a}_n| = \frac{\dot{s}^2}{\rho} = \sqrt{|\boldsymbol{a}|^2 - |\boldsymbol{a}_t|^2} = 6\,[\mathrm{m/s^2}] \ \ .$$

Moreover, the unit tangent vector to the particle path takes the form

$$\boldsymbol{e}_t = \cos(30^o)\,\boldsymbol{e}_\theta + \sin(30^o)\,\boldsymbol{e}_3 = \frac{\sqrt{3}}{2}\boldsymbol{e}_\theta + \frac{1}{2}\boldsymbol{e}_3 \ \ ,$$

so that the corresponding unit normal vector reduces to

$$\boldsymbol{e}_n \cdot \boldsymbol{e}_t = \boldsymbol{e}_n \cdot \boldsymbol{e}_3 = 0 \ \Rightarrow \ \ \boldsymbol{e}_n = -\boldsymbol{e}_r \ \ .$$

Notice that the minus sign is taken since $\boldsymbol{e}_n$ points toward the center of curvature.

Thus,

$$\boldsymbol{a} = |\boldsymbol{a}_t|\boldsymbol{e}_t + |\boldsymbol{a}_n|\boldsymbol{e}_n = -6\boldsymbol{e}_r + 4\sqrt{3}\,\boldsymbol{e}_\theta + 4\boldsymbol{e}_3 \ \ .$$

Now, recall that the acceleration can be expressed in terms of cylindrical-polar coordinates in the form

$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{e}_\theta + \ddot{x}_3\boldsymbol{e}_3 \ \ , \ \ r = 2\,[m] \ \ , \ \ \dot{r} = 0 \ \ , \ \ \ddot{r} = 0 \ \ ,$$

such that

$$-2\dot{\theta}^2 = -6 \ \Rightarrow \ \ \dot{\theta} = \sqrt{3}\,[\mathrm{rad/s}] \ \ , \ \ 2\ddot{\theta} = 4\sqrt{3} \ \Rightarrow \ \ \ddot{\theta} = 2\sqrt{3}\,[\mathrm{rad/s^2}] \ \ ,$$

$$\ddot{x}_3 = 4\,[\mathrm{m/s^2}] \ \ .$$

Next, using Fig. 1.1 it follows that

$$s = \frac{r\theta}{\cos(30^o)} \ \Rightarrow \ \ \dot{s} = \frac{r\dot{\theta}}{\cos(30^o)} = \frac{2\sqrt{3}}{\sqrt{3}/2} = 4\,[\mathrm{m/s}] \ \ .$$

Consequently,

$$\boldsymbol{v} = \dot{s}\boldsymbol{e}_t = 2\sqrt{3}\,\boldsymbol{e}_\theta + 2\boldsymbol{e}_3\,[\mathrm{m/s}] \ \ .$$

Problem 3

The cam shown in Fig. 3.3 is designed so that the center of the roller A which follows the contour moves on a limaçon defined by $r = b - c\cos(\beta)$, where $b > c$ and β is the angle between the line OB fixed to the limaçon and the slotted arm. The base vectors $\{\boldsymbol{e}_r, \boldsymbol{e}_\theta\}$ of the polar coordinate system are fixed to the slotted bar. Moreover, take $b = 100$ [mm] and $c = 75$ [mm].

At the instant when $\beta = 30^{\text{o}}$:

1. Determine the total acceleration of the roller A if the slotted arm revolves with a constant counterclockwise angular speed of 40 [rpm] while the limaçon stays fixed.
2. Determine the total acceleration of the roller A if the slotted arm stays fixed while the limaçon revolves with a constant clockwise angular speed of 30 [rpm].
3. Determine the total acceleration of the roller A if the slotted arm revolves with a constant counterclockwise angular speed of 40 [rpm] while the limaçon revolves with a constant clockwise angular speed of 30 [rpm].

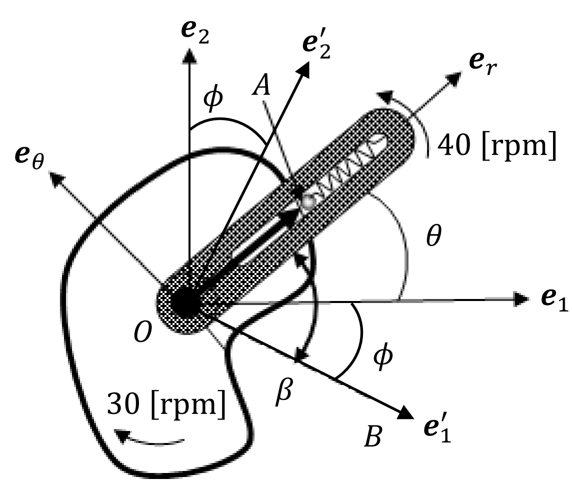

Figure 3.3

Solution:

Using the geometry in Fig. 3.3, the angle ϕ is related to the angles $\{\theta, \beta\}$ by

$$\phi = \beta - \theta \quad ,$$

such that

$$\dot{\beta} = \dot{\theta} + \dot{\phi} \quad .$$

Next, using this expression, the position, velocity and acceleration of the roller A expressed in terms of polar coordinates are given, respectively, by

$$\boldsymbol{x} = [b - c\cos(\beta)]\boldsymbol{e}_r \quad ,$$

$$\boldsymbol{v} = c(\dot{\phi} + \dot{\theta})\sin(\beta)\,\boldsymbol{e}_r + [b - c\cos(\beta)]\dot{\theta}\boldsymbol{e}_\theta \quad ,$$

$$\boldsymbol{a} = \left[c\left\{(\dot{\phi} + \dot{\theta})^2 + \dot{\theta}^2\right\}\cos(\beta) - b\dot{\theta}^2\right]\boldsymbol{e}_r + 2c(\dot{\phi} + \dot{\theta})\dot{\theta}\sin(\beta)\,\boldsymbol{e}_\theta \quad .$$

Case 1:

$$\dot{\theta} = 40\,[\text{rpm}] = \frac{4\pi}{3}\,[\text{rad}/s] \quad , \quad \dot{\phi} = 0 \quad , \quad \beta = 30^{\text{o}} \quad \Rightarrow$$

$$\boldsymbol{a} = 0.525\,\boldsymbol{e}_r + 1.316\,\boldsymbol{e}_\theta\,[\text{m/s}^2] \quad .$$

Case 2:

$$\dot{\theta} = 0 \quad , \quad \dot{\phi} = -30\,[\text{rpm}] = -\pi\,[\text{rad}/s] \quad , \quad \beta = 30^{\text{o}} \quad \Rightarrow$$

$$\boldsymbol{a} = 0.641\,\boldsymbol{e}_r\,[\text{m/s}^2] \quad .$$

Case 3:

$$\dot{\theta} = \frac{4\pi}{3}\,[\text{rad}/s] \quad , \quad \dot{\phi} = -\pi\,[\text{rad}/s] \quad , \quad \beta = 30^{\text{o}} \quad \Rightarrow$$

$$\boldsymbol{a} = -0.544\,\boldsymbol{e}_r + 0.329\,\boldsymbol{e}_\theta\,[\text{m/s}^2] \quad .$$

Problem 4

The hollow tube shown in Fig. 3.4 is inclined at an angle α to the vertical axis and it rotates along a circular path of radius R with a constant angular speed about the vertical axis. A particle P moves inside the tube under the control of an inextensible string which is held fixed at the point D. Moreover, the coordinate system $\boldsymbol{e}'_i$ is fixed to the tube, the distance traveled by the particle as it moves along the tube from the fixed point B is denoted by s, and the angle between the radial lines OC and OD is denoted by ϕ.

Initially, at the time $t = 0$, $\phi = 0$ and $s = 0$.

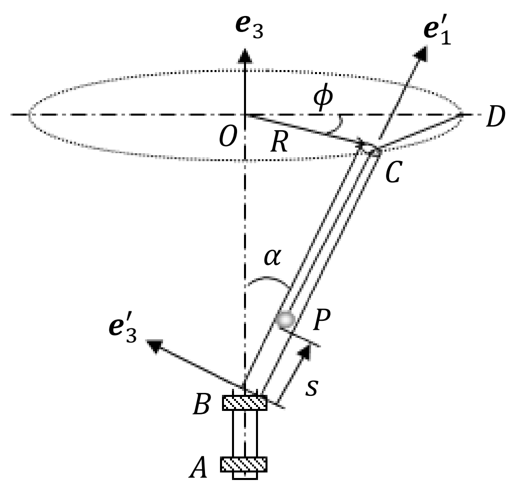

Figure 3.4

1. Determine the velocity of the particle P.
2. Determine the acceleration of the particle P.
3. Determine the velocity of the particle P along the tube.
4. Determine the acceleration of the particle P along the tube.

Solution:

The system $\boldsymbol{e}_i'$ is defined by

$$\dot{\boldsymbol{e}}_i' = \boldsymbol{\omega} \times \dot{\boldsymbol{e}}_i \quad , \quad \boldsymbol{\omega} = -\dot{\theta}\boldsymbol{e}_3 \quad , \quad \boldsymbol{e}_3 = \cos(\alpha)\,\boldsymbol{e}_1' + \sin(\alpha)\,\boldsymbol{e}_3' \quad .$$

The angular speeds $\dot{\theta}$ and $\dot{\phi}$ can be related using the velocity $\boldsymbol{v}_C$ of the upper end of the hollow tube, such that

$$\boldsymbol{x}_{C/B} = L\boldsymbol{e}_1' \;\; \Rightarrow \;\; \boldsymbol{v}_C = -\dot{\theta}[\cos(\alpha)\,\boldsymbol{e}_1' + \sin(\alpha)\,\boldsymbol{e}_3'] \times L\boldsymbol{e}_1' = -\dot{\theta}L\sin(\alpha)\,\boldsymbol{e}_2' \quad ,$$

$$\boldsymbol{x}_{C/O} = R[\sin(\phi)\,\boldsymbol{e}_1 + \cos(\phi)\,\boldsymbol{e}_2] \;\; \Rightarrow \;\; \boldsymbol{v}_C = R\dot{\phi}[\cos(\phi)\,\boldsymbol{e}_1 - \sin(\phi)\,\boldsymbol{e}_2] \quad ,$$

where L denotes the length of the tube, the unit vector $\boldsymbol{e}_2$ points rightward and the unit vector $\boldsymbol{e}_1$ is defined by $\boldsymbol{e}_1 = \boldsymbol{e}_2 \times \boldsymbol{e}_3$. Therefore,

$$\dot{\theta}L\sin(\alpha) = R\dot{\phi} \;\; \Rightarrow \;\; \dot{\theta} = \frac{R}{L\sin(\alpha)}\dot{\phi} \quad .$$

Now, using the geometry in Fig. 3.4 at the time $t = 0$, i.e. when the upper end C of the hollow tube coincides with the fixed point D, it follows that

$$\sin(\alpha) = \frac{R}{L} \quad .$$

Hence,

$$\dot{\theta} = \dot{\phi} \quad .$$

Next, the velocity of the particle P is given by

$$\boldsymbol{x}_{P/B} = \boldsymbol{x}_P = s\boldsymbol{e}_1' \;\; \Rightarrow \;\; \boldsymbol{v}_p = \frac{\delta \boldsymbol{x}_P}{\delta t} + \boldsymbol{\omega} \times \boldsymbol{x}_P = \dot{s}\boldsymbol{e}_1' - \dot{\phi}s\sin(\alpha)\,\boldsymbol{e}_2' \quad .$$

However,

$$s = |\boldsymbol{x}_{C/D}| = 2R\sin\left(\frac{\phi}{2}\right) \;\; \Rightarrow \;\; \dot{s} = R\dot{\phi}\cos\left(\frac{\phi}{2}\right) \quad ,$$

so that

$$\boldsymbol{v}_p = R\dot{\phi}\cos\left(\frac{\phi}{2}\right)\boldsymbol{e}_1' - 2R\dot{\phi}\sin(\alpha)\sin\left(\frac{\phi}{2}\right)\boldsymbol{e}_2' \quad .$$

Furthermore, using the Table 3.1,

	$\boldsymbol{e}_1'$	$\boldsymbol{e}_2'$	$\boldsymbol{e}_3'$
$\boldsymbol{\omega}$	$-\dot{\phi}\cos(\alpha)$	0	$-\dot{\phi}\sin(\alpha)$
$\boldsymbol{v}_P$	$R\dot{\phi}\cos\left(\frac{\phi}{2}\right)$	$-2R\dot{\phi}\sin(\alpha)\sin\left(\frac{\phi}{2}\right)$	0
$\delta\boldsymbol{v}_P/\delta t$	$-\frac{R\dot{\phi}^2}{2}\sin\left(\frac{\phi}{2}\right)$	$-R\dot{\phi}^2\sin(\alpha)\cos\left(\frac{\phi}{2}\right)$	0
$\boldsymbol{\omega}\times\boldsymbol{v}_P$	$-2R\dot{\phi}^2\sin^2(\alpha)\sin\left(\frac{\phi}{2}\right)$	$-R\dot{\phi}^2\sin(\alpha)\cos\left(\frac{\phi}{2}\right)$	$R\dot{\phi}^2\sin(2\alpha)\sin\left(\frac{\phi}{2}\right)$

Table 3.1

the acceleration of the particle P,

$$\boldsymbol{a}_P = \frac{\delta\boldsymbol{v}_P}{\delta t} + \boldsymbol{\omega}\times\boldsymbol{v}_P \quad ,$$

takes the form

$$\boldsymbol{a}_P = -R\dot{\phi}^2\sin\left(\frac{\phi}{2}\right)\left[\frac{1}{2} + 2\sin^2(\alpha)\right]\boldsymbol{e}_1' - 2R\dot{\phi}^2\sin(\alpha)\cos\left(\frac{\phi}{2}\right)\boldsymbol{e}_2'$$

$$+ R\dot{\phi}^2\sin(2\alpha)\sin\left(\frac{\phi}{2}\right)\boldsymbol{e}_3' \quad .$$

Next, the velocity of the particle P along the tube is given by

$$\frac{\delta}{\delta t}[(\boldsymbol{x}_P\cdot\boldsymbol{e}_i')\boldsymbol{e}_i'] = \frac{\delta}{\delta t}(s\boldsymbol{e}_1') = \dot{s}\boldsymbol{e}_1' = R\dot{\phi}\cos\left(\frac{\phi}{2}\right)\boldsymbol{e}_1' \quad .$$

Moreover, the acceleration of the particle P along the tube takes the form

$$\frac{\delta^2}{\delta t^2}[(\boldsymbol{x}_P\cdot\boldsymbol{e}_i')\boldsymbol{e}_i'] = \frac{\delta}{\delta t}\left[R\dot{\phi}\cos\left(\frac{\phi}{2}\right)\boldsymbol{e}_1'\right] = -\frac{R\dot{\phi}^2}{2}\sin\left(\frac{\phi}{2}\right)\boldsymbol{e}_1' \quad .$$

Problem Set 4 Solutions

Problem 1

The two ends C and D of the bar CD shown in Fig. 4.1 are confined to move in the rotating slots of the right-angled frame ABF, which is hinged at B to a car that moves to the right with a constant speed v_1. The angular speed of the frame about B is $\dot{\theta}$ and is constant for the interval of motion concerned. Moreover, the whole system is accelerated upward with a constant acceleration a_0.

Initially, at the time $t = 0$, $\gamma = \theta = 0^{\text{o}}$ and the acceleration of the system is zero.

1. Determine the velocity of the midpoint E of the bar CD.
2. Determine the velocity of E relative C.
3. Determine the acceleration of E.

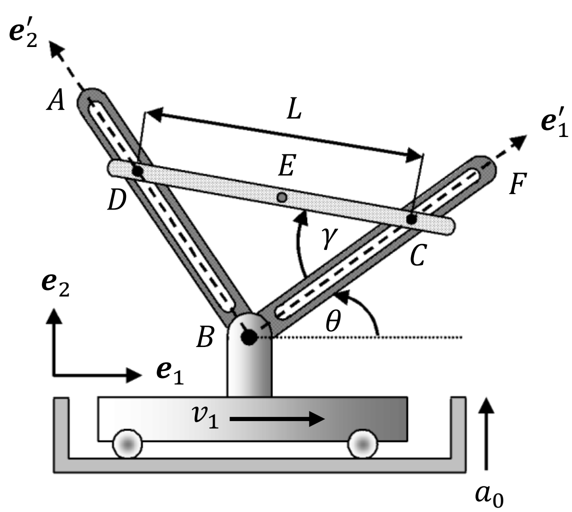

Figure 4.1

Solution:

The system $\boldsymbol{e}_i'$ rotates with the angular velocity $\boldsymbol{\omega}$, such that

$$\dot{\boldsymbol{e}}_i' = \boldsymbol{\omega} \times \boldsymbol{e}_i' \quad , \quad \boldsymbol{\omega} = \dot{\theta}\boldsymbol{e}_3 \quad .$$

Now, the position of E relative to B is given by

$$\boldsymbol{x}_{E/B} = \boldsymbol{x}_{C/B} + \boldsymbol{x}_{E/C} = L\cos(\gamma)\,\boldsymbol{e}_1' + \frac{L}{2}[-\cos(\gamma)\,\boldsymbol{e}_1' + \sin(\gamma)\,\boldsymbol{e}_2']$$

$$= \frac{L}{2}[\cos(\gamma)\,\boldsymbol{e}_1' + \sin(\gamma)\,\boldsymbol{e}_2'] \quad ,$$

so that the velocity of E relative to B becomes

$$\boldsymbol{v}_{E/B} = \frac{\delta \boldsymbol{x}_{E/B}}{\delta t} + \boldsymbol{\omega} \times \boldsymbol{x}_{E/B} = \frac{(\dot{\gamma} + \dot{\theta})L}{2}[-\sin(\gamma)\,\boldsymbol{e}_1' + \cos(\gamma)\,\boldsymbol{e}_2'] \quad .$$

Moreover, the acceleration and velocity of B take the forms

$$\boldsymbol{a}_B = a_0\boldsymbol{e}_2 \quad \Rightarrow \quad \boldsymbol{v}_B = \boldsymbol{v}_B(0) + a_0 t\boldsymbol{e}_2 = v_1\boldsymbol{e}_1 + a_0 t\boldsymbol{e}_2 \ .$$

Thus, the velocity of E reduces to

$$\boldsymbol{v}_E = \boldsymbol{v}_B + \boldsymbol{v}_{E/B} = v_1\boldsymbol{e}_1 + a_0 t\boldsymbol{e}_2 + \frac{(\dot{\gamma} + \dot{\theta})L}{2}[-\sin(\gamma)\,\boldsymbol{e}_1' + \cos(\gamma)\,\boldsymbol{e}_2'] \quad .$$

However,

$$\boldsymbol{e}_1' = \cos(\theta)\,\boldsymbol{e}_1 + \sin(\theta)\,\boldsymbol{e}_2 \quad , \quad \boldsymbol{e}_2' = -\sin(\theta)\,\boldsymbol{e}_1 + \cos(\theta)\,\boldsymbol{e}_2 \quad ,$$

such that

$$\boldsymbol{v}_E = \left[v_1 - \frac{(\dot{\gamma} + \dot{\theta})L}{2}\{\sin(\gamma)\cos(\theta) + \cos(\gamma)\sin(\theta)\}\right]\boldsymbol{e}_1$$

$$+ \left[a_0 t + \frac{(\dot{\gamma} + \dot{\theta})L}{2}\{\cos(\gamma)\cos(\theta) - \sin(\gamma)\sin(\theta)\}\right]\boldsymbol{e}_2 \quad .$$

Equivalently,

$$\boldsymbol{v}_E = \left[v_1 - \frac{(\dot{\gamma} + \dot{\theta})L}{2}\sin(\gamma + \theta)\right]\boldsymbol{e}_1 + \left[a_0 t + \frac{(\dot{\gamma} + \dot{\theta})L}{2}\cos(\gamma + \theta)\right]\boldsymbol{e}_2 \quad .$$

Furthermore, the acceleration E takes the form

$$\boldsymbol{a}_E = \dot{\boldsymbol{v}}_E = \left[-\frac{\left(\dot{\gamma}+\dot{\theta}\right)^2 L}{2}\cos(\gamma+\theta)\right]\boldsymbol{e}_1 + \left[a_0 - \frac{\left(\dot{\gamma}+\dot{\theta}\right)^2 L}{2}\sin(\gamma+\theta)\right]\boldsymbol{e}_2$$

Next, the position and velocity of C relative to B are given by

$$\boldsymbol{x}_{C/B} = L\cos(\gamma)\,\boldsymbol{e}'_1 \quad \Rightarrow \quad \boldsymbol{v}_{C/B} = \frac{\delta \boldsymbol{x}_{C/B}}{\delta t} + \boldsymbol{\omega}\times\boldsymbol{x}_{C/B} = -\dot{\gamma}L\sin(\gamma)\,\boldsymbol{e}'_1 + \dot{\theta}L\cos(\gamma)\,\boldsymbol{e}'_2 \quad .$$

Hence, the velocity of C reduces to

$$\boldsymbol{v}_C = \boldsymbol{v}_B + \boldsymbol{v}_{C/B} = v_1\boldsymbol{e}_1 + a_0 t\boldsymbol{e}_2 - \dot{\gamma}L\sin(\gamma)\,\boldsymbol{e}'_1 + \dot{\theta}L\cos(\gamma)\,\boldsymbol{e}'_2 \quad .$$

Using the transformation relations given previously, it follows that

$$\boldsymbol{v}_C = \left[v_1 - \dot{\gamma}L\sin(\gamma)\cos(\theta) - \dot{\theta}L\cos(\gamma)\sin(\theta)\right]\boldsymbol{e}_1$$

$$+ \left[a_0 t - \dot{\gamma}L\sin(\gamma)\sin(\theta) + \dot{\theta}L\cos(\gamma)\cos(\theta)\right]\boldsymbol{e}_2 \quad .$$

Consequently, the velocity of E relative to C becomes

$$\boldsymbol{v}_{E/C} = L\left[-\frac{\left(\dot{\gamma}+\dot{\theta}\right)}{2}\sin(\gamma+\theta) + \dot{\gamma}\sin(\gamma)\cos(\theta) + \dot{\theta}\cos(\gamma)\sin(\theta)\right]\boldsymbol{e}_1$$

$$+ L\left[\frac{\left(\dot{\gamma}+\dot{\theta}\right)}{2}\cos(\gamma+\theta) + \dot{\gamma}\sin(\gamma)\sin(\theta) - \dot{\theta}\cos(\gamma)\cos(\theta)\right]\boldsymbol{e}_2 \quad .$$

Problem 2

A car at latitude λ on the rotating earth drives straight north with a constant speed v, as shown in Fig. 4.2. The coordinate system $\boldsymbol{e}_i''$ is fixed to the earth which rotates about its axis $\boldsymbol{e}_2''$ once every 24 hours, and the coordinate system $\boldsymbol{e}_i'$ traces the motion of the car on the surface of the earth.

Determine the acceleration of the car.

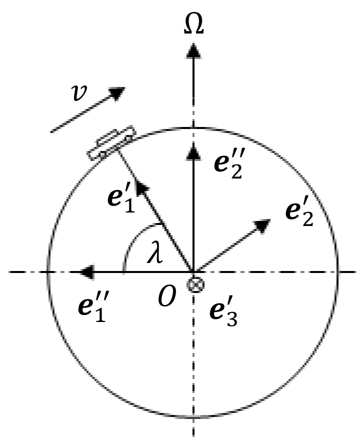

Figure 4.2

Solution:

The system $\{\boldsymbol{e}_i'', \boldsymbol{e}_i'\}$ rotate with the angular velocities $\{\boldsymbol{\Omega}, \boldsymbol{\omega}\}$, respectively, such that

$$\dot{\boldsymbol{e}}_i'' = \boldsymbol{\Omega} \times \boldsymbol{e}_i'' \quad , \quad \boldsymbol{\Omega} = \Omega \boldsymbol{e}_2'' \quad ; \quad \dot{\boldsymbol{e}}_i' = \boldsymbol{\omega} \times \boldsymbol{e}_i' \quad , \quad \boldsymbol{\omega} = \boldsymbol{\Omega} + \dot{\lambda} \boldsymbol{e}_2' = \Omega \boldsymbol{e}_2'' + \dot{\lambda} \boldsymbol{e}_2' \quad .$$

Also, these coordinate systems are related by

$$\boldsymbol{e}_1' = \cos(\lambda)\, \boldsymbol{e}_1'' + \sin(\lambda)\, \boldsymbol{e}_2'' \quad , \quad \boldsymbol{e}_2' = -\sin(\lambda)\, \boldsymbol{e}_1'' + \cos(\lambda)\, \boldsymbol{e}_2'' \quad , \quad \boldsymbol{e}_3' = \boldsymbol{e}_3'' \quad .$$

Next, the position $\boldsymbol{x}$ of the car relative to the fixed origin O should be expressed in terms of the coordinate system $\boldsymbol{e}_i''$ since the velocity of the car relative to the earth is known and given by

$$\frac{\delta[(\boldsymbol{x}\cdot\boldsymbol{e}_i'')\boldsymbol{e}_i'']}{\delta t} = v\boldsymbol{e}_2' = v[-\sin(\lambda)\,\boldsymbol{e}_1'' + \cos(\lambda)\,\boldsymbol{e}_2''] \quad .$$

To this end, denoting the radius of the earth by $R_\oplus$ it follows that

$$\boldsymbol{x} = R_\oplus \boldsymbol{e}_1' = R_\oplus[\cos(\lambda)\,\boldsymbol{e}_1'' + \sin(\lambda)\,\boldsymbol{e}_2''] \quad \Rightarrow$$

$$\frac{\delta[(\boldsymbol{x}\cdot\boldsymbol{e}_i'')\boldsymbol{e}_i'']}{\delta t} = \dot{\lambda} R_\oplus[-\sin(\lambda)\,\boldsymbol{e}_1'' + \cos(\lambda)\,\boldsymbol{e}_2''] \quad .$$

Therefore,

$$\dot{\lambda} = \frac{v}{R_\oplus} \quad .$$

Now, using Table 4.1,

	$\boldsymbol{e}_1''$	$\boldsymbol{e}_2''$	$\boldsymbol{e}_3''$
$\boldsymbol{\Omega}$	0	Ω	0
$\boldsymbol{x}$	$R_\oplus\cos(\lambda)$	$R_\oplus\sin(\lambda)$	0
$\delta\boldsymbol{x}/\delta t$	$-v\sin(\lambda)$	$v\cos(\lambda)$	0
$\boldsymbol{\Omega}\times\boldsymbol{x}$	0	0	$-\Omega R_\oplus\cos(\lambda)$
$\boldsymbol{v}$	$-v\sin(\lambda)$	$v\cos(\lambda)$	$-\Omega R_\oplus\cos(\lambda)$
$\delta\boldsymbol{v}/\delta t$	$-\frac{v^2}{R_\oplus}\cos(\lambda)$	$-\frac{v^2}{R_\oplus}\sin(\lambda)$	$\Omega v\sin(\lambda)$
$\boldsymbol{\Omega}\times\boldsymbol{v}$	$-\Omega^2 R_\oplus\cos(\lambda)$	0	$\Omega v\sin(\lambda)$
$\boldsymbol{a}$	$-\left(\frac{v^2}{R_\oplus}+\Omega^2\right)\cos(\lambda)$	$-\frac{v^2}{R_\oplus}\sin(\lambda)$	$2\Omega v\sin(\lambda)$

Table 4.1

the velocity and acceleration of the car are given, respectively, by

$$\boldsymbol{v} = v[-\sin(\lambda)\,\boldsymbol{e}_1'' + \cos(\lambda)\,\boldsymbol{e}_2''] - \Omega R_\oplus\cos(\lambda)\,\boldsymbol{e}_3'' \quad ,$$

$$\boldsymbol{a} = -\left(\frac{v^2}{R_\oplus}+\Omega^2\right)\cos(\lambda)\,\boldsymbol{e}_1'' - \frac{v^2}{R_\oplus}\sin(\lambda)\,\boldsymbol{e}_2'' + 2\Omega v\sin(\lambda)\,\boldsymbol{e}_3'' \quad .$$

Problem 3

Consider the assembly shown in Fig. 4.3. The motor turns the disk at the constant speed $\dot{\phi}$. The motor is also swiveling about the horizontal axis that passes through the point B at the constant speed $\dot{\theta}$. Simultaneously, the assembly is rotating about the vertical axis $\boldsymbol{e}_2''$ at the constant rate $\dot{\psi}$. The system $\boldsymbol{e}_i'$ is fixed to the shaft BC, such that $\{\boldsymbol{e}_1', \boldsymbol{e}_2', \boldsymbol{e}_2''\}$ are always in the same plane.

1. Determine the angular acceleration of the disk.
2. Determine the velocity and acceleration of the center C of the disk.

Next, consider the point P which is located at a distance R from the center C of the disk.

3. Determine the velocity and acceleration of P relative to C.
4. Detertmine the velocity and acceleration of P.

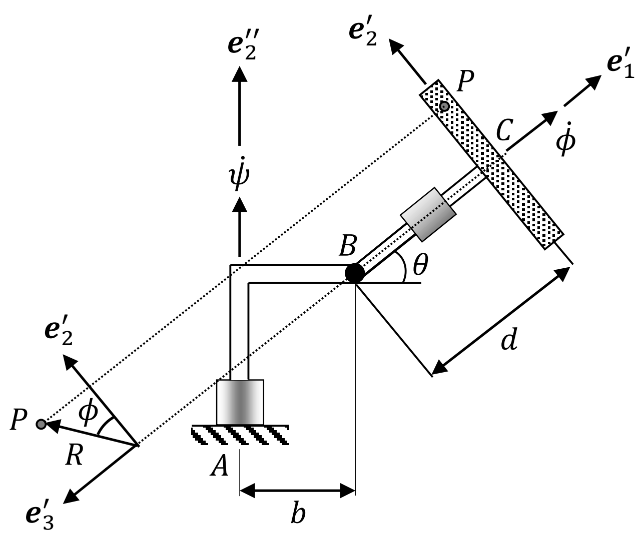

Figure 4.3

Solution:

The system $\{\boldsymbol{e}_i'', \boldsymbol{e}_i'\}$ rotate with the angular velocities $\{\boldsymbol{\Omega}, \boldsymbol{\Lambda}\}$, respectively, such that

$$\dot{\boldsymbol{e}}_i'' = \boldsymbol{\Omega} \times \boldsymbol{e}_i'' \quad , \quad \boldsymbol{\Omega} = \dot{\psi}\boldsymbol{e}_2'' \quad ; \quad \dot{\boldsymbol{e}}_i' = \boldsymbol{\Lambda} \times \boldsymbol{e}_i' \quad , \quad \boldsymbol{\Lambda} = \boldsymbol{\Omega} + \dot{\theta}\boldsymbol{e}_3' = \dot{\psi}\boldsymbol{e}_2'' + \dot{\theta}\boldsymbol{e}_3' \quad .$$

Also, these coordinate systems are related by (see Fig. 4.4)

$$\boldsymbol{e}_1' = \cos(\theta)\, \boldsymbol{e}_1'' + \sin(\theta)\, \boldsymbol{e}_2'' \quad , \quad \boldsymbol{e}_2' = -\sin(\theta)\, \boldsymbol{e}_1'' + \cos(\theta)\, \boldsymbol{e}_2'' \quad , \quad \boldsymbol{e}_3' = \boldsymbol{e}_3'' \quad .$$

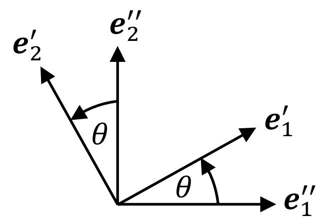

Figure 4.4

Therefore, the angular velocity $\boldsymbol{\omega}$ of the disk is given by

$$\boldsymbol{\omega} = \dot{\phi}\boldsymbol{e}_1' + \boldsymbol{\Lambda} = \dot{\phi}\boldsymbol{e}_1' + \dot{\psi}\boldsymbol{e}_2'' + \dot{\theta}\boldsymbol{e}_3' = \dot{\phi}\cos(\theta)\, \boldsymbol{e}_1'' + \left[\dot{\phi}\sin(\theta) + \dot{\psi}\right]\boldsymbol{e}_2'' + \dot{\theta}\boldsymbol{e}_3'' \quad .$$

Moreover, using Table 4.2,

	$\boldsymbol{e}_1''$	$\boldsymbol{e}_2''$	$\boldsymbol{e}_3''$
$\boldsymbol{\Omega}$	0	$\dot{\psi}$	0
$\boldsymbol{\omega}$	$\dot{\phi}\cos(\theta)$	$\dot{\phi}\sin(\theta) + \dot{\psi}$	$\dot{\theta}$
$\delta\boldsymbol{\omega}/\delta t$	$-\dot{\phi}\dot{\theta}\sin(\theta)$	$\dot{\phi}\dot{\theta}\cos(\theta)$	0
$\boldsymbol{\Omega} \times \boldsymbol{\omega}$	$\dot{\psi}\dot{\theta}$	0	$-\dot{\psi}\dot{\theta}\cos(\theta)$
$\dot{\boldsymbol{\omega}}$	$\dot{\theta}\left[\dot{\psi} - \dot{\phi}\sin(\theta)\right]$	$\dot{\phi}\dot{\theta}\cos(\theta)$	$-\dot{\psi}\dot{\theta}\cos(\theta)$

Table 4.2

the angular acceleration $\dot{\boldsymbol{\omega}}$ of the disk takes the form

$$\dot{\boldsymbol{\omega}} = \dot{\theta}\left[\dot{\psi} - \dot{\phi}\sin(\theta)\right]\boldsymbol{e}_1'' + \dot{\phi}\dot{\theta}\cos(\theta)\, \boldsymbol{e}_2'' - \dot{\psi}\dot{\theta}\cos(\theta)\, \boldsymbol{e}_3'' \quad .$$

Next, the position of the center C of the disk is given by

$$\boldsymbol{x}_C = \boldsymbol{x}_B + \boldsymbol{x}_{C/B} = b\boldsymbol{e}_1'' + d\boldsymbol{e}_1' = [b + d\cos(\theta)]\boldsymbol{e}_1'' + d\sin(\theta)\, \boldsymbol{e}_2''$$

Thus, using Table 4.3,

	$\boldsymbol{e}_1''$	$\boldsymbol{e}_2''$	$\boldsymbol{e}_3''$
$\boldsymbol{\Omega}$	0	$\dot{\psi}$	0
$\boldsymbol{x}_C$	$b + d\cos(\theta)$	$d\sin(\theta)$	0
$\delta\boldsymbol{x}_C/\delta t$	$-\dot{\theta}d\sin(\theta)$	$\dot{\theta}d\cos(\theta)$	0
$\boldsymbol{\Omega}\times\boldsymbol{x}_C$	0	0	$-\dot{\psi}[b + d\cos(\theta)]$
$\boldsymbol{v}_C$	$-\dot{\theta}d\sin(\theta)$	$\dot{\theta}d\cos(\theta)$	$-\dot{\psi}[b + d\cos(\theta)]$
$\delta\boldsymbol{v}_C/\delta t$	$-\dot{\theta}^2 d\cos(\theta)$	$-\dot{\theta}^2 d\sin(\theta)$	$\dot{\psi}\dot{\theta}d\sin(\theta)$
$\boldsymbol{\Omega}\times\boldsymbol{v}_C$	$-\dot{\psi}^2[b + d\cos(\theta)]$	0	$\dot{\psi}\dot{\theta}d\sin(\theta)$
$\boldsymbol{a}_C$	$-\dot{\psi}^2[b + d\cos(\theta)]$ $-\dot{\theta}^2 d\cos(\theta)$	$-\dot{\theta}^2 d\sin(\theta)$	$2\dot{\psi}\dot{\theta}d\sin(\theta)$

Table 4.3

the velocity and acceleration of C are given, respectively, by

$$\boldsymbol{v}_C = -\dot{\theta}d\sin(\theta)\,\boldsymbol{e}_1'' + \dot{\theta}d\cos(\theta)\,\boldsymbol{e}_2'' - \dot{\psi}[b + d\cos(\theta)]\boldsymbol{e}_3'' \quad ,$$

$$\boldsymbol{a}_C = -\left[\dot{\psi}^2\{b + d\cos(\theta)\} + \dot{\theta}^2 d\cos(\theta)\right]\boldsymbol{e}_1'' - \dot{\theta}^2 d\sin(\theta)\,\boldsymbol{e}_2'' + 2\dot{\psi}\dot{\theta}d\sin(\theta)\,\boldsymbol{e}_3'' \quad .$$

Next, the position of the point P relative to C takes the form

$$\boldsymbol{x}_{P/C} = R[\cos(\phi)\,\boldsymbol{e}_2' + \sin(\phi)\,\boldsymbol{e}_3']$$

$$= R[-\sin(\theta)\cos(\phi)\,\boldsymbol{e}_1'' + \cos(\phi)\cos(\theta)\,\boldsymbol{e}_2'' + \sin(\phi)\,\boldsymbol{e}_3''] \quad .$$

Equivalently,

$$\boldsymbol{x}_{P/C} = R\left[-\frac{1}{2}\{\sin(\theta+\phi) + \sin(\theta-\phi)\}\boldsymbol{e}_1'' + \frac{1}{2}\{\cos(\theta+\phi) + \cos(\theta-\phi)\}\boldsymbol{e}_2'' + \sin(\phi)\,\boldsymbol{e}_3''\right]$$

For convenience, denote

$$\alpha = \theta + \phi \quad , \quad \beta = \theta - \phi \quad ,$$

such that

$$\boldsymbol{x}_{P/C} = R\left[-\frac{1}{2}\{\sin(\alpha) + \sin(\beta)\}\boldsymbol{e}_1'' + \frac{1}{2}\{\cos(\alpha) + \cos(\beta)\}\boldsymbol{e}_2'' + \sin(\phi)\,\boldsymbol{e}_3''\right] \quad .$$

Consequently, using Table 4.4, the velocity and acceleration of P relative to C are given, respectively, by

$$\boldsymbol{v}_{P/C} = R\left[\dot{\psi}\sin(\phi) - \frac{\dot{\alpha}}{2}\cos(\alpha) - \frac{\dot{\beta}}{2}\cos(\beta)\right]\boldsymbol{e}_1'' - \frac{R}{2}\left[\dot{\alpha}\sin(\alpha) + \dot{\beta}\sin(\beta)\right]\boldsymbol{e}_2''$$

$$+ R\left[\dot{\phi}\cos(\phi) + \frac{\dot{\psi}}{2}\{\sin(\alpha) + \sin(\beta)\}\right]\boldsymbol{e}_3'' \quad ,$$

$$\boldsymbol{a}_{P/C} = R\left[\frac{\dot{\psi}^2}{2}\{\sin(\alpha) + \cos(\alpha)\} + \frac{\dot{\alpha}^2}{2}\sin(\alpha) + \frac{\dot{\beta}^2}{2}\sin(\beta) + 2\dot{\psi}\dot{\phi}\cos(\phi)\right]\boldsymbol{e}_1''$$

$$- \frac{R}{2}\left[\dot{\alpha}^2\cos(\alpha) + \dot{\beta}^2\cos(\beta)\right]\boldsymbol{e}_2''$$

$$+ R\left[\left(\dot{\psi}^2 - \dot{\phi}^2\right)\sin(\phi) + \dot{\psi}\dot{\alpha}\cos(\alpha) + \dot{\psi}\dot{\beta}\cos(\beta)\right]\boldsymbol{e}_3'' \quad ,$$

with,

$$\alpha = \theta + \phi \quad , \quad \beta = \theta - \phi \quad ; \quad \dot{\alpha} = \dot{\theta} + \dot{\phi} \quad , \quad \dot{\beta} = \dot{\theta} - \dot{\phi} \quad .$$

Next, using the expressions for $\{\boldsymbol{v}_C, \boldsymbol{v}_{P/C}, \boldsymbol{a}_C, \boldsymbol{a}_{P/C}\}$ obtained previously, the velocity and acceleration of P become

$$\boldsymbol{v}_P = \boldsymbol{v}_{P/C} + \boldsymbol{v}_C \quad ; \quad \boldsymbol{a}_P = \boldsymbol{a}_{P/C} + \boldsymbol{a}_C \quad .$$

	$\boldsymbol{e}_1''$	$\boldsymbol{e}_2''$	$\boldsymbol{e}_3''$
$\boldsymbol{\Omega}$	0	$\dot{\psi}$	0
$\boldsymbol{x}_{P/C}$	$-\frac{R}{2}\sin(\alpha) - \frac{R}{2}\sin(\beta)$	$\frac{R}{2}\cos(\alpha) + \frac{R}{2}\cos(\beta)$	$R\sin(\phi)$
$\delta\boldsymbol{x}_{P/C}/\delta t$	$-\frac{\dot{\alpha}R}{2}\cos(\alpha) - \frac{\dot{\beta}R}{2}\cos(\beta)$	$-\frac{\dot{\alpha}R}{2}\sin(\alpha) - \frac{\dot{\beta}R}{2}\sin(\beta)$	$\dot{\phi}R\cos(\phi)$
$\boldsymbol{\Omega}\times\boldsymbol{x}_{P/C}$	$\dot{\psi}R\sin(\phi)$	0	$\frac{\dot{\psi}R}{2}\sin(\alpha) + \frac{\dot{\psi}R}{2}\sin(\beta)$
$\boldsymbol{v}_{P/C}$	$-\frac{\dot{\alpha}R}{2}\cos(\alpha) - \frac{\dot{\beta}R}{2}\cos(\beta) + \dot{\psi}R\sin(\phi)$	$-\frac{\dot{\alpha}R}{2}\sin(\alpha) - \frac{\dot{\beta}R}{2}\sin(\beta)$	$\dot{\phi}R\cos(\phi) + \frac{\dot{\psi}R}{2}\sin(\alpha) + \frac{\dot{\psi}R}{2}\sin(\beta)$
$\delta\boldsymbol{v}_{P/C}/\delta t$	$\frac{\dot{\alpha}^2R}{2}\sin(\alpha) + \frac{\dot{\beta}^2R}{2}\sin(\beta) + \dot{\psi}\dot{\phi}R\cos(\phi)$	$-\frac{\dot{\alpha}^2R}{2}\cos(\alpha) - \frac{\dot{\beta}^2R}{2}\cos(\beta)$	$-\dot{\phi}^2R\sin(\phi) + \frac{\dot{\psi}\dot{\alpha}R}{2}\cos(\alpha) + \frac{\dot{\psi}\dot{\beta}R}{2}\cos(\beta)$
$\boldsymbol{\Omega}\times\boldsymbol{v}_{P/C}$	$\dot{\psi}\dot{\phi}R\cos(\phi) + \frac{\dot{\psi}^2R}{2}[\sin(\alpha) + \sin(\beta)]$	0	$\frac{\dot{\psi}\dot{\alpha}R}{2}\cos(\alpha) + \frac{\dot{\psi}\dot{\beta}R}{2}\cos(\beta) + \dot{\psi}^2R\sin(\phi)$
$\boldsymbol{a}_{P/C}$	$\frac{\dot{\alpha}^2R}{2}\sin(\alpha) + \frac{\dot{\beta}^2R}{2}\sin(\beta) + 2\dot{\psi}\dot{\phi}R\cos(\phi) + \frac{\dot{\psi}^2R}{2}[\sin(\alpha) + \cos(\alpha)]$	$-\frac{\dot{\alpha}^2R}{2}\cos(\alpha) - \frac{\dot{\beta}^2R}{2}\cos(\beta)$	$-\dot{\phi}^2R\sin(\phi) + \dot{\psi}\dot{\alpha}R\cos(\alpha) + \dot{\psi}\dot{\beta}R\cos(\beta) + \dot{\psi}^2R\sin(\phi)$

Table 4.4

Problem Set 5 Solutions

Problem 1

End A of the rigid link AB is confined to move in the negative $\boldsymbol{e}_1$ direction while end B is confined to move along the vertical axis. Determine the component $\boldsymbol{\omega}_n$ normal to AB of the angular velocity of the link as it passes the position shown in Fig. 5.1 with the speed $v_A = 0.3$ [m/s].

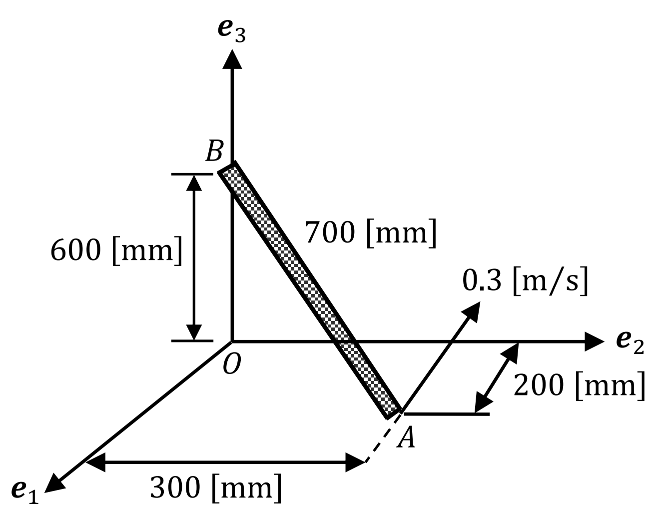

Figure 5.1

Solution:

The position and velocity of B relative to A are given, respectively, by

$$\boldsymbol{x}_{B/A} = 0.6\boldsymbol{e}_3 - (0.2\boldsymbol{e}_1 + 0.3\boldsymbol{e}_2) = -0.2\boldsymbol{e}_1 - 0.3\boldsymbol{e}_2 + 0.6\boldsymbol{e}_3 \text{ [m]} \quad ,$$

$$\boldsymbol{v}_{B/A} = v_B\boldsymbol{e}_3 - (-0.3\boldsymbol{e}_1) = \boldsymbol{v}_{B/A} = 0.3\boldsymbol{e}_1 + v_B\boldsymbol{e}_3 \text{ [m/s]} \quad .$$

Now, recall that $\boldsymbol{v}_{B/A}$ is perpendicular to $\boldsymbol{x}_{B/A}$, i.e.,

$$\boldsymbol{v}_{B/A} \cdot \boldsymbol{x}_{B/A} = 0 \quad .$$

Therefore,

$$(0.3\boldsymbol{e}_1 + v_B\boldsymbol{e}_3) \cdot (-0.2\boldsymbol{e}_1 - 0.3\boldsymbol{e}_2 + 0.6\boldsymbol{e}_3) = 0 \quad \Rightarrow \quad v_B = 0.1 \text{ [m/s]} \quad .$$

Furthermore,

$$\boldsymbol{v}_{B/A} = 0.3\boldsymbol{e}_1 + 0.1\boldsymbol{e}_3 \text{ [m/s]} \quad .$$

Consequently, $\boldsymbol{\omega}_n$ takes the form

$$\boldsymbol{\omega}_n = \frac{\boldsymbol{e}_{B/A} \times \boldsymbol{v}_{B/A}}{|\boldsymbol{x}_{B/A}|} = \frac{\boldsymbol{x}_{B/A} \times \boldsymbol{v}_{B/A}}{|\boldsymbol{x}_{B/A}|^2} = \frac{1}{49}(-3\boldsymbol{e}_1 + 20\boldsymbol{e}_2 + 9\boldsymbol{e}_3) \text{ [rad/s]} \quad .$$

Problem 2

Determine the angular velocity of the telescoping link BC for the position shown in Fig. 5.2, where the driving links AB and CD have the angular velocities indicate.

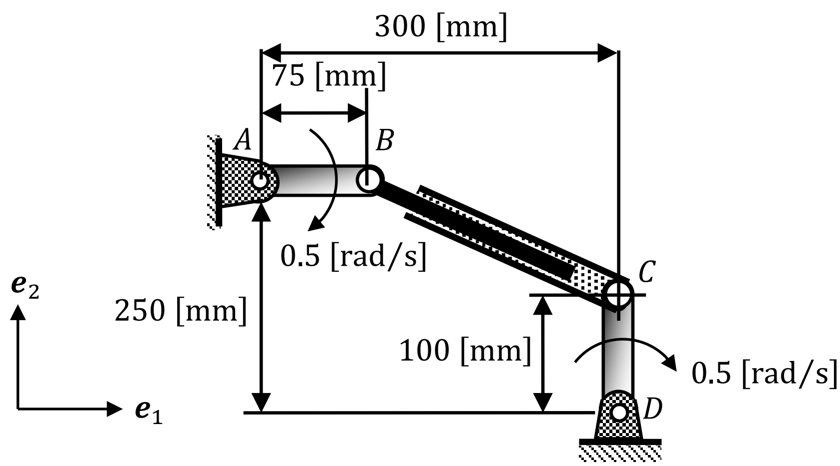

Figure 5.2

Solution:

The velocities of B and C are given, respectively, by

$$\boldsymbol{v}_B = \boldsymbol{v}_{B/A} = \boldsymbol{\omega}^{AB} \times \boldsymbol{x}_{B/A} = -0.5\boldsymbol{e}_3 \times 0.075\boldsymbol{e}_1 = -0.0375\boldsymbol{e}_2 \text{ [rad/s]} \quad ,$$

$$\boldsymbol{v}_C = \boldsymbol{v}_{C/D} = \boldsymbol{\omega}^{CD} \times \boldsymbol{x}_{C/D} = -0.5\boldsymbol{e}_3 \times 0.1\boldsymbol{e}_2 = 0.05\boldsymbol{e}_1 \text{ [rad/s]} \quad .$$

Thus, the angular velocity $\boldsymbol{\omega}^{BC}$ of the telescoping link BC takes the form

$$\boldsymbol{\omega}^{BC} = \frac{\boldsymbol{x}_{B/C} \times \boldsymbol{v}_{B/C}}{\left|\boldsymbol{x}_{B/C}\right|^2} \quad , \quad \boldsymbol{x}_{B/C} = -0.225\boldsymbol{e}_1 + 0.15\boldsymbol{e}_2 \text{ [m]} \quad ,$$

$$\boldsymbol{v}_{B/C} = -(0.05\boldsymbol{e}_1 + 0.0375\boldsymbol{e}_2) \text{ [rad/s]} \quad \Rightarrow$$

$$\boldsymbol{\omega}^{BC} = 0.218\boldsymbol{e}_3 \text{ [rad/s]} \quad .$$

Problem 3

The slotted wheel of radius $R = 60$ [cm] shown in Fig. 5.3 rolls on the horizontal plane in a circle of radius $L = 60$ [cm]. The wheel shaft BC is pivoted about an axis through the point B at one end, and is driven by the vertical shaft at the constant rate $\dot{\phi} = 4$ [rad/s] about the vertical axis. The slider P moves in the slot and its radial distance relative to the center of the disk is denoted by $s(t)$. The system $\{\boldsymbol{e}_i'', \boldsymbol{e}_i'\}$ are fixed to BC and the wheel, respectively, and they are always in the same plane with θ denoting the angle between the axes $\boldsymbol{e}_1'$ and $\boldsymbol{e}_1''$.

1. Determine the angular velocity of the disk.
2. Determine the angular velocity of the disk for the position $\theta = 30^{\mathrm{o}}$.
3. Determine the velocity and acceleration of the slider P.

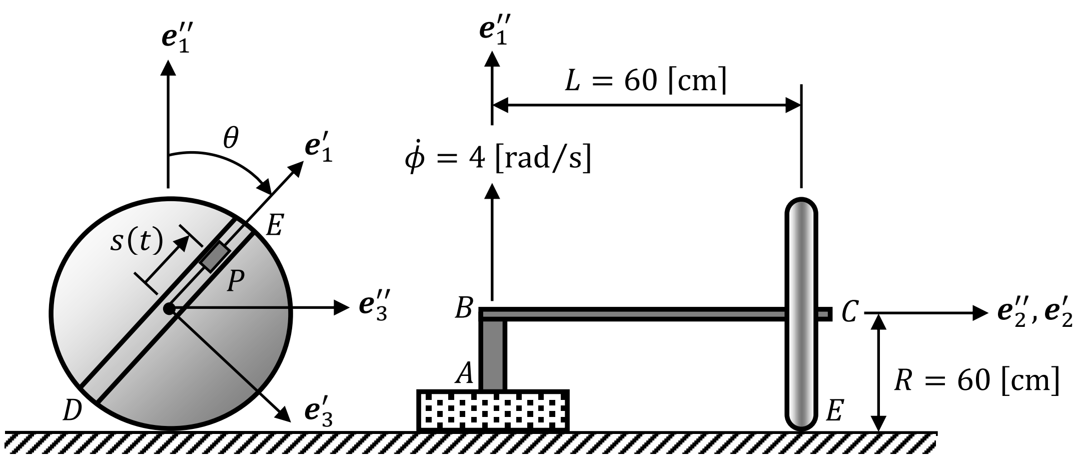

Figure 5.3

Solution:

Let $\{\boldsymbol{\omega}, \boldsymbol{\Omega}\}$ denote the angular velocities of the wheel and its shaft, respectively, such that

$$\dot{\boldsymbol{e}}_i'' = \boldsymbol{\Omega} \times \boldsymbol{e}_i'' \quad , \quad \boldsymbol{\Omega} = \dot{\phi}\boldsymbol{e}_1'' \quad ,$$

$$\dot{\boldsymbol{e}}_i' = \boldsymbol{\omega} \times \boldsymbol{e}_i' \quad , \quad \boldsymbol{\omega} = \boldsymbol{\Omega} + \dot{\psi}\boldsymbol{e}_2'' = \dot{\phi}\boldsymbol{e}_1'' + \dot{\psi}\boldsymbol{e}_2'' \quad ,$$

where $\dot{\psi}$ is angular speed of the wheel relative to BC.

Next, since the wheel rolls without slipping on the horizontal plane, it follows that the velocity $\boldsymbol{v}_{E/E'}$ of the point E fixed to the disk relative to the point E' fixed to the horizontal plane vanishes, such that

$$\boldsymbol{v}_{E/E'} = \boldsymbol{0} \quad , \quad \boldsymbol{v}_{E'} = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{v}_E = \boldsymbol{0} \quad .$$

Moreover, $\boldsymbol{v}_E$ can be expressed as

$$\boldsymbol{v}_E = \boldsymbol{v}_C + \boldsymbol{\omega} \times \boldsymbol{x}_{E/C} \quad ,$$

where,

$$\boldsymbol{v}_C = \boldsymbol{v}_{C/B} = \boldsymbol{\Omega} \times \boldsymbol{x}_{C/B} = \dot{\phi}e_1'' \times L\boldsymbol{e}_2'' = \dot{\phi}L\boldsymbol{e}_3'' \quad .$$

Therefore,

$$\boldsymbol{v}_E = \dot{\phi}L\boldsymbol{e}_3'' + \left(\dot{\phi}\boldsymbol{e}_1'' + \dot{\psi}\boldsymbol{e}_2''\right) \times (-R\boldsymbol{e}_1'') = \left(\dot{\phi}L + \dot{\psi}R\right)\boldsymbol{e}_3'' = \boldsymbol{0} \quad \Rightarrow$$

$$\dot{\psi} = -\frac{\dot{\phi}L}{R} = -4 \text{ [rad/s]} \quad .$$

Substituting this value into $\boldsymbol{\omega}$ yields

$$\boldsymbol{\omega} = 4(\boldsymbol{e}_1'' - \boldsymbol{e}_2'') \text{ [rad/s]} \quad .$$

Furthermore, expressing $\boldsymbol{\omega}$ in terms of the coordinate system $\boldsymbol{e}_i'$, it follows that

$$\boldsymbol{\omega} = 4[\cos(\theta)\,\boldsymbol{e}_1' - \boldsymbol{e}_2' - \sin(\theta)\,\boldsymbol{e}_3'] \text{ [rad/s]} \quad .$$

Hence, the value $\boldsymbol{\omega}(30^{\text{o}})$ of $\boldsymbol{\omega}$ at the position $\theta = 30^{\text{o}}$ is given by

$$\boldsymbol{\omega}(30^{\mathrm{o}}) = 4\left(\frac{\sqrt{3}}{2}\boldsymbol{e}_1' - \boldsymbol{e}_2' - \frac{1}{2}\boldsymbol{e}_3'\right)\ [\mathrm{rad/s}] \quad .$$

Next, the velocity $\boldsymbol{v}_P$ and acceleration $\boldsymbol{a}_P$ of the slider P take the forms

$$\boldsymbol{v}_P = \boldsymbol{v}_C + \boldsymbol{v}_{P/C} \quad , \quad \boldsymbol{v}_C = \dot{\phi} L \boldsymbol{e}_3'' \quad ;$$

$$\boldsymbol{a}_P = \boldsymbol{a}_C + \boldsymbol{a}_{P/C} \quad , \quad \boldsymbol{a}_C = \dot{\phi} L\left(\dot{\phi}\boldsymbol{e}_1'' \times \boldsymbol{e}_3''\right) = -\dot{\phi}^2 L \boldsymbol{e}_2'' \quad .$$

where the velocity $\boldsymbol{v}_{P/C}$ and acceleration $\boldsymbol{a}_{P/C}$ of P relative to C are given in Table 5.1.

	$\boldsymbol{e}_1''$	$\boldsymbol{e}_2''$	$\boldsymbol{e}_3''$
$\boldsymbol{\Omega}$	$\dot{\phi}$	0	0
$\boldsymbol{x}_{P/C}$	$s\cos(\theta)$	$s\sin(\theta)$	0
$\delta\boldsymbol{x}_{P/C}/\delta t$	$\dot{s}\cos(\theta) - \dot{\theta}s\sin(\theta)$	$\dot{s}\sin(\theta) + \dot{\theta}s\cos(\theta)$	0
$\boldsymbol{\Omega} \times \boldsymbol{x}_{P/C}$	0	0	$\dot{\phi}s\sin(\theta)$
$\boldsymbol{v}_{P/C}$	$\dot{s}\cos(\theta) - \dot{\theta}s\sin(\theta)$	$\dot{s}\sin(\theta) + \dot{\theta}s\cos(\theta)$	$\dot{\phi}s\sin(\theta)$
$\delta\boldsymbol{v}_{P/C}/\delta t$	$\left(\ddot{s} - \dot{\theta}^2 s\right)\cos(\theta)$ $- \left(\ddot{\theta}s + 2\dot{\theta}\dot{s}\right)\sin(\theta)$	$\left(\ddot{s} - \dot{\theta}^2 s\right)\sin(\theta)$ $+ \left(\ddot{\theta}s + 2\dot{\theta}\dot{s}\right)\cos(\theta)$	$\dot{\phi}\left[\dot{s}\sin(\theta)\right.$ $\left.+ \dot{\theta}s\cos(\theta)\right]$
$\boldsymbol{\Omega} \times \boldsymbol{v}_{P/C}$	0	$-\dot{\phi}^2 s\sin(\theta)$	$\dot{\phi}\left[\dot{s}\sin(\theta)\right.$ $\left.+ \dot{\theta}s\cos(\theta)\right]$
$\boldsymbol{a}_{P/C}$	$\left(\ddot{s} - \dot{\theta}^2 s\right)\cos(\theta)$ $- \left(\ddot{\theta}s + 2\dot{\theta}\dot{s}\right)\sin(\theta)$	$\left[\ddot{s} - \left(\dot{\theta}^2 + \dot{\phi}^2\right)s\right]\sin(\theta)$ $+ \left(\ddot{\theta}s + 2\dot{\theta}\dot{s}\right)\cos(\theta)$	$2\dot{\phi}\left[\dot{s}\sin(\theta)\right.$ $\left.+ \dot{\theta}s\cos(\theta)\right]$

Table 5.1

Problem 4

The hollow curved member OE shown in Fig. 5.4 rotates counterclockwise at a constant rate $\dot{\phi} = 2$ [rad/s], and the pin A causes the link BC to rotate as well. For the instant when

$$\theta = 30^{o} \quad , \quad \beta = 45^{o} \quad , \quad H = 280 \text{ [mm]} \quad , \quad L = 120 \text{ [mm]} \quad ,$$

where β is the angle between the vertical axis and the tangent to OE at A, determine the velocity of end B of the link BC.

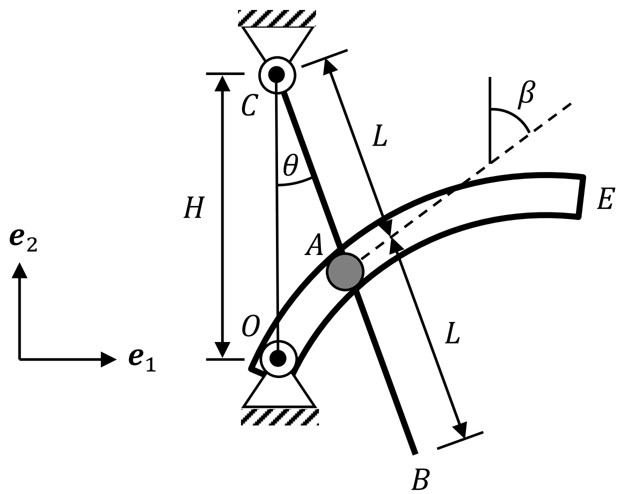

Figure 5.4

Solution:

The velocity of the pin A fixed to the link BC is given by

$$\boldsymbol{x}_{A/C} = L[\sin(\theta)\,\boldsymbol{e}_1 - \cos(\theta)\,\boldsymbol{e}_2] \quad \Rightarrow \quad \boldsymbol{v}_A = \dot{\theta}L[\cos(\theta)\,\boldsymbol{e}_1 + \sin(\theta)\,\boldsymbol{e}_2] \quad .$$

On the other hand, the velocity $\boldsymbol{v}_{A'}$ of the point A' fixed to the curved member OE which is instantaneously in contact with A takes the form

$$\boldsymbol{x}_{A'/O} = \boldsymbol{x}_{C/O} + \boldsymbol{x}_{A'/C} = L\sin(\theta)\,\boldsymbol{e}_1 + [H - L\cos(\theta)]\boldsymbol{e}_2 \quad \Rightarrow$$

$$\boldsymbol{v}_{A'} = \dot{\phi}\boldsymbol{e}_3 \times \boldsymbol{x}_{A/O} = \dot{\phi}[-\{H - L\cos(\theta)\}\boldsymbol{e}_1 + L\sin(\theta)\,\boldsymbol{e}_2] \quad .$$

Next, since the pin A is confined to move along OE, it follows that

$$\boldsymbol{v}_{A/A'} \cdot \boldsymbol{n} = 0 \quad ,$$

where $\boldsymbol{n}$ is the unit normal to OE at A. Therefore, with the help of the unit tangent $\boldsymbol{t}$ to OE at A, i.e.,

$$\boldsymbol{t} = \cos(\beta)\,\boldsymbol{e}_1 + \sin(\beta)\,\boldsymbol{e}_2 \quad ,$$

it follows that

$$\boldsymbol{n} = \boldsymbol{e}_3 \times \boldsymbol{t} = -\sin(\beta)\,\boldsymbol{e}_1 + \cos(\beta)\,\boldsymbol{e}_2 \quad .$$

Consequently, the angular speed $\dot{\theta}$ of the link BC is given by

$$-\left[\dot{\theta}L\cos(\theta) + \dot{\phi}\{H - L\cos(\theta)\}\right]\sin(\beta) + \left[\dot{\theta}L\sin(\theta) - \dot{\phi}L\sin(\theta)\right]\cos(\beta) = 0 \quad \Rightarrow$$

$$\dot{\theta} = -\dot{\phi}\left[\frac{\sin(\theta) + \left\{\frac{H}{L} - \cos(\theta)\right\}\tan(\beta)}{\cos(\theta)\tan(\beta) - \sin(\theta)}\right] = -10.75\ [\mathrm{rad/s}] \quad .$$

Next, the velocity $\boldsymbol{v}_B$ of end B of the link BC takes the form

$$\boldsymbol{x}_{B/C} = 2L[\sin(\theta)\,\boldsymbol{e}_1 - \cos(\theta)\,\boldsymbol{e}_2] \quad \Rightarrow$$

$$\boldsymbol{v}_B = 2\dot{\theta}L[\cos(\theta)\,\boldsymbol{e}_1 + \sin(\theta)\,\boldsymbol{e}_2] = -2.234\boldsymbol{e}_1 - 1.290\boldsymbol{e}_2\ [\mathrm{m/s}] \quad .$$

Problem Set 6 Solutions

Problem 1

Fig. 6.1 shows an astronaut training facility. The drum swivels about the horizontal axis $\boldsymbol{e}_1''$ that passes through the hinge A at the rate $\dot{\beta}$. The training room is located inside the drum and it rotates about the axis $\boldsymbol{e}_1'$ at the rate $\dot{\psi}$. Simultaneously, the training facility rotates about the vertical axis $\boldsymbol{e}_2''$ at the rate Ω . At the instant when

$$\beta = 0 \quad , \quad \dot{\beta} = 0.9 \text{ [rad/s]} \quad , \quad \Omega = 0.2 \text{ [rad/s]} \quad , \quad \dot{\psi} = 0.9 \text{ [rad/s]} \quad ,$$

determine the angular velocity and acceleration of the training room.

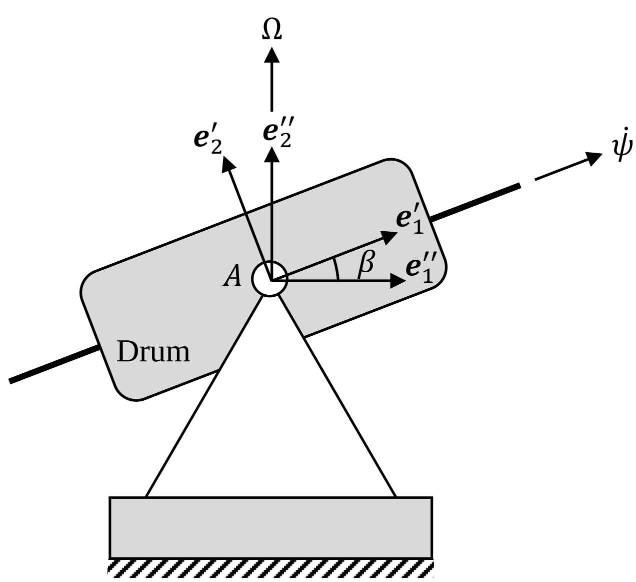

Figure 6.1

Solution:

Let $\{\boldsymbol{\Omega}, \boldsymbol{\omega}\}$ denote the angular velocities of the coordinate systems $\{\boldsymbol{e}_i'', \boldsymbol{e}_i'\}$, such that

$$\dot{\boldsymbol{e}}_i'' = \boldsymbol{\Omega} \times \boldsymbol{e}_i'' \quad , \quad \boldsymbol{\Omega} = \Omega \boldsymbol{e}_2'' \quad ; \quad \dot{\boldsymbol{e}}_i' = \boldsymbol{\omega} \times \boldsymbol{e}_i' \quad , \quad \boldsymbol{\omega} = \boldsymbol{\Omega} + \dot{\beta}\boldsymbol{e}_3' = \Omega \boldsymbol{e}_2'' + \dot{\beta}\boldsymbol{e}_3' \quad .$$

Moreover, these coordinate systems are related by

$$\boldsymbol{e}_1' = \cos(\beta)\,\boldsymbol{e}_1'' + \sin(\beta)\,\boldsymbol{e}_2'' \quad , \quad \boldsymbol{e}_2' = -\sin(\beta)\,\boldsymbol{e}_1'' + \cos(\beta)\,\boldsymbol{e}_2'' \quad , \quad \boldsymbol{e}_3' = \boldsymbol{e}_3'' \quad .$$

Next, the angular velocity $\boldsymbol{\Lambda}$ of the training room takes the form

$$\boldsymbol{\Lambda} = \boldsymbol{\omega} + \dot{\psi}\boldsymbol{e}_1' = \dot{\psi}\boldsymbol{e}_1' + \Omega\boldsymbol{e}_2'' + \dot{\beta}\boldsymbol{e}_3' = \dot{\psi}\cos(\beta)\,\boldsymbol{e}_1'' + \left[\Omega + \dot{\psi}\sin(\beta)\right]\boldsymbol{e}_2'' + \dot{\beta}\boldsymbol{e}_3'' \quad .$$

Substituting the values given previously, it follows that

$$\boldsymbol{\Omega} = 0.9\boldsymbol{e}_1'' + 0.2\boldsymbol{e}_2'' + 0.9\boldsymbol{e}_3'' \ [\mathrm{rad/s}] \quad .$$

Now, using Table 6.1,

	$\boldsymbol{e}_1''$	$\boldsymbol{e}_2''$	$\boldsymbol{e}_3''$
$\boldsymbol{\Omega}$	0	Ω	0
$\boldsymbol{\Lambda}$	$\dot{\psi}\cos(\beta)$	$\Omega + \dot{\psi}\sin(\beta)$	$\dot{\beta}$
$\delta\boldsymbol{\Lambda}/\delta t$	$\ddot{\psi}\cos(\beta) - \dot{\psi}\dot{\beta}\sin(\beta)$	$\dot{\Omega} + \ddot{\psi}\sin(\beta) + \dot{\psi}\dot{\beta}\cos(\beta)$	$\ddot{\beta}$
$\boldsymbol{\Omega} \times \boldsymbol{\Lambda}$	$\Omega\dot{\beta}$	0	$-\Omega\dot{\psi}\cos(\beta)$
$\dot{\boldsymbol{\Lambda}}$	$\ddot{\psi}\cos(\beta) - \dot{\psi}\dot{\beta}\sin(\beta) + \Omega\dot{\beta}$	$\dot{\Omega} + \ddot{\psi}\sin(\beta) + \dot{\psi}\dot{\beta}\cos(\beta)$	$\ddot{\beta} - \Omega\dot{\psi}\cos(\beta)$

Table 6.1

the angular velocity $\dot{\boldsymbol{\Lambda}}$ of the training room becomes

$$\dot{\boldsymbol{\Lambda}} = 0.18\boldsymbol{e}_1'' + 0.81\boldsymbol{e}_2'' - 0.18\boldsymbol{e}_3'' \ [\mathrm{rad/s^2}] \quad .$$

where use has been made of the values given previously.

Problem 2

The 20 [kg] block A is placed on top of the 100 [kg] block B, as shown in Fig. 6.2. Block A is being pulled horizontally by a rope with a pull magnitude of P. If the coefficient of static and kinetic friction between the two blocks are both essentially the same value of 0.5, and the horizontal plane is frictionless:

1. Plot the acceleration of each block as a function of P.
2. Determine the acceleration of each block for $P = 60$ [N] and $P = 40$ [N].

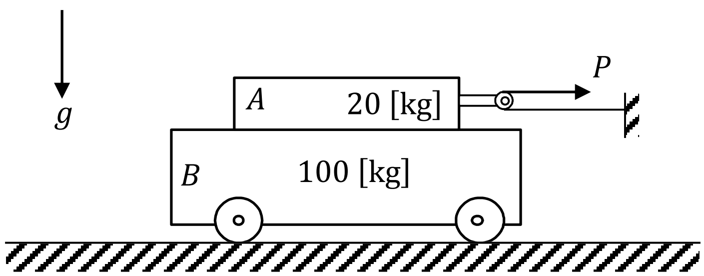

Figure 6.2

Solution:

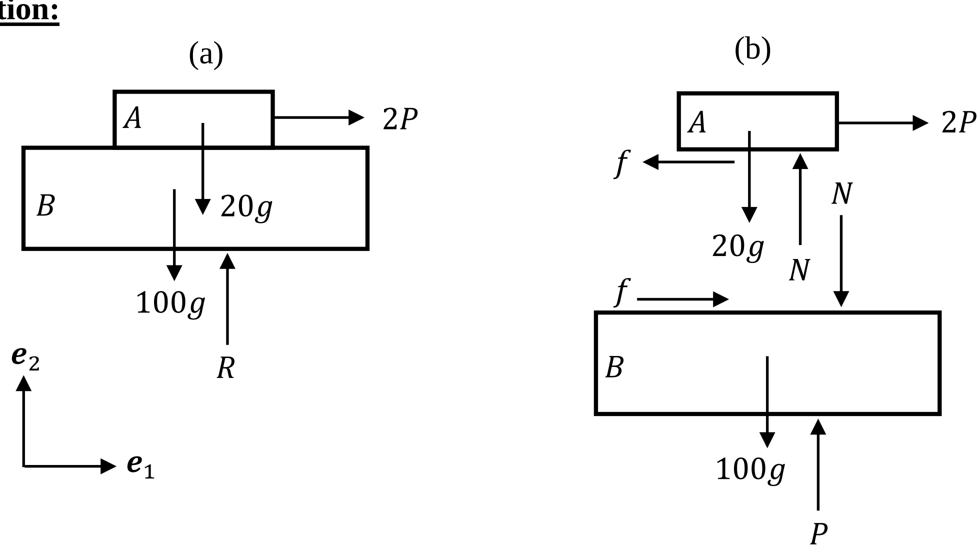

Figure 6.3

The two blocks move with the same acceleration before block A starts slipping on top of block B, as shown in Fig. 6.3a. Denoting the acceleration of the system by a it follows from the balance of linear momentum that

$$2P\boldsymbol{e}_1 + (R - 120g)\boldsymbol{e}_2 = 120a\boldsymbol{e}_1 \quad \Rightarrow$$

$$R = 1200 \text{ [N]} \quad , \quad a = P/60 \approx 0.0167P \text{ [m/s}^2\text{]} \quad .$$

Next, Fig. 6.3b shows the free body diagram of each block when A is slipping on top of B. Denoting the accelerations of $\{A, B\}$ by $\{a_A, a_B\}$, respectively, it follows from the balance of linear momentum that

$$(2P - f)\boldsymbol{e}_1 + (N - 20g)\boldsymbol{e}_2 = 20a_A \quad , \quad f = \mu N \quad \Rightarrow$$

$$N = 20g = 200 \text{ [N]} \quad , \quad a_A = 0.1P - 4.905 \text{ [m/s}^2\text{]} \quad ,$$

and,

$$f\boldsymbol{e}_1 + (P - N - 100g)\boldsymbol{e}_2 = 100a_B \quad , \quad f = \mu N \quad \Rightarrow$$

$$P = 1200 \text{ [N]} \quad , \quad a_B = 0.981 \text{ [m/s}^2\text{]} \quad .$$

Now, the continuity of the accelerations yields

$$a_A = a \quad \text{and} \quad a_B = a \quad \Rightarrow \quad P = 58.86 \text{ [N]} \quad .$$

The plots of $\{a_A, a_B\}$ as functions of P are shown in Fig. 6.4.

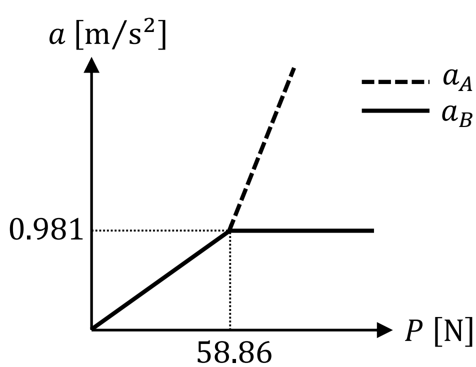

Fig. 6.4

Next, assuming that A does not slip on top of B, it follows from the balance of linear for each block that

$$(2P - f)\boldsymbol{e}_1 + (N - 20g)\boldsymbol{e}_2 = 20a \quad \Rightarrow \quad N = 196.2 \text{ [N]} \quad , \quad 2P - f = 20a \quad ,$$

and,

$$f\boldsymbol{e}_1 + (P - N - 100g)\boldsymbol{e}_2 = 100a \quad \Rightarrow \quad P = 1177.2 \text{ [N]} \quad , \quad f = 100a \quad .$$

Solving these equations for f yields

$$f = 5P/3 \approx 1.667P \quad .$$

Moreover, since friction is static in this case,

$$f \leq \mu_s N \quad \Rightarrow \quad P \leq 58.86 \text{ [N]} \quad .$$

Consequently, for $P = 60$ [N], block A slips on top of block B, so that

$$a_A = 0.1(60) - 4.905 = 1.095 \text{ [m/s}^2\text{]} \quad , \quad a_B = 0.01\mu N = 0.981 \text{ [m/s}^2\text{]} \quad .$$

Also, for $P = 40$ [N], block A does not slip on top of block B, so that

$$a_A = a_B = a = 0.667 \text{ [m/s}^2\text{]} \quad .$$

Problem 3

The sliders A and B are connected by a light rigid bar of length $l = 0.5$ [m] and move in the slots shown in Fig. 6.5. The slider A is being pulled horizontally by a constant force of magnitude $P = 40$ $[N]$. For the position where $x_A = 0.4$ [m], the velocity of A is given by $v_A = 0.9$ $[\mathrm{m/s}]$ to the right. At this instant:

1. Determine the acceleration of each slider.
2. Determine the force in the bar.

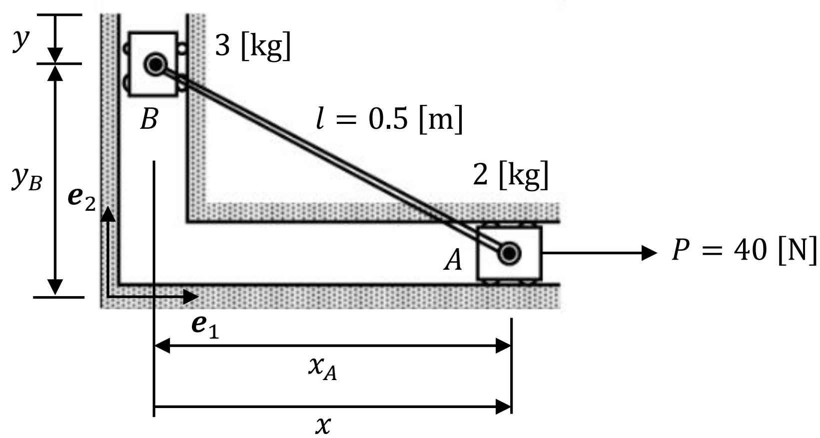

Figure 6.5

Solution:

The velocities of A and B are given, respectively, by

$$\boldsymbol{v}_A = \dot{x}\boldsymbol{e}_1 \quad , \quad \boldsymbol{v}_B = -\dot{y}\boldsymbol{e}_2 \quad ,$$

Moreover, $\{\dot{x}, \dot{y}\}$ are related to $\{\dot{x}_A, \dot{y}_B\}$ by

$$\dot{x} = \dot{x}_A \quad , \quad \dot{y} = -\dot{y}_B \quad ; \quad \ddot{x} = \ddot{x}_A \quad , \quad \ddot{y} = -\ddot{y}_B \quad .$$

Moreover, using the geometry in Fig. 6.5, then

$$x_A^2 + y_B^2 = l^2 \quad \Rightarrow \quad \dot{y}_B = -\frac{x_A \dot{x}_A}{y_B} = -\frac{x_A \dot{x}_A}{\sqrt{l^2 - x_A^2}} \quad , \quad \ddot{y}_B = -\frac{\dot{x}_A^2 + \dot{y}_B^2 + x_A \ddot{x}_A}{y_B} \quad ,$$

where,

$$x_A = 0.4 \text{ [m]} \quad , \quad \dot{x}_A = \dot{x} = 0.9 \text{ [m/s]} \quad ; \quad \dot{y}_B = -1.2 \text{ [m/s]} \quad \Rightarrow \quad \dot{y} = 1.2 \text{ [m/s]} \quad ,$$

$$\ddot{y}_B = -\frac{\dot{x}_A^2 + \dot{y}_B^2 + x_A \ddot{x}_A}{y_B} = -7.5 - \frac{4}{3}\ddot{x}_A \quad \Rightarrow \quad \ddot{y} = 7.5 + \frac{4}{3}\ddot{x}_A \quad .$$

Next, using the free body diagrams shown in Fig. 6.6 for each block,

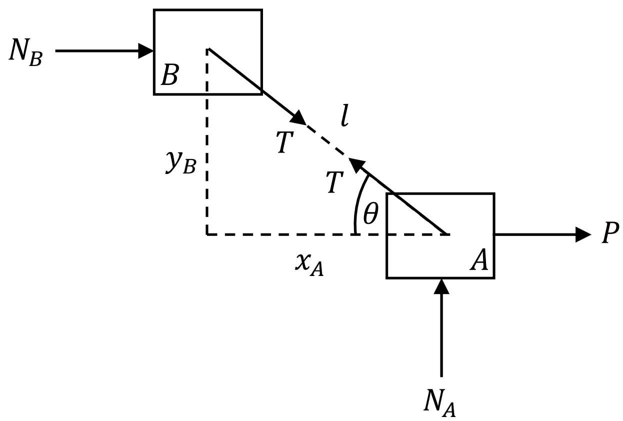

Figure 6.6

it follows that

$$P - T\cos(\theta) = m_A \ddot{x} \quad , \qquad -T\sin(\theta) = -m_B \ddot{y} \quad ,$$

where,

$$\cos(\theta) = \frac{x_A}{l} = 0.8 \quad , \quad \sin(\theta) = \frac{y_B}{l} = 0.6 \quad .$$

Solving the three equations

$$\ddot{y} = 7.5 + \frac{4}{3}\ddot{x}_A \quad , \quad 40 - 0.8\,T = 2\ddot{x} \quad , \quad 0.6\,T = 3\ddot{y} \quad ,$$

for $\{\ddot{x}, \ddot{y}, T\}$ yields

$$T = 46.6 \text{ [N]} \quad , \quad \ddot{x} = 1.36 \text{ [m/s}^2\text{]} \quad , \quad \ddot{y} = 9.32 \text{ [m/s}^2\text{]} \quad .$$

Problem 4

The small ball of mass m, shown in Fig. 6.7, is attached to a light bar of length L which swivels about the horizontal axis through B at the constant rate $\dot{\beta}$. Simultaneously, the vertical bar rotates about the vertical axis with a constant angular speed $\dot{\phi}$.

1. Determine the acceleration of the ball.
2. Determine the tension T in the bar.
3. Determine the shear force exerted on the bar by the ball.

Express your answers in terms of $\{L, \beta, \dot{\beta}, \dot{\phi}, g\}$.

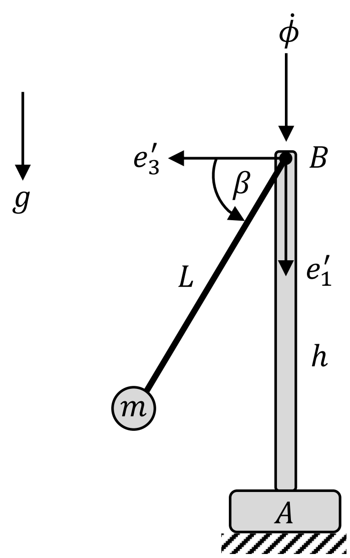

Figure 6.7

Solution:

The angular velocity of the coordinate system $\boldsymbol{e}_i'$ is denoted by $\boldsymbol{\omega}$, such that

$$\dot{\boldsymbol{e}}_i' = \boldsymbol{\omega} \times \boldsymbol{e}_i' \quad , \quad \boldsymbol{\omega} = -\dot{\phi}\boldsymbol{e}_1' \quad .$$

Now, the position of the ball is

$$\boldsymbol{x} = L[\sin(\beta)\,\boldsymbol{e}'_1 + \cos(\beta)\,\boldsymbol{e}'_3] \quad .$$

Thus, using Table 6.2, the velocity and acceleration of the ball take, respectively, the forms

$$\boldsymbol{v} = \dot{\beta}L\cos(\beta)\,\boldsymbol{e}'_1 - \dot{\phi}L\cos(\beta)\,\boldsymbol{e}'_2 - \dot{\beta}L\sin(\beta)\,\boldsymbol{e}'_3 \quad ,$$

$$\boldsymbol{a} = -\dot{\beta}^2L\sin(\beta)\,\boldsymbol{e}'_1 + 2\dot{\phi}\dot{\beta}L\sin(\beta)\,\boldsymbol{e}'_2 - \left(\dot{\beta}^2 + \dot{\phi}^2\right)L\cos(\beta)\,\boldsymbol{e}'_3 \quad .$$

	$\boldsymbol{e}'_1$	$\boldsymbol{e}'_2$	$\boldsymbol{e}'_3$
$\boldsymbol{\omega}$	$\dot{\phi}$	0	0
$\boldsymbol{x}$	$L\sin(\beta)$	0	$L\cos(\beta)$
$\delta\boldsymbol{x}/\delta t$	$\dot{\beta}L\cos(\beta)$	0	$-\dot{\beta}L\sin(\beta)$
$\boldsymbol{\Omega}\times\boldsymbol{x}$	0	$-\dot{\phi}L\cos(\beta)$	0
$\boldsymbol{v}$	$\dot{\beta}L\cos(\beta)$	$-\dot{\phi}L\cos(\beta)$	$-\dot{\beta}L\sin(\beta)$
$\delta\boldsymbol{v}/\delta t$	$-\dot{\beta}^2L\sin(\beta)$	$\dot{\phi}\dot{\beta}L\sin(\beta)$	$-\dot{\beta}^2L\cos(\beta)$
$\boldsymbol{\Omega}\times\boldsymbol{v}$	0	$\dot{\phi}\dot{\beta}L\sin(\beta)$	$-\dot{\phi}^2L\cos(\beta)$
$\boldsymbol{a}$	$-\dot{\beta}^2L\sin(\beta)$	$2\dot{\phi}\dot{\beta}L\sin(\beta)$	$-\left(\dot{\beta}^2+\dot{\phi}^2\right)L\cos(\beta)$

Table 6.2

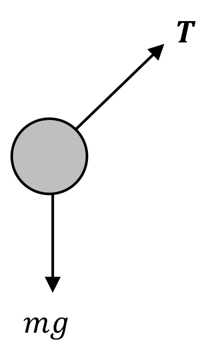

Figure 6.8

Next, using the free body diagram shown in Fig. 6.8 it follows that

$$\boldsymbol{T} + mg\boldsymbol{e}'_1 = T'_i\boldsymbol{e}'_i = m\left[-\dot{\beta}^2L\sin(\beta)\,\boldsymbol{e}'_1 + 2\dot{\phi}\dot{\beta}L\sin(\beta)\,\boldsymbol{e}'_2 - \left(\dot{\beta}^2 + \dot{\phi}^2\right)L\cos(\beta)\,\boldsymbol{e}'_3\right] \quad .$$

Thus,

$$T'_1 = -m\left[g + \dot{\beta}^2L\sin(\beta)\right] \quad , \quad T'_2 = 2m\dot{\phi}\dot{\beta}L\sin(\beta) \quad , \quad T'_3 = -m\left(\dot{\beta}^2 + \dot{\phi}^2\right)L\cos(\beta) \quad .$$

Furthermore, the force $\boldsymbol{R}$ exerted on the bar by the ball is given by

$$\boldsymbol{R} = -\boldsymbol{T} = m\left[g + \dot{\beta}^2 L\sin(\beta)\right]\boldsymbol{e}'_1 - 2m\dot{\phi}\dot{\beta}L\sin(\beta)\,\boldsymbol{e}'_2 + m\left(\dot{\beta}^2 + \dot{\phi}^2\right)L\cos(\beta)\,\boldsymbol{e}'_3 \quad .$$

Therefore, the shear force $\boldsymbol{S}$ exerted on the bar by the ball takes the form

$$\boldsymbol{S} = \boldsymbol{R} - (\boldsymbol{R}\cdot\boldsymbol{e}_b)\boldsymbol{e}_b \quad , \quad \boldsymbol{e}_b = \sin(\beta)\,\boldsymbol{e}'_1 + \cos(\beta)\,\boldsymbol{e}'_3 \ ,$$

where $\boldsymbol{e}_b$ in is the unit direction of $\boldsymbol{x}$.

Problem Set 7 Solutions

Problem 1

The two springs of stiffness $800\ [\mathrm{N/m}]$ and unstretched length of $0.3\ [\mathrm{m}]$ are attached to the collar of mass $10\ [\mathrm{kg}]$, which slides with negligible friction on the fixed vertical shaft under the action of gravity, as shown in Fig. 7.1. The collar is released from rest at the top position.

1. Determine the distance traveled by the collar along the vertical shaft.
2. Determine the velocity of the collar as it covers half of that distance.

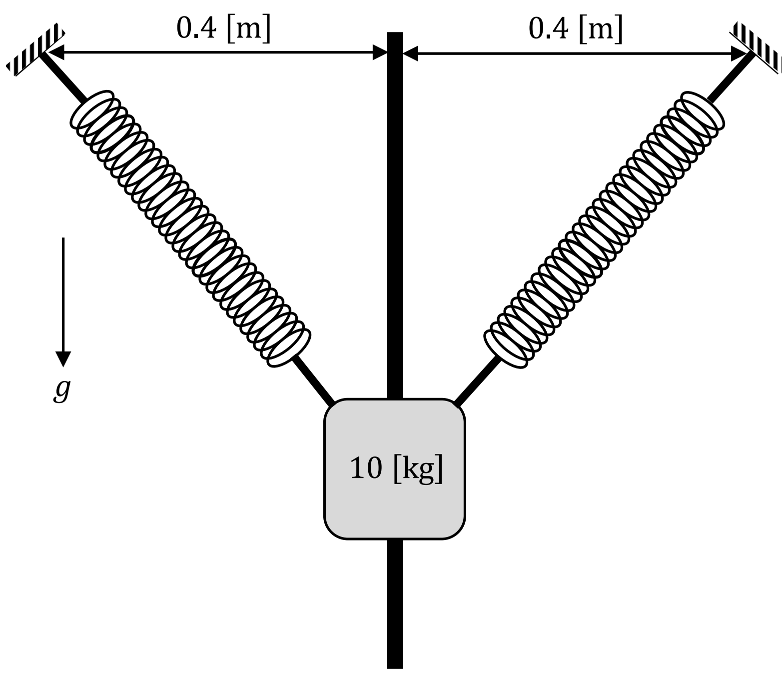

Figure 7.1

Solution:

The free body diagram of the collar is shown in Fig. 7.2.

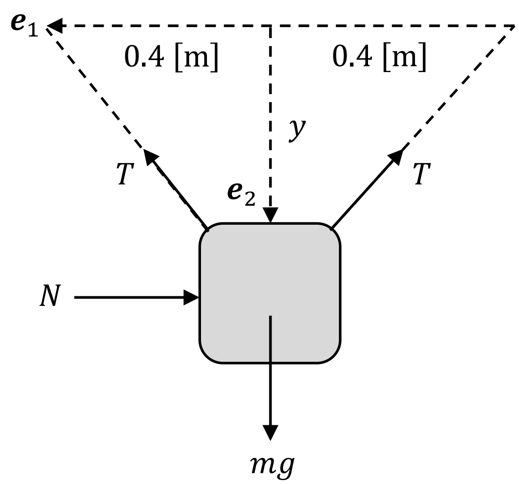

Figure 7.2

The velocity of the collar is given by

$$\boldsymbol{v} = \dot{y}\boldsymbol{e}_2 \quad .$$

The resultant force acting on the collar takes the form

$$\boldsymbol{F} = \boldsymbol{F}_g + \boldsymbol{F}_e + \overline{\boldsymbol{F}} \quad , \quad \boldsymbol{F}_g = mg\boldsymbol{e}_2 \quad , \quad \boldsymbol{F}_e = -2T\left(\frac{y}{\sqrt{0.4^2 + y^2}}\right)\boldsymbol{e}_2 \quad , \quad \overline{\boldsymbol{F}} = N\boldsymbol{e}_1 \quad .$$

Now, using the balance of energy, it follows that

$$\overline{U}_{2/1} = \int_{t_1}^{t_2} (\overline{\boldsymbol{F}} \cdot \boldsymbol{v})\, dt = 0 = \Delta T + \Delta V_g + \Delta V_e \quad ,$$

where,

$$\Delta T = \frac{1}{2}m\dot{y}^2 = 5\dot{y}^2 \quad , \quad \Delta V_g = -(mg\boldsymbol{e}_2) \cdot y\boldsymbol{e}_2 = -mgy = -98.1y \quad ,$$

$$\Delta V_e = \frac{1}{2}k[(l_2 - l_0)^2 - (l_1 - l_0)^2] = 400\left[\left(\sqrt{0.4^2 + y^2} - 0.3\right)^2 - (0.4 - 0.3)^2\right] \quad .$$

Thus,

$$5\dot{y}^2 - 98.1y + 400\left[\left(\sqrt{0.16+y^2} - 0.3\right)^2 - 0.01\right] = 0 \quad .$$

Next, the maximum distance traveled by the collar before bouncing back up corresponds to the condition $\dot{y} = 0$, so that

$$-98.1y_{\max} + 400\left[\left(\sqrt{0.16+y_{\max}^2} - 0.3\right)^2 - 0.01\right] = 0 \;\Rightarrow\; y_{\max} = 0.551\ [\mathrm{m}] \quad .$$

Using the same previous arguments with

$$y = \frac{y_{\max}}{2} = 0.2755\ [\mathrm{m}] \quad ,$$

it follows from the balance of energy that

$$5\dot{y}^2 - 98.1\left(\frac{1}{2}y_{\max}\right) + 400\left[\left(\sqrt{0.16+\left(\frac{1}{2}y_{\max}\right)^2} - 0.3\right)^2 - 0.01\right] = 0 \;\Rightarrow$$

$$\dot{y} = 1.856\ [\mathrm{m/s}] \quad .$$

Problem 2

The 10 [kg] bead A is released from rest in the position shown in Fig. 7.3 and slides freely up the fixed circular rod AB of radius $a = 2.4$ [m] under the action of gravity and a constant force $P = 250$ [N]. Then, the bead slides on the rough horizontal rod BC with a kinetic friction of 0.5 under the action of gravity alone. Later, the bead sticks to a spring of stiffness k at the left end C of the rod BC.

1. Determine the work done by the force P on the bead from A to B.
2. Determine the velocity of the bead as it passes through the point B.
3. Determine the work done by friction on the bead from B to C.
4. Determine the value of the spring's stiffness k when it is maximally compressed by 10 [cm].

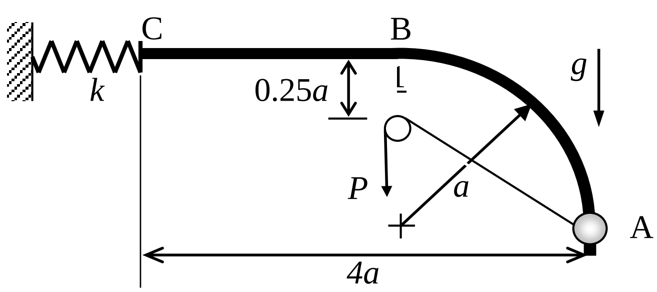

Figure 7.3

Solution:

The free body diagrams of the bead are shown in Fig. 7.4.

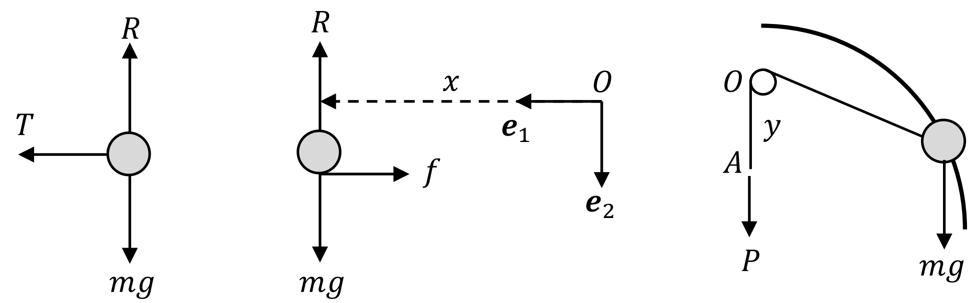

Figure 7.4

Using Fig. 7.4 (right), the work done by the force P on the bead from A to B is given by

$$\bar{U}_{B/A}^{(P)} = \int_0^{t_B} (\boldsymbol{F}_P \cdot \boldsymbol{v}_A)dt = \int_0^{t_B} (P\boldsymbol{e}_2 \cdot \dot{y}\boldsymbol{e}_2)dt = P[y(t_B) - y(0)] \quad ,$$

where,

$$y(0) = 0 \quad , \quad y(t_B) = L - \frac{a}{4} = \sqrt{\left(a - \frac{a}{4}\right)^2 + a^2} - \frac{a}{4} = a \quad .$$

In the last expression, L denotes the length of the inextensible rope. Thus,

$$\bar{U}_{B/A}^{(P)} = Pa = (250)(2.4) = 600 \text{ [J]} \quad .$$

Next, the velocity $\dot{x}_B$ of the bead at B can be calculated using the balance of energy from A to B, so that

$$\bar{U}_{B/A}^{(P)} = Pa = \Delta T + \Delta V_g = \frac{1}{2}m\dot{x}_0^2 - (mg\boldsymbol{e}_2) \cdot (-a\boldsymbol{e}_2) = \frac{1}{2}m\dot{x}_B^2 + mga \quad \Rightarrow$$

$$\dot{x}_B = \sqrt{2a\left(\frac{P}{m} - g\right)} = 8.54 \text{ [m/s]} \quad .$$

Now, using Fig. 7.4 (middle), the work done by the force f on the bead from B to C is given by

$$\bar{U}^{(f)}_{C/B} = \int_{t_B}^{t_C} [-\mu mg\boldsymbol{e}_1 \cdot \dot{x}\boldsymbol{e}_1]dt = -\mu mg(\Delta x) = -\mu mg(3a) = -353.16 \text{ [J]} \quad .$$

Moreover, the velocity $\dot{x}_C$ of the bead at C can be calculated using the balance of energy from B to C, so that

$$\bar{U}^{(f)}_{C/B} = -\mu mg(3a) = \Delta T + \Delta V_g = \frac{1}{2}m(\dot{x}_C^2 - \dot{x}_B^2) \;\; \Rightarrow$$

$$\dot{x}_C = \sqrt{\dot{x}_B^2 - 6\mu ga} = 1.516 \text{ [m/s]} \quad .$$

Finally, using Fig. 7.4 (left), the balance of energy yields

$$\bar{U}_{C/D} = \int_{t_C}^{t_D} [R\boldsymbol{e}_2 \cdot \dot{x}\boldsymbol{e}_1]dt = 0 = \Delta T + \Delta V_g + \Delta V_e = \frac{1}{2}m(\dot{x}_D^2 - \dot{x}_C^2) + \frac{1}{2}k\delta^2 \quad ,$$

where δ denotes the compression in the spring. Now, at maximum compression, $\dot{x}_D = 0$ and $\delta = \delta_{\max} = 0.1$ [m], so that

$$k = \frac{m\dot{x}_C^2}{\delta_{\max}^2} \approx 2.3 \text{ [kN/m]} \quad .$$

Problem 3

The small ball of mass m is attached to an inextensible rope of length L, as shown in Fig. 7.5. Initially, at the time $t = 0$, $\theta(0) = \theta_0$, $\dot{\theta}(0) = 0$, and the particle is given a velocity of $\boldsymbol{v}(0) = -v_0 \boldsymbol{e}'_3$. Just afterwards, the rope begins swiveling about the horizontal axis through B at the rate $\dot{\theta}$, and the vertical bar begins rotating about the vertical axis at the rate $\dot{\phi}$. The system $\boldsymbol{e}''_i$ is fixed to the vertical bar and it lies in the plane containing the system $\boldsymbol{e}'_i$.

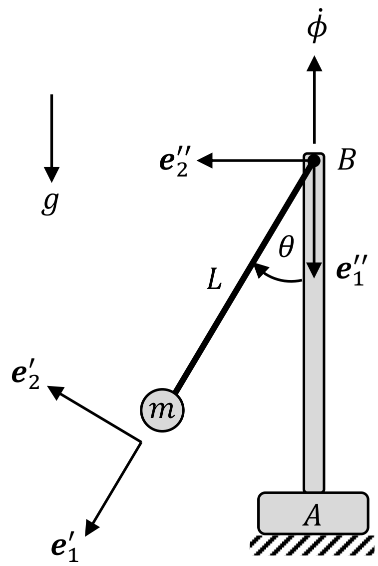

Figure 7.5

1. Is the linear momentum of the ball conserved in the $\boldsymbol{e}'_3$ direction?
2. Is the angular momentum of the ball about B conserved in the $\boldsymbol{e}''_3$ direction?
3. Is the angular momentum of the ball about B conserved in the $\boldsymbol{e}''_1$ direction?
4. Does the rope do work on the ball?
5. Determine the kinetic energy of the ball.
6. Determine the values of $\dot{\theta}$ and $\dot{\phi}$ in terms of $\{L, \theta, \theta_0, v_0, g\}$.

7. Determine the absolute acceleration of the ball.

8. Determine the tension in the rope.

Solution:

The systems $\{\boldsymbol{e}'_i, \boldsymbol{e}''_i\}$ rotate, respectively, with angular velocities $\{\boldsymbol{\omega}, \boldsymbol{\Omega}\}$, such that

$$\dot{\boldsymbol{e}}''_i = \boldsymbol{\Omega} \times \boldsymbol{e}''_i \quad , \quad \boldsymbol{\Omega} = -\dot{\phi}\boldsymbol{e}''_1 \quad ; \quad \dot{\boldsymbol{e}}'_i = \boldsymbol{\omega} \times \boldsymbol{e}'_i \quad , \quad \boldsymbol{\omega} = \boldsymbol{\Omega} + \dot{\theta}\boldsymbol{e}''_3 = -\dot{\phi}\boldsymbol{e}''_1 + \dot{\theta}\boldsymbol{e}''_3 \quad .$$

Next, the free body diagram of the ball is shown in Fig. 7.6.

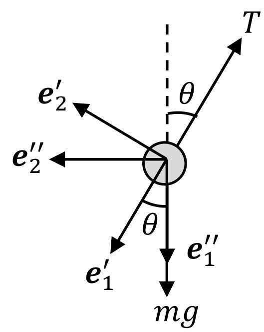

Figure 7.6

The resultant force acting on the particle is given by

$$\boldsymbol{F} = [mg\cos(\theta) - T]\boldsymbol{e}'_1 - mg\sin(\theta)\,\boldsymbol{e}'_2 = [mg - T\cos(\theta)]\boldsymbol{e}''_1 - T\sin(\theta)\,\boldsymbol{e}''_2 \quad ,$$

so that

$$\boldsymbol{F} \cdot \boldsymbol{e}'_3 = \boldsymbol{F} \cdot \boldsymbol{e}''_3 = 0 \quad .$$

Since the direction $\boldsymbol{e}'_3$ (or $\boldsymbol{e}''_3$) is not fixed in space, the linear momentum $\boldsymbol{G} \cdot \boldsymbol{e}'_3$ of the particle in this direction is not necessarily conserved.

Next, the resultant moment acting on the particle about O takes the form

$$\boldsymbol{M}_O = \boldsymbol{x} \times \boldsymbol{F} = -mgL\sin(\theta)\,\boldsymbol{e}'_3 = -mgL\sin(\theta)\,\boldsymbol{e}''_3 \quad ,$$

such that

$$\boldsymbol{M}_O \cdot \boldsymbol{e}''_1 = 0 \quad , \quad \boldsymbol{M}_O \cdot \boldsymbol{e}''_3 = 0 \quad .$$

Since the direction $\boldsymbol{e}_3''$ is not fixed in space, the angular momentum $\boldsymbol{H}_O \cdot \boldsymbol{e}_3''$ of the particle about O in this direction is not necessarily conserved. However, since the direction $\boldsymbol{e}_1''$ is fixed in space, the angular momentum $\boldsymbol{H}_O \cdot \boldsymbol{e}_1''$ of the particle about O in this direction is conserved.

Now, the position and velocity of the particle are given, respectively, by

$$\boldsymbol{x} = L\boldsymbol{e}_1' = L\cos(\theta)\,\boldsymbol{e}_1'' + L\sin(\theta)\,\boldsymbol{e}_2'' \quad ,$$

$$\boldsymbol{v} = \frac{\delta \boldsymbol{x}}{\delta t} + \boldsymbol{\Omega} \times \boldsymbol{x} = -\dot{\theta} L\sin(\theta)\,\boldsymbol{e}_1'' + \dot{\theta} L\cos(\theta)\,\boldsymbol{e}_2'' - \dot{\phi} L\sin(\theta)\,\boldsymbol{e}_3'' \quad .$$

Thus, the angular momentum $\boldsymbol{H}_O$ of the particle about O becomes

$$\boldsymbol{H}_O = \boldsymbol{x} \times m\boldsymbol{v} = -mL^2\dot{\phi}\sin(\theta)\,\boldsymbol{e}_1'' + \frac{1}{2} mL^2\dot{\phi}\sin(2\theta)\,\boldsymbol{e}_2'' + mL^2\dot{\theta}\boldsymbol{e}_3'' \ ,$$

such that

$$\boldsymbol{H}_O \cdot \boldsymbol{e}_1'' = -mL^2\dot{\phi}\sin(\theta) \quad ,$$

$$\boldsymbol{H}_O(0) \cdot \boldsymbol{e}_1'' = [\boldsymbol{x}(0) \times (-mv_0\boldsymbol{e}_3'')] \cdot \boldsymbol{e}_1'' = -mv_0 L\sin(\theta_0) \quad \Rightarrow$$

$$-mL^2\dot{\phi}\sin(\theta) = -mv_0 L\sin(\theta_0) \quad \Rightarrow \quad \dot{\phi} = \frac{v_0\sin(\theta_0)}{L\sin(\theta)} \quad .$$

Next, the work done by the rope on the particle takes the form

$$\bar{U}_{2/1} = \int_0^t \boldsymbol{F}_T \cdot \boldsymbol{v}\, dt \quad , \quad \boldsymbol{F}_T = -T\cos(\theta)\,\boldsymbol{e}_1'' - T\sin(\theta)\,\boldsymbol{e}_2'' \quad \Rightarrow$$

$$\bar{U}_{2/1} = \int_0^t \left[\dot{\theta} L\sin(\theta)\,T\cos(\theta) - \dot{\theta} L\cos(\theta)\,T\sin(\theta)\right] dt = 0 \quad .$$

Therefore, the mechanical energy of the particle is conserved.

Next, using the balance of energy, it follows that

$$(T_2 - T_1) + \left(V_{g2} - V_{g1}\right) = 0 \quad ,$$

where,

$$T_2 - T_1 = \frac{1}{2}m[\boldsymbol{v}\cdot\boldsymbol{v} - \boldsymbol{v}(0)\cdot\boldsymbol{v}(0)] = \frac{1}{2}mL^2\left[\dot{\theta}^2 + \dot{\phi}^2\sin^2(\theta) - \frac{v_0^2}{L^2}\right] \ ,$$

$$V_{g2} - V_{g1} = -mg\boldsymbol{e}_1''\cdot[\boldsymbol{x} - \boldsymbol{x}(0)] = -mgL[\cos(\theta) - \cos(\theta_0)] \ ,$$

such that

$$\dot{\theta}^2 = \frac{v_0^2}{L^2} + \frac{2g}{L}[\cos(\theta) - \cos(\theta_0)] - \dot{\phi}^2\sin^2(\theta) \ .$$

Substituting the value of $\dot{\phi}$ into this equation yields

$$\dot{\theta} = \left[\left\{\frac{v_0}{L}\cos(\theta)\right\}^2 + \frac{2g}{L}\{\cos(\theta) - \cos(\theta_0)\}\right]^{1/2} \ .$$

The absolute acceleration of the particle takes the form

$$\boldsymbol{a} = \frac{\delta\boldsymbol{v}}{\delta t} + \boldsymbol{\Omega}\times\boldsymbol{v}$$

$$= -L\left[\dot{\theta}^2\cos(\theta) + \ddot{\theta}\sin(\theta)\right]\boldsymbol{e}_1'' + L\left[-\left(\dot{\theta}^2 + \dot{\phi}^2\right)\sin(\theta) + \ddot{\theta}\cos(\theta)\right]\boldsymbol{e}_2''$$

$$- L\left[2\dot{\theta}\dot{\phi}\cos(\theta) + \ddot{\phi}\sin(\theta)\right]\boldsymbol{e}_3'' \ .$$

Consequently,

$$\boldsymbol{F}\cdot\boldsymbol{e}_2'' = m\boldsymbol{a}\cdot\boldsymbol{e}_2'' \ \Rightarrow \ T = mL\left[\dot{\theta}^2 + \dot{\phi}^2 - \ddot{\theta}\cot(\theta)\right] \ ,$$

where,

$$\ddot{\theta} = -\frac{\sin(\theta)}{L}\left[\frac{v_0^2}{L}\cos(\theta) - g\right]$$

Problem 4

An object of mass $m = 2$ [kg] moves on the inside of a smooth conical dish of radius $R = 3$ $[m]$ and edge length of $Y = 5$ [m] while being attached to a vertical spring of stiffness $k = 300$ $[\mathrm{N/m}]$, as shown in Fig. 7.7. At the time $t = 0$, $x(0) = 4$ [m], the spring is unstretched and the object is given a velocity $v_0 = 3$ $[\mathrm{m}/s]$ tangent to the horizontal rim of the surface of the dish.

1. Write down the equation of motion of the object.
2. Determine the minimal distance traveled by the particle relative to the bottom end of the dish.
3. Determine the velocity of the particle at that distance.

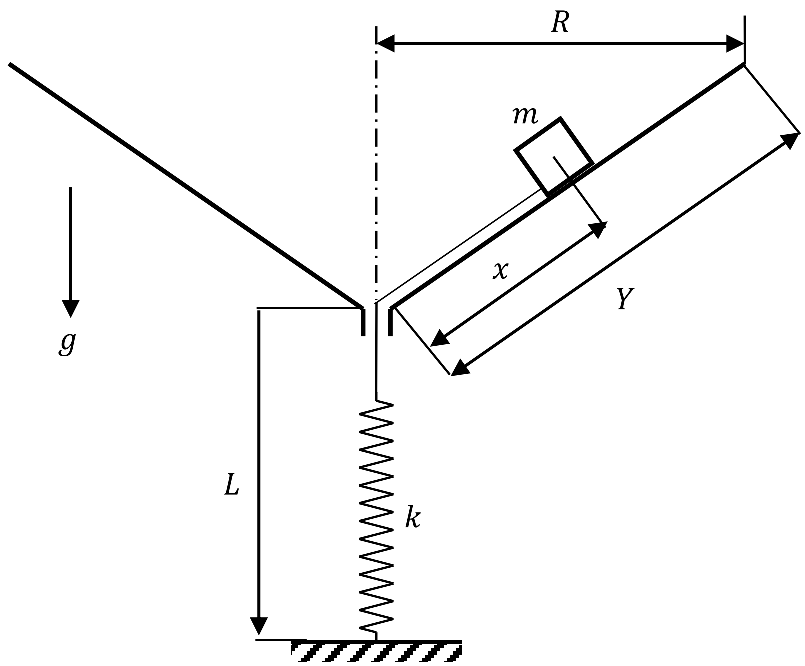

Figure 7.7

Solution:

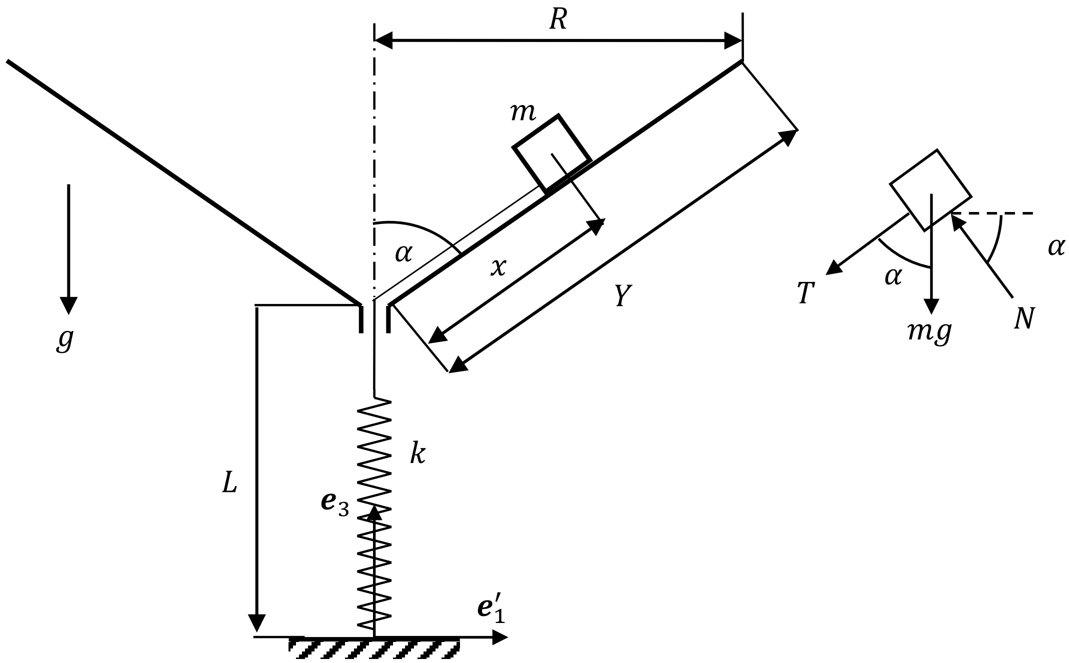

Figure 7.8

The angle α defined in Fig. 7.8 is given by

$$\sin(\alpha) = \frac{R}{Y} = \frac{3}{5} \quad , \quad \cos(\alpha) = \frac{\sqrt{Y^2 - R^2}}{Y} = \frac{4}{5} \quad .$$

Also, the system $\boldsymbol{e}'_i$ traces the motion of the object, such that

$$\dot{\boldsymbol{e}}'_i = \boldsymbol{\omega} \times \boldsymbol{e}'_i \quad , \quad \boldsymbol{\omega} = \dot{\theta} \boldsymbol{e}_3 \quad .$$

Now, the position and velocity of the object are given, respectively, by

$$\boldsymbol{x} = L\boldsymbol{e}_3 + x[\sin(\alpha)\,\boldsymbol{e}'_1 + \cos(\alpha)\,\boldsymbol{e}_3] = x\sin(\alpha)\,\boldsymbol{e}'_1 + [L + x\cos(\alpha)]\boldsymbol{e}_3 \quad ,$$

$$\boldsymbol{v} = \dot{\boldsymbol{x}} = \dot{x}\sin(\alpha)\,\boldsymbol{e}'_1 + \dot{\theta}x\sin(\alpha)\,\boldsymbol{e}'_2 + \dot{x}\cos(\alpha)\,\boldsymbol{e}_3 \quad .$$

Next, using the free body diagram of the object, shown in Fig.7.8 (right), the resultant force acting on the object takes the form

$$\boldsymbol{F} = \boldsymbol{F}_g + \boldsymbol{F}_e + \overline{\boldsymbol{F}} \;\; ; \quad \boldsymbol{F}_g = -mg\boldsymbol{e}_3 \;\; ; \quad \boldsymbol{F}_e = -T[\sin(\alpha)\,\boldsymbol{e}'_1 + \cos(\alpha)\,\boldsymbol{e}_3] \;\; ,$$

$$T=\frac{1}{2}k(l-L)=\frac{1}{2}kx \quad ; \quad \bar{\boldsymbol{F}}=N[-\cos(\alpha)\,\boldsymbol{e}'_1+\sin(\alpha)\,\boldsymbol{e}_3] \quad ,$$

Thus, the balance of energy equation yields

$$\bar{U}_{2/1}=\int_0^t \bar{\boldsymbol{F}}\cdot\boldsymbol{v}\,dt=0=(T_2-T_1)+\left(V_{g2}-V_{g1}\right)+(V_{e2}-V_{e1}) \quad ,$$

$$T_2-T_1=\frac{1}{2}m(\boldsymbol{v}\cdot\boldsymbol{v}-v_0^2)=\frac{1}{2}m\left[\dot{x}^2+\dot{\theta}^2x^2\sin^2(\alpha)\right] \quad ;$$

$$V_{g2}-V_{g1}=-(-mg\boldsymbol{e}_3)\cdot(\boldsymbol{x}-\boldsymbol{x}_0)=mg(x-x_0)\cos(\alpha) \quad ;$$

$$V_{e2}-V_{e1}=\frac{1}{2}k[(l_2-L)^2-(l_1-L)^2]=\frac{1}{2}k(x^2-x_0^2) \quad \Rightarrow$$

$$\dot{x}^2+\dot{\theta}^2x^2\sin^2(\alpha)+\left[2g\cos(\alpha)+\frac{k}{m}(x+x_0)\right](x-x_0)=0$$

Next, the resultant moment acting on the object about the origin takes the form

$$\boldsymbol{M}_O=\boldsymbol{x}\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{e}'_1 & \boldsymbol{e}'_2 & \boldsymbol{e}_3\\ x\sin(\alpha) & 0 & L+x\cos(\alpha)\\ -T\sin(\alpha)-N\cos(\alpha) & 0 & -mg-T\cos(\alpha)+N\sin(\alpha)\end{vmatrix} \quad \Rightarrow$$

$$\boldsymbol{M}_O\cdot\boldsymbol{e}'_1=0 \quad , \quad \boldsymbol{M}_O\cdot\boldsymbol{e}_3=0 \quad .$$

This shows that the angular momentum of the object about the origin in the fixed vertical $\boldsymbol{e}_3$ direction is conserved. Therefore,

$$\boldsymbol{H}_O\cdot\boldsymbol{e}_3=\boldsymbol{H}_O(0)\cdot\boldsymbol{e}_3 \quad ,$$

where,

$$\boldsymbol{H}_O\cdot\boldsymbol{e}_3=\boldsymbol{e}_3\cdot\boldsymbol{x}\times m\boldsymbol{v}=\begin{vmatrix}0 & 0 & 1\\ x\sin(\alpha) & 0 & L+x\cos(\alpha)\\ m\dot{x}\sin(\alpha) & m\dot{\theta}x\sin(\alpha) & m\dot{x}\cos(\alpha)\end{vmatrix}=m\dot{\theta}x^2\sin^2(\alpha)\,,$$

$$\boldsymbol{H}_O(0)\cdot\boldsymbol{e}_3=m\dot{\theta}_0x_0^2\sin^2(\alpha) \quad \Rightarrow \quad \dot{\theta}x^2=\dot{\theta}_0x_0^2 \quad .$$

Now,

$$\boldsymbol{v}(0)=v_0\boldsymbol{e}'_2=\dot{x}_0\sin(\alpha)\,\boldsymbol{e}'_1+\dot{\theta}_0x_0\sin(\alpha)\,\boldsymbol{e}'_2+\dot{x}_0\cos(\alpha)\,\boldsymbol{e}_3 \quad \Rightarrow$$

$$\dot{x}_0 = 0 \quad , \quad \dot{\theta}_0 = \frac{v_0}{x_0 \sin(\alpha)} \quad .$$

Hence,

$$\dot{\theta} = \frac{\dot{\theta}_0 x_0^2}{x^2} = \frac{x_0 v_0}{x^2 \sin(\alpha)} \quad .$$

Substituting this expression and the given data in the resulting equation of the balance of energy, it follows that

$$\dot{x}^2 + \dot{\theta}^2 x^2 \sin^2(\alpha) + \left[2g\cos(\alpha) + \frac{k}{m}(x + x_0)\right](x - x_0) = 0 \quad .$$

The minimum value of the position x of the object on the surface of the dish is obtained by the requirement $\dot{x} = 0$, such that

$$\frac{240}{x^2} + (616 + 150x)(x - 4) = 0 \quad \Rightarrow \quad x_{\min} \approx 0.313 \text{ [m]} \quad .$$

Furthermore, the velocity of the object at $x = x_{\min}$ reduces to

$$\boldsymbol{v}(x_{\min}) = \dot{\theta} x_{\min} \sin(\alpha)\, \boldsymbol{e}'_2 = \frac{x_0 v_0}{x_{\min}} \boldsymbol{e}'_2 \approx 38.3 \boldsymbol{e}'_2 \text{ [m/s]} \quad .$$

Problem Set 8 Solutions

Problem 1

Figure 8.1 shows a particle of mass m, which is attached to a spring of stiffness k and free length r_0, and placed on a frictionless table. At the time $t = 0$, the spring's length is r_0 and the particle is given a velocity v_0 in the direction perpendicular to the spring.

1. Determine the equation of motion of the particle.
2. Are the linear momentum, angular momentum about the fixed point O and mechanical energy of the particle conserved?
3. Describe the motion of the particle.

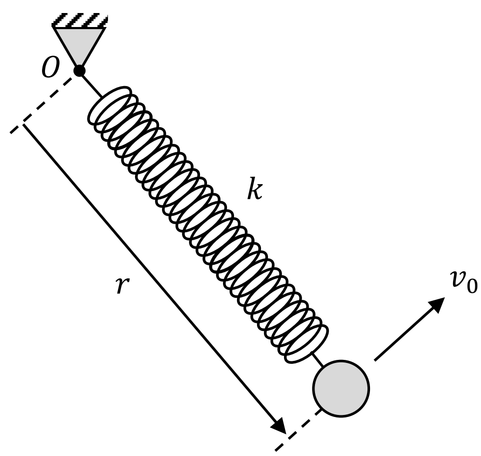

Figure 8.1

Solution

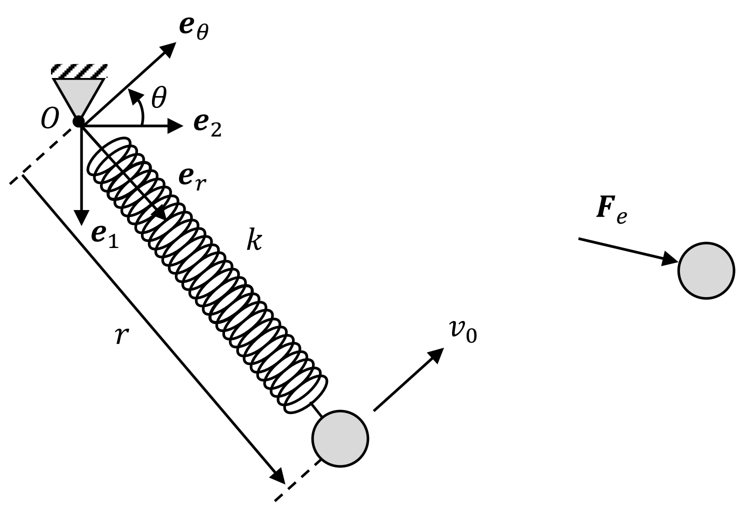

Figure 8.2

Using Fig. 8.2 (left), it follows that the position $\boldsymbol{x}$ and velocity $\boldsymbol{v}$ of the particle expressed in terms of polar coordinates are given, respectively, by

$$\boldsymbol{x} = r\boldsymbol{e}_r \quad , \quad \boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta \quad .$$

Next, since the only force acting on the particle is due to the spring [see Fig. 8.2 (right)], the energy of the particle is conserved so that

$$(T_2 - T_1) + (V_{e2} - V_{e1}) = 0 \quad ,$$

where,

$$T_2 - T_1 = \frac{1}{2}m(\boldsymbol{v}\cdot\boldsymbol{v} - v_0^2) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 - v_0^2\right) \quad ,$$

$$V_{e2} - V_{e1} = V_{e2} = \frac{1}{2}k(r - r_0)^2 \quad .$$

Hence,

$$\dot{r}^2 + r^2\dot{\theta}^2 + \frac{k}{m}(r - r_0)^2 = v_0^2 \quad .$$

Now, the moment exerted on the particle about the origin O vanishes. Therefore, the angular momentum of the particle about O is conserved, such that

$$\boldsymbol{H}_O(t) = \boldsymbol{H}_O(0) \quad ,$$

where,

$$\boldsymbol{H}_O(t) = \boldsymbol{x} \times m\boldsymbol{v} = r^2\dot{\theta}\boldsymbol{e}_3 \quad , \quad \boldsymbol{H}_O(0) = r_0\boldsymbol{e}_1 \times v_0\boldsymbol{e}_2 = r_0 v_0 \boldsymbol{e}_3 \quad .$$

Thus,

$$r^2\dot{\theta} = r_0 v_0 \quad \Rightarrow \quad \dot{\theta} = \frac{r_0 v_0}{r^2} \quad .$$

Substituting this expression into resulting equation of the conservation of energy, then

$$\dot{r}^2 + \left(\frac{r_0 v_0}{r}\right)^2 + \frac{k}{m}(r - r_0)^2 = v_0^2 \quad .$$

Thus, by taking the time derivative of this equation, it follows that the motion of the particle is governed by

$$2\dot{r}\ddot{r} + 2\left(\frac{r_0 v_0}{r}\right)\left(-\frac{r_0 v_0}{r^2}\dot{r}\right) + \frac{2k}{m}\dot{r} = 0 \quad \Rightarrow$$

$$\ddot{r} - \frac{r_0^2 v_0^2}{r^3} + \frac{k}{m}(r - r_0) = 0 \quad , \quad r(0) = r_0 \quad , \quad \dot{r}(0) = 0 \quad .$$

The particle performs a circular motion with radial oscillation with its angular momentum about the origin O being conserved. However, with the help of Fig. 8.2 (right), then the resultant force acting on the particle is given by

$$\boldsymbol{F} = \boldsymbol{F}_e = -T\boldsymbol{e}_r = -k(r - r_0)\boldsymbol{e}_r \quad .$$

This shows that the linear momentum of the particle is not conserved in the direction $\boldsymbol{e}_r$. Moreover, $\boldsymbol{F} \cdot \boldsymbol{e}_\theta = 0$ does not ensure that the linear momentum of the particle is conserved in this direction since $\boldsymbol{e}_\theta$ is not fixed in space.

Problem 2

Figure 8.3 shows a small ball of mass m which is attached to a rigid bar AB with length L and negligible. The bar is attached at its end A to a cart of mass M, which moves horizontally along a frictionless track. Moreover, the bar rotates freely about the vertical axis passing through A. At the time $t = 0$, $\theta(0) = 0$, the velocity of the cart is v_0 and the angular velocity of the bar is ω_0.

1. Determine the velocity of the cart when $\theta = \pi$.
2. Determine the angular velocity of the bar when $\theta = \pi$.
3. Determine the maximum and minimum angular velocities of the bar.
4. Determine the maximum and minimum velocities of the cart.

Express your answers in terms of $\{m, M, R, v_0, \omega_0\}$.

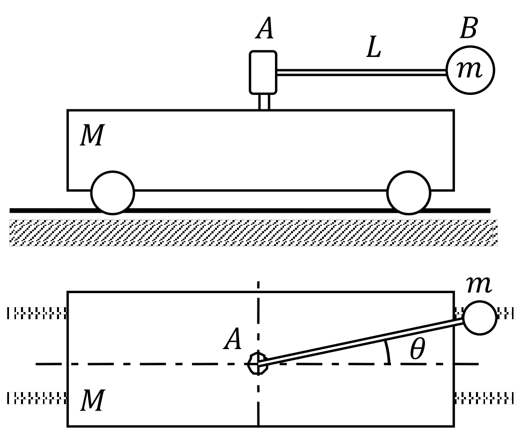

Figure 8.3

Solution:

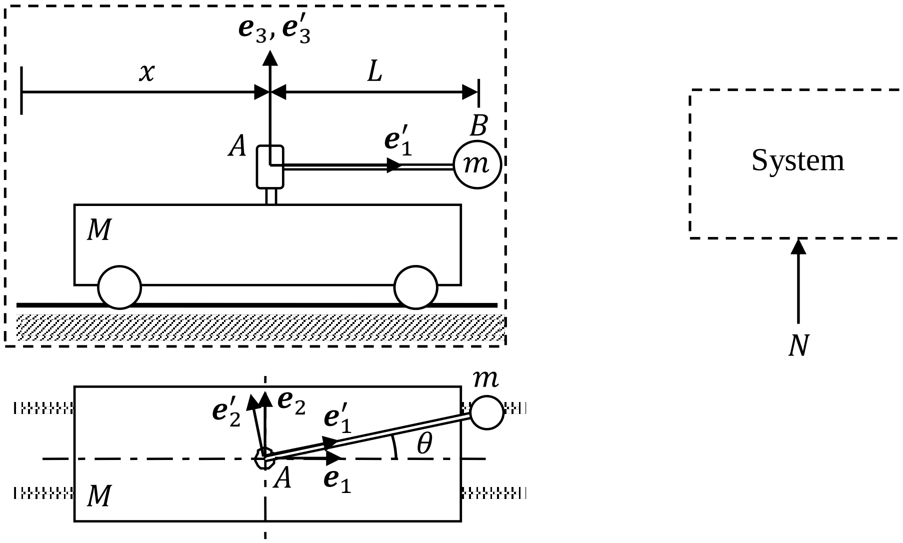

Figure 8.4

The system $\boldsymbol{e}_i'$ rotates with angular velocity $\boldsymbol{\omega}$, such that

$$\dot{\boldsymbol{e}}_i' = \boldsymbol{\omega} \times \boldsymbol{e}_i' \quad , \quad \boldsymbol{\omega} = \dot{\theta}\boldsymbol{e}_3' \quad .$$

Using Fig. 8.4 (right), then the resultant force acting on the system is given by

$$\boldsymbol{F} = N\boldsymbol{e}_2 \quad \Rightarrow \quad \boldsymbol{F} \cdot \boldsymbol{e}_1 = 0 \quad .$$

This shows that the linear momentum of the system in the fixed horizontal direction $\boldsymbol{e}_1$ is conserved, such that

$$M\boldsymbol{v}_A \cdot \boldsymbol{e}_1 + m\boldsymbol{v}_B \cdot \boldsymbol{e}_1 = M\boldsymbol{v}_A(0) \cdot \boldsymbol{e}_1 + m\boldsymbol{v}_B(0) \cdot \boldsymbol{e}_1 \quad ,$$

where,

$$\boldsymbol{v}_A = \dot{x}\boldsymbol{e}_1 \quad , \quad \boldsymbol{v}_A(0) = v_0\boldsymbol{e}_1 \quad ,$$

$$\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{\omega} \times \boldsymbol{x}_{B/A} = \dot{x}\boldsymbol{e}_1 + \dot{\theta}\boldsymbol{e}_3' \times L\boldsymbol{e}_1' = \dot{x}\boldsymbol{e}_1 + \dot{\theta}L\boldsymbol{e}_2' \quad \Rightarrow$$

$$\boldsymbol{v}_B = \left[\dot{x} - \dot{\theta}L\sin(\theta)\right]\boldsymbol{e}_1 + \dot{\theta}L\cos(\theta)\,\boldsymbol{e}_2 \quad , \quad \boldsymbol{v}_B(0) = v_0\boldsymbol{e}_1 + \dot{\theta}_0 L\boldsymbol{e}_2 \quad .$$

Therefore,

$$(M+m)\dot{x} - m\dot{\theta}L\sin(\theta) = (M+m)v_0 \quad .$$

So, the velocity of the cart when $\theta = \pi$ reduces to

$$\dot{x}(\pi) = v_0 \quad .$$

Next, since the only force acting on the system is the normal force, which is perpendicular to the velocity of the point where it acts, the energy of the system is conserved. Thus,

$$T_2 - T_1 = 0 \quad ,$$

where,

$$T_1 = \frac{1}{2}M[\boldsymbol{v}_A(0)\cdot\boldsymbol{v}_A(0)] + \frac{1}{2}m[\boldsymbol{v}_B(0)\cdot\boldsymbol{v}_B(0)] = \frac{1}{2}(M+m)v_0^2 + \frac{1}{2}m\omega_0^2L^2 \quad ,$$

$$T_2 = \frac{1}{2}M(\boldsymbol{v}_A\cdot\boldsymbol{v}_A) + \frac{1}{2}m(\boldsymbol{v}_B\cdot\boldsymbol{v}_B) = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m\left[\dot{\theta}^2L^2 - 2\dot{x}\dot{\theta}L\sin(\theta)\right] \quad ,$$

such that

$$(M+m)\dot{x}^2 + m\left[\dot{\theta}^2L^2 - 2\dot{x}\dot{\theta}L\sin(\theta)\right] = (M+m)v_0^2 + m\omega_0^2L^2 \quad .$$

So, the angular velocity of the bar when $\theta = \pi$ reduces to

$$\dot{\theta}(\pi) = \omega_0 \quad .$$

Now, solving the resulting equations of the conservation of energy and linear momentum in $\boldsymbol{e}_1$ for $\{\dot{x}, \dot{\theta}\}$, it follows that

$$\dot{\theta}^2 = f(\theta) \quad , \quad f(\theta) = \omega_0^2\left[1 - \frac{m}{M+m}\sin^2(\theta)\right]^{-1} \quad .$$

The critical values of $\dot{\theta}$ are obtained by requiring that $df(\theta)/d\theta = 0$. Thus,

$$\frac{df(\theta)}{d\theta} = \frac{m\omega_0^2\sin(2\theta)}{(M+m)\left[1 - \frac{m}{M+m}\sin^2(\theta)\right]} = 0 \quad \Rightarrow \quad \sin(2\theta) = 0 \quad \Rightarrow \quad \theta = \left\{0, \pm\frac{\pi}{2}\right\} \quad .$$

This shows that

$$\dot{\theta}_{\min} = \dot{\theta}(0) = \omega_0 \quad , \quad \dot{\theta}_{\max} = \dot{\theta}\left(\pm\frac{\pi}{2}\right) = \omega_0\sqrt{1 + \frac{m}{M}} \quad .$$

Furthermore,

$$\dot{x}_{\min} = \dot{x}\left(-\frac{\pi}{2}\right) = v_0 - \frac{mL\omega_0}{\sqrt{M(M+m)}} \quad , \quad \dot{x}_{\max} = \dot{x}\left(\frac{\pi}{2}\right) = v_0 + \frac{mL\omega_0}{\sqrt{M(M+m)}} \quad .$$

Problem 3

A tennis ball of mass m is released from rest at a height of 1600 [mm] above the ground, as shown in Fig. 8.5.

1. Determine the minimum coefficient of restitution for which the ball rises to a height of 1100 [mm] after the collision with the ground.
2. Determine the maximum energy lost in this case.

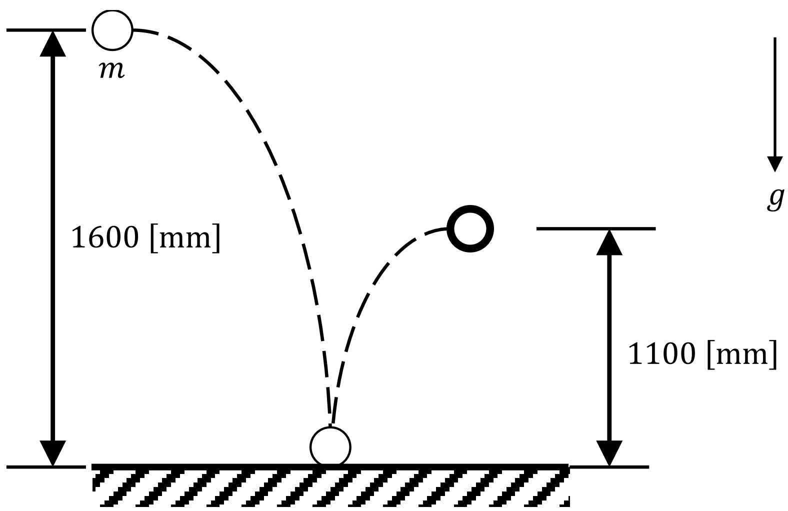

Figure 8.5

Solution:

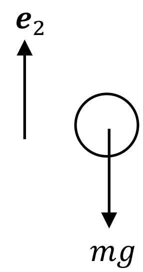

Figure 8.6

Just before impact, the energy of the particle is conserved since only the gravitational force acts on it (see Fig. 8.6). Hence,

$$(T_1 - T_0) + (V_{g1} - V_{g0}) = 0 \quad ,$$

where,

$$T_1 - T_0 = T_1 = \frac{1}{2} m v_{m1}^2 \quad , \quad V_{g1} - V_{g0} = mg\boldsymbol{e}_2 \cdot (0 - 1600)\boldsymbol{e}_2 = -1.6mg \quad .$$

Therefore, the velocity of the particle just before impact is given by

$$v_{m1} = \sqrt{3.2g} \;\; [m/s] \quad .$$

Next, using the coefficient of restitution e, with the subscript ‘s’ denoting the fixed horizontal surface, it follows that

$$e = \frac{(\boldsymbol{v}_{m2} - \boldsymbol{v}_{s2}) \cdot \boldsymbol{e}_2}{(\boldsymbol{v}_{s1} - \boldsymbol{v}_{m1}) \cdot \boldsymbol{e}_2} = -\frac{v_{m2}}{v_{m1}} = -\frac{v_{m2}}{\sqrt{3.2g}} \quad \Rightarrow \quad v_{m2} = -e\sqrt{3.2g} \quad .$$

After impact, the energy of the particle is conserved for similar arguments as before. Hence,

$$(T_3 - T_2) + (V_{g3} - V_{g2}) = 0 \quad ,$$

where,

$$T_3 - T_2 = -T_2 = -1.6mge^2 \quad , \quad V_{g3} - V_{g2} = mg\boldsymbol{e}_2 \cdot (1.1 - 0)\boldsymbol{e}_2 = 1.1mg \quad .$$

Hence,

$$1.6mge^2 + 1.1mg = 0 \quad \Rightarrow \quad e = \sqrt{\frac{1.1}{1.6}} \approx 0.829 \quad .$$

Furthermore, the energy lost during impact is given by

$$\Delta T = T_2 - T_1 = \frac{1}{2}m(v_{m2}^2 - v_{m1}^2) = \frac{1}{2}m(3.2ge^2 - 3.2g) = -\frac{1}{2}mg \quad .$$

Problem 3

Figure 8.7 shows a particle of mass m_1 which is attached to the ceiling through an inextensible string of length l_1. Moreover, a particle of mass m_2 is attached to m_1 through an inextensible string of length l_2. At the time $t = 0$, m_2 is released from rest at a distance l_1 below the ceiling and the string l_2 is unstretched. At the instant when $\cos(\alpha) = 0.8$ and $\sin(\alpha) = 0.6$, the string l_2 becomes taut.

Determine the velocities of the particles just after impact, when the string l_2 becomes taut.

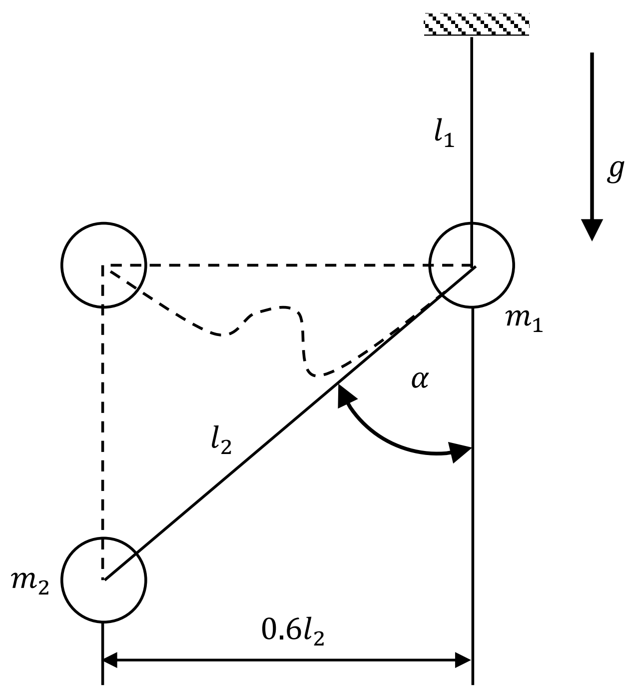

Figure 8.7

Solution:

Let A and B denote the particles of masses m_1 and m_2, respectively.

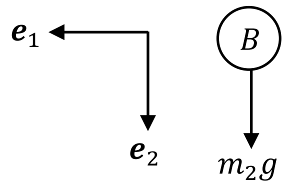

Figure 8.8

Fig. 8.8 shows that before impact, the energy of B is conserved since only the gravitational force acts on it. Hence,

$$(T_1 - T_0) + \left(V_{g1} - V_{g0}\right) = 0 \quad ,$$

where,

$$T_1 - T_0 = T_1 = \frac{1}{2} m_2 v_{B1}^2 \quad , \quad V_{g2} - V_{g1} = -m_2 g \boldsymbol{e}_2 \cdot l_2 \cos(\alpha)\, \boldsymbol{e}_2 = -m_2 g l_2 \cos(\alpha) \quad ,$$

such that the velocity of the B just before impact is given by

$$v_{B1} = \sqrt{2 g l_2 \cos(\alpha)} = \sqrt{1.6 g l_2} \quad .$$

Next, Fig. 8.9 shows the free body diagrams of A and B just after impact.

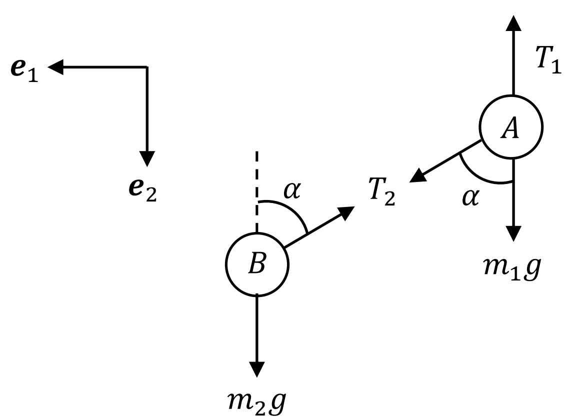

Figure 8.9

Now, the velocities of the particles just after impact are given by

$$\boldsymbol{v}_{A2} = (\boldsymbol{v}_{A2} \cdot \boldsymbol{e}_1)\boldsymbol{e}_1 \quad , \quad \boldsymbol{v}_{B2} = (\boldsymbol{v}_{B2} \cdot \boldsymbol{e}_1)\boldsymbol{e}_1 + (\boldsymbol{v}_{B2} \cdot \boldsymbol{e}_2)\boldsymbol{e}_2 \quad .$$

Since the gravitational forces are not impulsive (see Fig. 8.9), it follows from the balance of linear impulse-momentum of each particle that

$$\hat{T}_2 \sin(\alpha)\, \boldsymbol{e}_1 + \left[\hat{T}_2 \cos(\alpha) - \hat{T}_1\right]\boldsymbol{e}_2 = m_1(\boldsymbol{v}_{A2} \cdot \boldsymbol{e}_1)\boldsymbol{e}_1 \quad \Rightarrow$$

$$\boldsymbol{v}_{A2} \cdot \boldsymbol{e}_1 = \frac{1}{m_1}\hat{T}_2 \sin(\alpha) \quad , \quad \hat{T}_1 = \hat{T}_2 \cos(\alpha) \quad ,$$

$$-\hat{T}_2 \sin(\alpha)\, \boldsymbol{e}_1 - \hat{T}_2 \cos(\alpha)\, \boldsymbol{e}_2 = m_2\left[(\boldsymbol{v}_{B2} \cdot \boldsymbol{e}_1)\boldsymbol{e}_1 + \left\{(\boldsymbol{v}_{B2} \cdot \boldsymbol{e}_2) - \sqrt{1.6 g l_2}\right\}\boldsymbol{e}_2\right] \quad \Rightarrow$$

$$\boldsymbol{v}_{B2} \cdot \boldsymbol{e}_1 = -\frac{1}{m_2}\hat{T}_2 \sin(\alpha) \quad , \quad \boldsymbol{v}_{B2} \cdot \boldsymbol{e}_2 = \sqrt{1.6 g l_2} - \frac{1}{m_2}\hat{T}_2 \cos(\alpha) \quad .$$

Next, assuming that the strings remain taut just after impact, then

$$\boldsymbol{v}_{B2/A2} \cdot \boldsymbol{x}_{B2/A2} = 0 \quad ,$$

such that

$$-\left(\frac{1}{m_2} + \frac{1}{m_1}\right)\hat{T}_2 l_2 \sin^2(\alpha) + \left[\sqrt{1.6 g l_2} - \frac{1}{m_2}\hat{T}_2 \cos(\alpha)\right] l_2 \cos(\alpha) = 0 \quad \Rightarrow$$

$$\hat{T}_2 = \frac{m_1 m_2 \cos(\alpha)}{m_1 + m_2 \sin^2(\alpha)}\sqrt{1.6 g l_2} \quad .$$

Consequently,

$$\hat{T}_1 = \frac{m_1 m_2 \cos^2(\alpha)}{m_1 + m_2 \sin^2(\alpha)}\sqrt{1.6 g l_2} = \frac{16 m_1 m_2}{25 m_1 + 9 m_2} \quad ,$$

$$\boldsymbol{v}_{A2} \cdot \boldsymbol{e}_1 = \left[\frac{m_2 \cos(\alpha)\sin(\alpha)}{m_1 + m_2 \sin^2(\alpha)}\right]\sqrt{1.6 g l_2} = \left(\frac{12 m_2}{25 m_1 + 9 m_2}\right)\sqrt{1.6 g l_2} \quad ,$$

$$\boldsymbol{v}_{B2} \cdot \boldsymbol{e}_1 = -\frac{m_1 \cos(\alpha)\sin(\alpha)}{m_1 + m_2 \sin^2(\alpha)}\sqrt{1.6 g l_2} = -\left(\frac{12 m_1}{25 m_1 + 9 m_2}\right)\sqrt{1.6 g l_2} \quad ,$$

$$\boldsymbol{v}_{B2} \cdot \boldsymbol{e}_2 = \left[\frac{m_1 + m_2}{m_1 + m_2 \sin^2(\alpha)}\right]\sin^2(\alpha)\sqrt{1.6 g l_2} = 9\left(\frac{m_1 + m_2}{25 m_1 + 9 m_2}\right)\sqrt{1.6 g l_2} \quad .$$

Problem Set 9 Solutions

Problem 1

Fig. 9.1 shows two particles of masses $m_1 = m$ and $m_2 = 2m$ connected by a spring of stiffness k and free length L. The particles are initially at rest. At the time $t = 0$, a third particle of mass $m_3 = 3m$, traveling with speed v in a direction perpendicular to the spring, strikes m_1. The coefficient of restitution at impact is given by e.

1. Determine the velocity of each mass just after impact.
2. Determine the angular velocity of the line connecting m_1 and m_2 as a function of the distance $x(t)$ between these particles.
3. Determine the differential equation associated with $x(t)$.

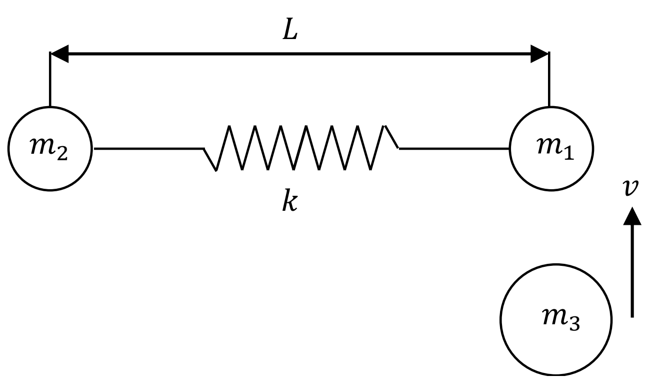

Figure 9.1

Solution:

Fig. 9.2 shows the free body diagram of the whole system at impact.

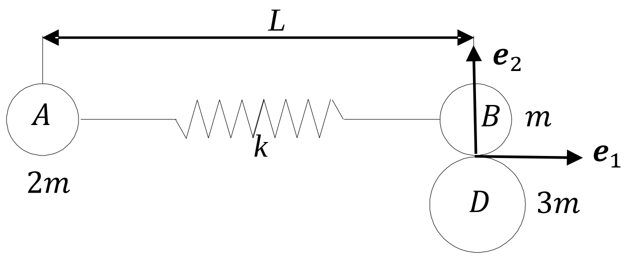

Figure 9.2

Using the definition of the coefficient of restitution, it follows that

$$e = \frac{(\boldsymbol{v}_{B2} - \boldsymbol{v}_{D2}) \cdot \boldsymbol{e}_2}{(\boldsymbol{v}_{D1} - \boldsymbol{v}_{B1}) \cdot \boldsymbol{e}_2} = \frac{(\boldsymbol{v}_{B2} \cdot \boldsymbol{e}_2) - (\boldsymbol{v}_{D2} \cdot \boldsymbol{e}_2)}{v} \quad \Rightarrow \quad (\boldsymbol{v}_{B2} \cdot \boldsymbol{e}_2) - (\boldsymbol{v}_{D2} \cdot \boldsymbol{e}_2) = ev \quad .$$

Fig. 9.3 shows the free body diagrams of each particle at impact.

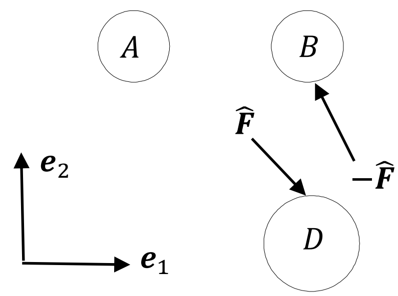

Figure 9.3

Since no forces act on A during impact,

$$\boldsymbol{v}_{A2} = \boldsymbol{0} \quad .$$

Now, assuming that B and D are smooth, then the impulsive force $\hat{\boldsymbol{F}}$ has no component in the $\boldsymbol{e}_1$ direction

$$\hat{\boldsymbol{F}} \cdot \boldsymbol{e}_1 = 0 \quad \Rightarrow \quad \boldsymbol{G}_2 \cdot \boldsymbol{e}_1 = \boldsymbol{G}_1 \cdot \boldsymbol{e}_1 \quad .$$

such that

$$\boldsymbol{v}_{D2} \cdot \boldsymbol{e}_1 = 0 \quad , \quad \boldsymbol{v}_{B2} \cdot \boldsymbol{e}_1 = 0 \quad .$$

This shows that the velocities of B and D just after impact are given, respectively, by

$$\boldsymbol{v}_{B2} = v_{B2}\boldsymbol{e}_2 \ \ , \ \ \boldsymbol{v}_{D2} = v_{D2}\boldsymbol{e}_2 \ \ .$$

Moreover, using these results, the coefficient of restitution equation reduces to

$$v_{B2} - v_{D2} = ev \ \ .$$

Next, Fig. 9.2 also shows that the linear momentum of the system is conserved during impact since no external forces act on it. Hence,

$$mv_{B2} + 3mv_{D2} = 3mv \ \ \Rightarrow \ \ v_{B2} + 3v_{D2} = 3v \ \ .$$

Solving the last two equations for $\{v_{B2}, v_{D2}\}$ yields

$$v_{B2} = \frac{3}{4}(1+e)v \ , \ v_{D2} = \frac{1}{4}(3-e)v \ \ .$$

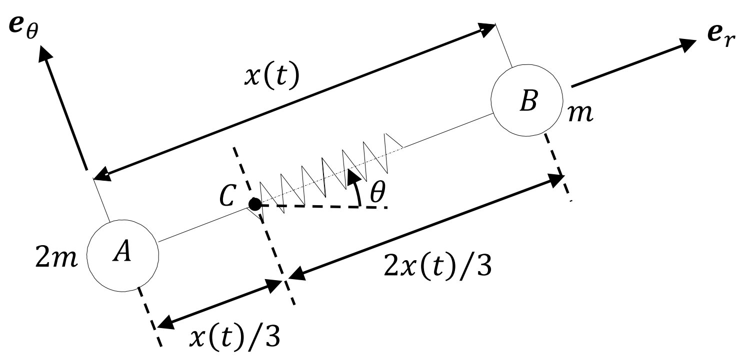

Figure 9.4

The location of the center of mass C of the system consisting of the particles A and B relative to A is shown in Fig. 9.4 and given by

$$\boldsymbol{x}_{C/A} = \frac{\sum_{i=A}^{B} m_i \boldsymbol{x}_{i/A}}{\sum_{i=A}^{B} m_i} = \frac{2m(\boldsymbol{0}) + mx(t)\boldsymbol{e}_r}{3m} = \frac{1}{3}x(t)\boldsymbol{e}_r \ \ .$$

Moreover, since no external forces act on this system, its linear momentum is conserved so that the velocity $\boldsymbol{v}_C$ of the center of mass C is conserved as well. Hence,

$$\boldsymbol{v}_C = \boldsymbol{v}_C(t_2) = \frac{2m\boldsymbol{v}_{A2} + m\boldsymbol{v}_{B2}}{3m} = \frac{1}{4}(1+e)v\boldsymbol{e}_2 \ \ .$$

Furthermore, for the same reason, the angular momentum about the center of mass C is conserved. Thus,

$$\boldsymbol{H}_C(t_3) = \boldsymbol{H}_C(t_2) \ , \ \ \boldsymbol{H}_C(t) = \sum_{i=A}^{B} \boldsymbol{x}_{i/C}(t) \times m_i \boldsymbol{v}_{i/C}(t) \ \ ,$$

where,

$$\boldsymbol{x}_{A/C} = -\frac{x}{3}\boldsymbol{e}_r \ , \ \ \boldsymbol{v}_{A/C} = -\frac{\dot{x}}{3}\boldsymbol{e}_r - \frac{x\dot{\theta}}{3}\boldsymbol{e}_\theta \ \ ,$$

$$\boldsymbol{x}_{B/C} = \frac{2x}{3}\boldsymbol{e}_r \ , \ \ \boldsymbol{v}_{B/C} = \frac{2\dot{x}}{3}\boldsymbol{e}_r + \frac{2x\dot{\theta}}{3}\boldsymbol{e}_\theta \ \ ,$$

such that,

$$\boldsymbol{H}_C(t_2) = \sum_{i=A}^{B} \boldsymbol{x}_{i/C}(t_2) \times m_i \boldsymbol{v}_{i/C}(t_2) = \boldsymbol{x}_{B/C}(t_2) \times m\boldsymbol{v}_{B/C}(t_2)$$

$$= \frac{2L}{3}\boldsymbol{e}_1 \times m\left[\frac{3}{4}(1+e)v\boldsymbol{e}_2\right] = \frac{1}{2}(1+e)mvL\boldsymbol{e}_3 \ \ ,$$

$$\boldsymbol{H}_C(t_3) = \sum_{i=A}^{B} \boldsymbol{x}_{i/C}(t_3) \times m_i \boldsymbol{v}_{i/C}(t_3)$$

$$= -\frac{x}{3}\boldsymbol{e}_r \times 2m\left(-\frac{\dot{x}}{3}\boldsymbol{e}_r - \frac{x\dot{\theta}}{3}\boldsymbol{e}_\theta\right) + \frac{2x}{3}\boldsymbol{e}_r \times m\left(\frac{2\dot{x}}{3}\boldsymbol{e}_r + \frac{2x\dot{\theta}}{3}\boldsymbol{e}_\theta\right)$$

$$= \frac{2mx^2\dot{\theta}}{3}\boldsymbol{e}_3 \ \ .$$

Equating both of these expressions yields

$$\dot{\theta} = \frac{3(1+e)vL}{4x^2} \ .$$

Again, for the same reason, the energy of the system is conserved, so that

$$T_3 = T_2 \ \ .$$

Moreover, the velocities of A and B take the forms

$$\boldsymbol{v}_A = \boldsymbol{v}_C + \boldsymbol{v}_{A/C} \quad , \quad \boldsymbol{v}_B = \boldsymbol{v}_C + \boldsymbol{v}_{B/C} \quad ,$$

where,

$$T_2 = \frac{1}{2} m v_{B2}^2 = \frac{9}{32}(1+e)^2 m v^2 \quad , \quad T_3 = \frac{1}{2}(2m)(\boldsymbol{v}_A \cdot \boldsymbol{v}_A) + \frac{1}{2} m (\boldsymbol{v}_B \cdot \boldsymbol{v}_B) \quad .$$

With the help of the transformation relation

$$\boldsymbol{e}_2 = \sin(\theta)\, \boldsymbol{e}_r + \cos(\theta)\, \boldsymbol{e}_\theta \quad ,$$

it follows that

$$\boldsymbol{v}_A = \left[\frac{1}{4}(1+e) v \sin(\theta) - \frac{\dot{x}}{3}\right] \boldsymbol{e}_r + \left[\frac{1}{4}(1+e) v \cos(\theta) - \frac{x\dot{\theta}}{3}\right] \boldsymbol{e}_\theta \quad ,$$

$$\boldsymbol{v}_B = \left[\frac{1}{4}(1+e) v \sin(\theta) + \frac{2\dot{x}}{3}\right] \boldsymbol{e}_r + \left[\frac{1}{4}(1+e) v \cos(\theta) + \frac{2x\dot{\theta}}{3}\right] \boldsymbol{e}_\theta \quad .$$

Thus,

$$T_3 = \frac{3m}{32}(1+e)^2 v^2 \sin^2(\theta) + \frac{m\dot{x}^2}{3} + \frac{3m}{32}(1+e)^2 v^2 \cos^2(\theta) + \frac{m x^2 \dot{\theta}^2}{3} \quad .$$

Now, equating the expressions of T_2 and T_3 yields

$$\dot{x}^2 + x^2 \dot{\theta}^2 = \frac{9}{16}(1+e)^2 v^2 \quad .$$

Substituting the expression of $\dot{\theta}$ obtained previously into this equation gives

$$\dot{x}^2 + \frac{9}{16}(1+e)^2 v^2 \left(\frac{L^2}{x^2} - 1\right) = 0 \quad .$$

Solving this equation for $x(t)$, it follows that

$$\frac{x}{\sqrt{x^2 - L^2}} dx = \frac{3}{4}(1+e) v \, dt \;\Rightarrow\; \sqrt{x^2 - L^2} = \frac{3}{4}(1+e) v t \;\Rightarrow$$

$$x(t) = \sqrt{L^2 + \left[\frac{3}{4}(1+e) v t\right]^2}$$

Problem 2

The upper end B of the bar AB, having a length of L and mass of m, is connected to the fixed point C by an inextensible rope, as shown in Fig. 9.5. At the time $t = 0$, the rope is cut, with $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$.

1. Determine the angular speed $\dot{\theta}$ of the bar as a function of θ.
2. Determine the reaction forces at A.

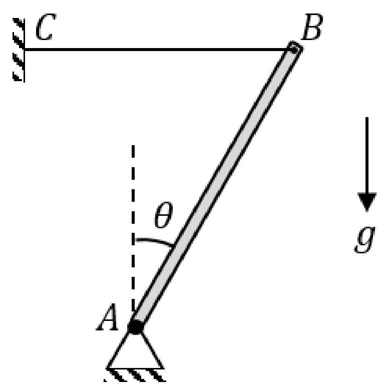

Figure 9.5

Solution:

Fig. 9.6 shows the free body diagram of the bar just after the rope is cut.

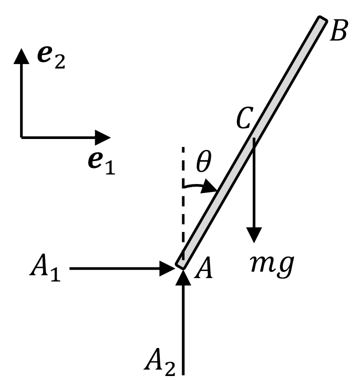

Figure 9.6

Since the reaction force $\{A_1, A_2\}$ at the fixed point A don't do work on the bar, its energy is conserved. Therefore,

$$(T_2 - T_1) + (V_{g2} - V_{g1}) = 0 \quad ,$$

where,

$$V_{g2} - V_{g1} = -(-mg\boldsymbol{e}_2) \cdot \left[\frac{L}{2}\cos(\theta) - \frac{L}{2}\cos(\theta_0)\right]\boldsymbol{e}_2 = -\frac{mgL}{2}[\cos(\theta_0) - \cos(\theta)] \quad ,$$

$$T_2 - T_1 = T_2 = \frac{1}{2}m(\bar{\boldsymbol{v}} \cdot \bar{\boldsymbol{v}}) + \frac{1}{2}\bar{I}\dot{\theta}^2 \quad ,$$

$$\bar{\boldsymbol{v}} = \dot{\theta}\boldsymbol{e}_3 \times \frac{L}{2}[\sin(\theta)\,\boldsymbol{e}_1 + \cos(\theta)\,\boldsymbol{e}_2] = \frac{\dot{\theta}L}{2}[-\cos(\theta)\,\boldsymbol{e}_1 + \sin(\theta)\,\boldsymbol{e}_2] \quad \Rightarrow$$

$$T_2 = \frac{1}{2}m\left(\frac{\dot{\theta}L}{2}\right)^2 + \frac{1}{2}\left(\frac{mL^2}{12}\right)\dot{\theta}^2 = \frac{mL^2}{6}\dot{\theta}^2 \quad ,$$

such that

$$\frac{mL^2}{6}\dot{\theta}^2 - \frac{mgL}{2}[\cos(\theta_0) - \cos(\theta)] = 0 \quad \Rightarrow$$

$$\dot{\theta}^2 = \frac{3g}{L}[\cos(\theta_0) - \cos(\theta)]$$

Also, since the bar rotates clockwise, the angular velocity is given by $\dot{\theta} = -\sqrt{\dot{\theta}^2}$.

Next, the balance of linear momentum of the bar yields

$$A_1\boldsymbol{e}_1 + (A_2 - mg)\boldsymbol{e}_2 = m\dot{\bar{\boldsymbol{v}}} \quad ,$$

$$\dot{\bar{\boldsymbol{v}}} = \frac{L}{2}\left[\{\dot{\theta}^2\sin(\theta) - \ddot{\theta}\cos(\theta)\}\boldsymbol{e}_1 + \{\ddot{\theta}\sin(\theta) + \dot{\theta}^2\cos(\theta)\}\boldsymbol{e}_2\right] \quad .$$

Therefore,

$$A_1 = \frac{mL}{2}\left[\dot{\theta}^2\sin(\theta) - \ddot{\theta}\cos(\theta)\right] \quad , \quad A_2 = m\left[g + \frac{L}{2}\{\ddot{\theta}\sin(\theta) + \dot{\theta}^2\cos(\theta)\}\right] \quad .$$

Furthermore, using the balance of angular momentum about the fixed point A gives

$$\boldsymbol{M}_A = \frac{L}{2}[\sin(\theta)\,\boldsymbol{e}_1 + \cos(\theta)\,\boldsymbol{e}_2] \times (-mg\boldsymbol{e}_2) = -\frac{mgL}{2}\sin(\theta)\,\boldsymbol{e}_3 = \dot{\boldsymbol{H}}_A \quad ,$$

where,

$$\boldsymbol{H}_A = \boldsymbol{H}_C + \boldsymbol{x}_{C/A} \times m\bar{\boldsymbol{v}} = \frac{mL^2\dot{\theta}}{12}\boldsymbol{e}_3 + \frac{mL^2\dot{\theta}}{4}\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta) & \sin(\theta) & 0 \end{vmatrix} = \frac{mL^2\dot{\theta}}{3}\boldsymbol{e}_3 \;\; \Rightarrow$$

$$\dot{\boldsymbol{H}}_A = \frac{mL^2\ddot{\theta}}{3}\boldsymbol{e}_3 \quad .$$

Thus,

$$\ddot{\theta} = -\frac{3g}{2L}\sin(\theta) \quad .$$

Substituting the expressions of $\{\dot{\theta}^2, \ddot{\theta}\}$ into the expressions of the reaction forces $\{A_1, A_2\}$, it follows that

$$A_1 = \frac{3mg}{2}\sin(\theta)\left[\cos(\theta_0) - \frac{1}{2}\cos(\theta)\right] \quad ,$$

$$A_2 = \frac{mg}{2}\left[3\cos(\theta_0)\cos(\theta) + \frac{3}{2}\sin^2(\theta) - 1\right] \quad .$$

Problem 3

Fig. 9.7 shows a cylinder of mass m and radius R which is being pulled to the right by a constant horizontal force P at its center C. Initially, at the time $t = 0$, $x(0) = 0$, $\dot{x}(0) = -\omega_0 R$, $\theta(0) = 0$ and $\dot{\theta}(0) = \omega_0 > 0$.

Determine $x(t)$ and $\theta(t)$ and the magnitude of the friction force between the cylinder and the ground for the following cases:

1. $P = 0$.
2. $P = 2\mu mg$.
3. $P = 4\mu mg$.

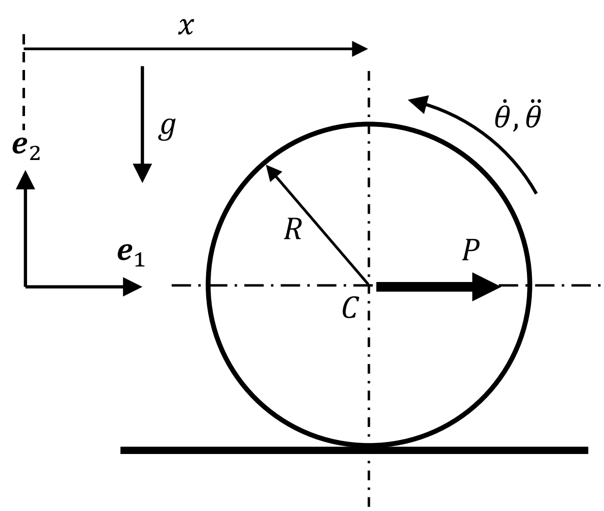

Figure 9.7

Solution:

The free body diagram of the cylinder is shown in Fig. 9.8.

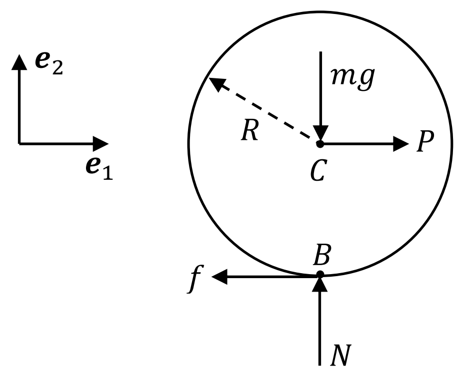

Figure 9.8

Assuming that the cylinder rolls without slipping along the ground, then

$$\boldsymbol{v}_{C/B} = \boldsymbol{v}_C = \dot{x}\boldsymbol{e}_1 = \boldsymbol{\omega} \times \boldsymbol{x}_{C/B} = \omega\boldsymbol{e}_3 \times R\boldsymbol{e}_2 = -\omega R\boldsymbol{e}_1 \quad \Rightarrow \quad \dot{x} = -\omega R \ \ , \ \ \ddot{x} = -\dot{\omega}R \ \ .$$

Moreover, the balance equations of linear momentum and angular momentum about the center C yield

$$N = mg \ \ , \ \ P - f = m\ddot{x} \ \ ,$$

$$-Rf = I_C\dot{\omega} = \frac{1}{2}mR^2\dot{\omega} \ \ \Rightarrow \ \ f = -\frac{1}{2}mR\dot{\omega} = \frac{1}{2}m\ddot{x} \ \ .$$

Therefore,

$$f = \frac{1}{3}P \ \ , \ \ \ddot{x} = \frac{2}{3}\frac{P}{m} \ \ , \ \ \dot{\omega} = -\frac{2}{3}\frac{P}{mR} \ \ .$$

Now, in rolling without slipping, the friction force is static. Thus,

$$|f| \le \mu|N| \ \ \Rightarrow \ \ P \le 3\mu mg \ \ .$$

Case 1: $P = 0$

Since $P < 3\mu mg$, the cylinder rolls without slipping along the ground. Thus,

$$\ddot{x} = 0 \ \ \Rightarrow \ \ x(t) = x(0) + \dot{x}(0)t = -\omega_0 Rt \ \ ,$$

$$\dot{\omega} = 0 \quad \Rightarrow \quad \theta(t) = \theta(0) + \dot{\theta}(0)t = \omega_0 t \quad .$$

Case 2: $P = 2\mu mg$

Since $P < 3\mu mg$, the cylinder rolls without slipping along the ground. Thus,

$$\ddot{x} = \frac{2}{3}\frac{P}{m} = \frac{4\mu g}{3} \quad \Rightarrow \quad x(t) = x(0) + \dot{x}(0)t + \frac{1}{2}\left(\frac{4\mu g}{3}\right)t^2 = \left(\frac{2\mu g t}{3} - \omega_0 R\right)t \quad ,$$

$$\dot{\omega} = -\frac{2}{3}\frac{P}{mR} = -\frac{4\mu g}{3R} \quad \Rightarrow \quad \theta(t) = \theta(0) + \dot{\theta}(0)t + \frac{1}{2}\left(-\frac{4\mu g}{3R}\right)t^2 = \left(\omega_0 - \frac{2\mu g}{3R}\right)t \quad .$$

Case 3: $P = 4\mu mg$

Since $P > 3\mu mg$, the cylinder slips on the ground. Thus,

$$f = \mu N = \mu mg \quad .$$

Furthermore, the equations of motion are given by

$$P - \mu mg = m\ddot{x} \quad \Rightarrow \quad \ddot{x} = 3\mu g \quad , \quad -R\mu mg = \frac{1}{2}mR^2\dot{\omega} \quad \Rightarrow \quad \dot{\omega} = -\frac{2\mu g}{R} \quad .$$

Thus,

$$x(t) = x(0) + \dot{x}(0)t + \frac{1}{2}(3\mu g)t^2 = \left(\frac{3\mu g t}{2} - \omega_0 R\right)t \quad ,$$

$$\theta(t) = \theta(0) + \dot{\theta}(0)t + \frac{1}{2}\left(-\frac{2\mu g}{R}\right)t^2 = \left(\omega_0 - \frac{\mu g t}{R}\right)t \quad .$$

Problem 4

A bowling ball of mass m and radius R is thrown onto the ground with a velocity v_0 that is essentially horizontal. The friction coefficient between the ball and the ground is μ. Initially, at the time $t = 0$, $\theta(0) = 0$ and $\dot{\theta}(0) = 0$.

Determine the distance traveled by the ball before it starts rolling without slipping on the ground.

Solution:

The free body diagram of the ball is shown in Fig. 9.9.

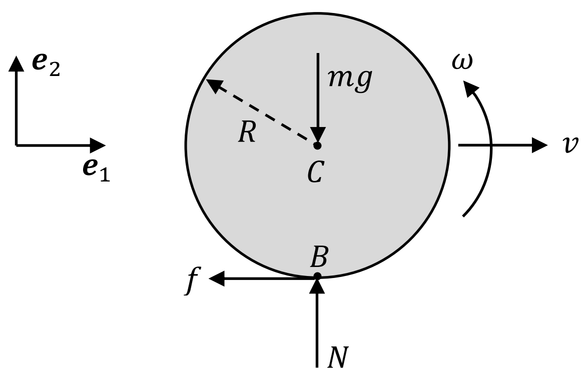

Figure 9.9

Since the ball slips on the ground, the friction force f acting on it at the contact point B with the ground is given by

$$f = \mu N \quad .$$

Now, using the balance of linear momentum and angular momentum about the center C of the ball, it follows that

$$N = mg \quad , \quad -f = -\mu mg = m\dot{v} \;\; \Rightarrow \;\; \dot{v} = -\mu g \;\; \Rightarrow \;\; v = v(0) - \mu g t = v_0 - \mu g t \quad ;$$

$$-Rf = -\mu mgR = \bar{I}\ddot{\theta} = \frac{2}{5}mR^2\dot{\theta} \;\; \Rightarrow \;\; \ddot{\theta} = -\frac{5\mu g}{2R} \;\; \Rightarrow \;\; \dot{\theta} = \dot{\theta}(0) - \frac{5\mu g t}{2R} = -\frac{5\mu g t}{2R} \quad .$$

Next, when the ball starts rolling without slipping, the velocity of the contact point B with the ground vanishes, $\boldsymbol{v}_B = 0$. Thus, the time spent up to this point is calculated using the kinematic condition for rolling without slipping, namely

$$\boldsymbol{v}_C = \boldsymbol{v}_{C/B} = \boldsymbol{\omega} \times \boldsymbol{x}_{C/B} = -R\omega\boldsymbol{e}_1 \;\;\Rightarrow\;\; v_0 - \mu g t = -R\left(-\frac{5\mu g t}{2R}\right)t \;\;\Rightarrow\;\; t = \frac{2v_0}{7\mu g} \;\; .$$

Moreover, the distance traveled up to this point is given by

$$x = x(0) + v_0 t - \frac{1}{2}\mu g t^2 = v_0\left(\frac{2v_0}{7\mu g}\right) - \frac{1}{2}\mu g\left(\frac{2v_0}{7\mu g}\right)^2 = \frac{12v_0^2}{49\mu g} \;\; .$$

Problem Set 10 Solutions

Problem 1

Consider the assembly shown in Fig. 10.1. The hanging block of mass m_1 is attached to the cylinder of center B, mass m_2 and radius r_2 by an inextensible cord, wrapped at a radius r_1 and passes over a drum of center A, mass m_3 and radius R.

It is assumed that the cord does not slip on the drum and the cylinder. Moreover, the coefficient of friction between the cylinder and the ground is μ.

1. Assuming that the cylinder rolls without slipping along the ground, determine the acceleration of the block.
2. Assuming that
   $\{\mu = 0.3, m_1 = m, m_2 = m/2\,, r_1 = r, r_2 = 2r, \bar{I}_2 = 6mr^2, \bar{I}_3 = 3mr^2, R = r_1 + r_2\}$,
   show that the cylinder slips along the ground. Also, determine the acceleration of the block.

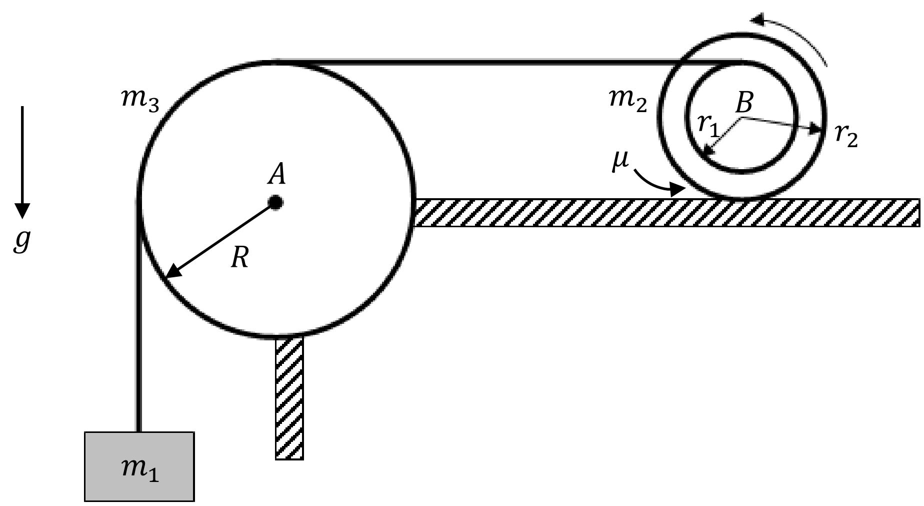

Figure 10.1

Solution:

The free body diagrams of each part is shown in Fig. 10.2.

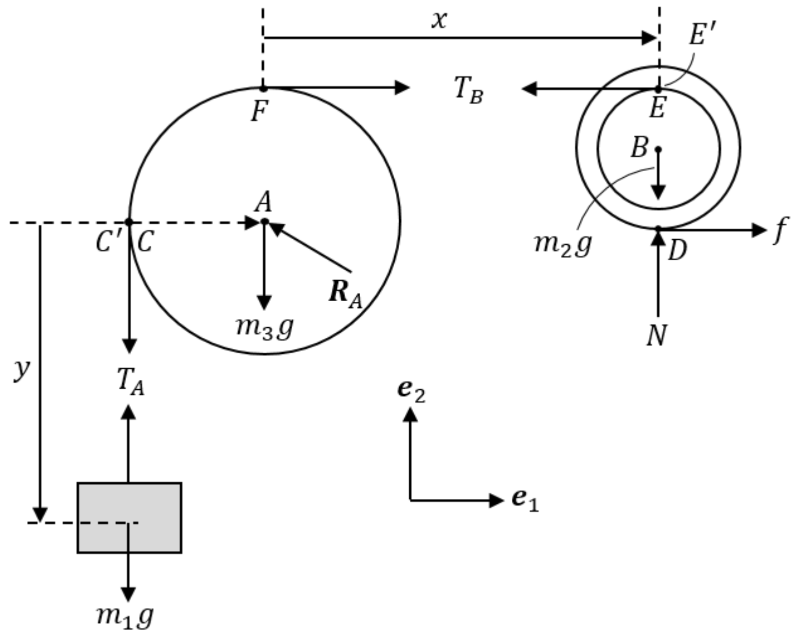

Figure 10.2

The balance of linear momentum equations of the block and cylinder are given by

$$T_A - m_1 g = -m_1 \ddot{y} \quad ; \quad N = m_2 g \quad , \quad f - T_B = m_2 \ddot{x} \quad .$$

Moreover, the balance of angular momentum equations of the drum and cylinder about their center of mass yield

$$(T_A - T_B)R = I_A \dot{\omega}_A \quad , \quad f r_2 + T_B r_1 = I_B \dot{\omega}_B \quad .$$

Next, since the cylinder rolls without slipping on the ground

$$\boldsymbol{v}_D = \dot{x}\boldsymbol{e}_1 + \omega_B \boldsymbol{e}_3 \times (-r_2 \boldsymbol{e}_2) = (\dot{x} + \omega_B r_2)\boldsymbol{e}_1 = \boldsymbol{0} \quad \Rightarrow \quad \ddot{x} = -\dot{\omega}_B r_2 \quad .$$

Now, since the cord does not slip on the drum and the cylinder

$$\boldsymbol{v}_{C'} = \boldsymbol{v}_C = \omega_A \boldsymbol{e}_3 \times (-R\boldsymbol{e}_1) = -\omega_A R \boldsymbol{e}_2 \quad , \quad \boldsymbol{v}_{F'} = \boldsymbol{v}_F = \omega_A \boldsymbol{e}_3 \times R\boldsymbol{e}_2 = -\omega_A R \boldsymbol{e}_1 \quad ,$$

$$\boldsymbol{v}_{E'} = \boldsymbol{v}_E = \dot{x}\boldsymbol{e}_1 + \omega_B \boldsymbol{e}_3 \times r_1 \boldsymbol{e}_2 = (\dot{x} - \omega_B r_1)\boldsymbol{e}_1 \quad .$$

Also, the inextensibility of the cord yields

$$\boldsymbol{v}_{C'} = -\dot{y}\boldsymbol{e}_2 \quad , \quad \boldsymbol{v}_{F'} = \boldsymbol{v}_{E'} \quad .$$

Thus,

$$\ddot{y} = \dot{\omega}_A R \quad , \quad \ddot{x} = \dot{\omega}_B r_1 - \dot{\omega}_A R \quad .$$

In summary, the equations of motion to be solved are given by

$$T_A - m_1 g = -m_1 \ddot{y} \quad , \quad f - T_B = m_2 \ddot{x} \quad , \quad (T_A - T_B)R = I_A \dot{\omega}_A \quad , \quad f r_2 + T_B r_1 = I_B \dot{\omega}_B \quad ,$$

$$\ddot{x} = -\dot{\omega}_B r_2 \quad , \quad \ddot{y} = \dot{\omega}_A R \quad , \quad \ddot{x} = \dot{\omega}_B r_1 - \dot{\omega}_A R \quad .$$

Consequently,

$$\ddot{y} = \left[\frac{m_1 R^2 (r_1 + r_2)^2}{\{I_A + (m_1 + m_2)R^2\} r_2^2 + 2(I_A + m_1 R^2) r_1 r_2 + (I_B + m_1 r_1^2) R^2 + I_A r_1^2}\right] g \quad .$$

Next, since the cylinder rolls without slipping on the ground, the frictional force satisfies

$$|f| \leq |\mu N| \quad .$$

where $N = m_2 g$ and f is obtained by solving the equations of motion, such that

$$f = \left[\frac{(I_B - m_2 r_1 r_2) R^2}{\{I_A + (m_1 + m_2)R^2\} r_2^2 + 2(I_A + m_1 R^2) r_1 r_2 + (I_B + m_1 r_1^2) R^2 + I_A r_1^2}\right] m_1 g \quad .$$

Therefore,

$$\mu \geq \left[\frac{(I_B - m_2 r_1 r_2) R^2}{\{I_A + (m_1 + m_2)R^2\} r_2^2 + 2(I_A + m_1 R^2) r_1 r_2 + (I_B + m_1 r_1^2) R^2 + I_A r_1^2}\right] \frac{m_1}{m_2} \quad .$$

Substituting the given data into this inequality yields

$$\mu \geq \frac{1}{2} \quad ,$$

which shows that if $\mu = 1/3$, the cylinder slips along the ground. Also, in this case, the frictional force is given by $f = \mu N = m_2 g/3$ and the equations of motion reduce to

$$T_A - m_1 g = -m_1 \ddot{y} \quad , \quad \frac{1}{3} m_2 g - T_B = m_2 \ddot{x} \quad , \quad (T_A - T_B)R = I_A \dot{\omega}_A \quad ,$$

$$\frac{1}{3} m_2 g r_2 + T_B r_1 = I_B \dot{\omega}_B \quad , \quad \ddot{y} = \dot{\omega}_A R \quad , \quad \ddot{x} = \dot{\omega}_B r_1 - \dot{\omega}_A R \quad .$$

Solving this system of equations for $\ddot{y}$, it follows that

$$\ddot{y} = \left[\frac{\{3m_1 m_2 r_1^2 + m_2^2 r_1 r_2 + (3m_1 - m_2)I_B\}R^2}{\{I_A + (m_1 + m_2)R^2\}I_B + m_2 r_1^2 (I_A + m_1 R^2)}\right]\frac{g}{3} \quad .$$

Substituting the given data into this expression gives

$$\ddot{y} = \frac{17}{35}g \approx 0.486g \quad .$$

Problem 2

Figure 10.3 shows a cylinder of center C, mass m_1 and radius R which is placed on a stationary box of mass m_2. The coefficient of friction between the cylinder and the box is μ. At the time $t = 0$, a block of mass m_1, moving freely with a leftward velocity of v, strikes the box and sticks to it.

1. Assuming that $\mu = 0$, determine the velocities of the box and the center C of the cylinder just after impact.
2. Assuming that $\mu > 0$ and the cylinder slips on the box during impact, determine the velocities of the box and the center C of the cylinder just after impact.
3. Using your answers in part 2, determine the time it takes for the cylinder to begin rolling without slipping on the box.
4. Assuming that $\mu \to \infty$, determine the velocities of the box and the center C of the cylinder just after impact.

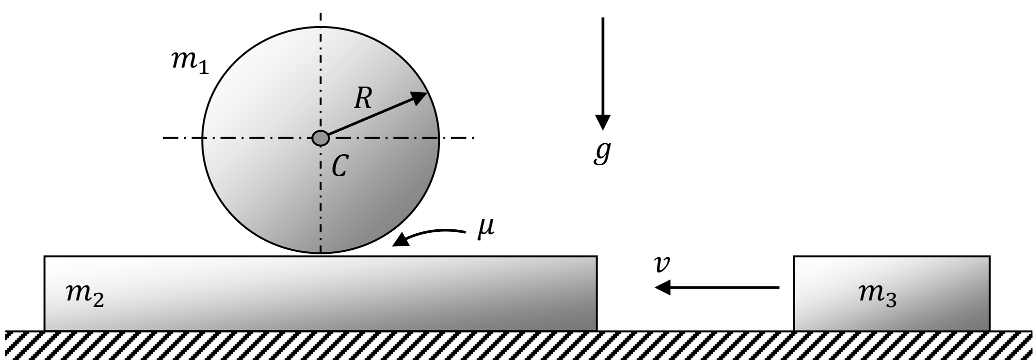

Figure 10.3

Solution:

The free body diagrams of the cylinder and the system consisting of the box and the block at impact are shown in Fig. 10.4.

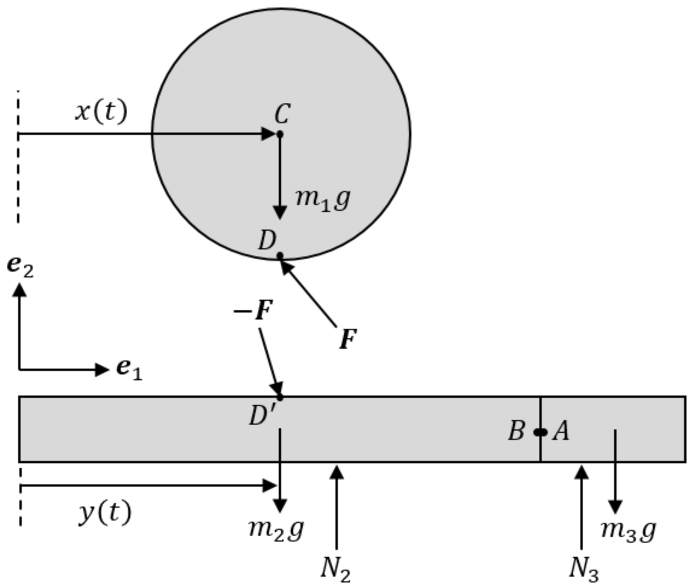

Figure 10.4

Since the block sticks to the box, their velocities $\{\boldsymbol{v}_{A2}, \boldsymbol{v}_{B2}\}$ just after impact are equal

$$\boldsymbol{v}_{A2} = \boldsymbol{v}_{B2} = u\boldsymbol{e}_1 \quad .$$

Next, for a finite value of μ, the impulsive force $\widehat{\boldsymbol{F}} = \int_{t_1}^{t_2} \boldsymbol{F} dt$ at impact $(t_2 \to t_1)$ vanishes. Thus, the balance equations of linear impulse-momentum and angular-impulse momentum about the center of mass C of the cylinder give

$$\widehat{\boldsymbol{F}} = m_1(\boldsymbol{v}_{C2} - \boldsymbol{v}_{C1}) = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{v}_{C2} = \boldsymbol{v}_{C1} = \boldsymbol{0} \quad ,$$

$$\widehat{\boldsymbol{M}}_C = I_C(\omega_2 - \omega_1)\boldsymbol{e}_3 = \boldsymbol{0} \quad \Rightarrow \quad \omega_2 = \omega_1 = 0 \quad .$$

Moreover, the balance of linear impulse-momentum equation of the system consisting of the box and the block yields

$$-\widehat{\boldsymbol{F}} = m_2(\boldsymbol{v}_{B2} - \boldsymbol{v}_{B1}) + m_3(\boldsymbol{v}_{A2} - \boldsymbol{v}_{A1}) = \boldsymbol{0} \quad \Rightarrow$$

$$m_2 u\boldsymbol{e}_1 + m_3[u\boldsymbol{e}_1 - (-v\boldsymbol{e}_1)] = \boldsymbol{0} \quad \Rightarrow \quad u = -\left(\frac{m_3}{m_2+m_3}\right)v \quad ,$$

such that

$$\boldsymbol{v}_{A2} = \boldsymbol{v}_{B2} = -\left(\frac{m_3}{m_2+m_3}\right)v\boldsymbol{e}_1 \quad .$$

Now, if the cylinder slips along the box, then

$$\boldsymbol{F} = -\mathrm{sgn}(\boldsymbol{v}_{slip}) f\boldsymbol{e}_1 + N\boldsymbol{e}_2 \quad , \quad f = \mu N \quad ,$$

$$\mathrm{sgn}(\boldsymbol{v}_{slip}) = \mathrm{sgn}(\boldsymbol{v}_{D/D'}) = \mathrm{sgn}(-\boldsymbol{v}_{D'}) = \mathrm{sgn}(-\boldsymbol{v}_B) = +1 \quad .$$

Therefore, the balance of linear momentum of the cylinder yields

$$N = m_1 g \quad , \quad -\mu m_1 g = m_1\ddot{x} \quad \Rightarrow \quad \ddot{x} = -\mu g \quad , \quad \dot{x}(0) = 0 \quad \Rightarrow \quad \dot{x} = -\mu g t \quad .$$

Furthermore, using the balance of angular momentum of the cylinder about its center of mass C, it follows that

$$-\mu m_1 g R = I_C\dot{\omega} \quad , \quad \omega(0) = 0 \quad \Rightarrow \quad \omega = -\frac{\mu m_1 g R}{I_C}t = -\frac{\mu m_1 g R}{\frac{1}{2}m_1 R^2}t = -\frac{2\mu g}{R}t \quad .$$

Next, at the instant when the cylinder starts rolling without slipping on the box, then

$$\boldsymbol{v}_C = -\mu g t\boldsymbol{e}_1 = \boldsymbol{v}_D + \boldsymbol{\omega}\times\boldsymbol{x}_{C/D} = \boldsymbol{v}'_D - \frac{2\mu g}{R}t\boldsymbol{e}_3 \times R\boldsymbol{e}_2 \quad \Rightarrow \quad \boldsymbol{v}'_D = -3\mu g t\boldsymbol{e}_1 \quad .$$

where the velocity $\boldsymbol{v}_{D'}$ of the box is calculated using the balance of linear momentum, such that

$$-\boldsymbol{F} + (N_2 - m_2 g)\boldsymbol{e}_2 = \mu m_1 g\boldsymbol{e}_1 + (N_2 - m_1 g - m_2 g)\boldsymbol{e}_2 = m_2\ddot{y}\boldsymbol{e}_1 \quad \Rightarrow$$

$$N_2 = (m_1+m_2)g \quad , \quad \ddot{y} = \frac{\mu m_1 g}{m_2} \quad , \quad \dot{y}(0) = u = -\left(\frac{m_3}{m_2+m_3}\right)v \quad \Rightarrow$$

$$\dot{y} = -\left(\frac{m_3}{m_2+m_3}\right)v + \frac{\mu m_1 g}{m_2}t \quad \Rightarrow \quad \boldsymbol{v}'_D = \left[-\left(\frac{m_3}{m_2+m_3}\right)v + \frac{\mu m_1 g}{m_2}t\right]\boldsymbol{e}_1 \quad .$$

Hence, the time it takes for the cylinder to begin rolling without slipping is given by

$$-\left(\frac{m_3}{m_2+m_3}\right)v+\frac{\mu m_1 g}{m_2}t=-3\mu g t \quad\Rightarrow\quad t=\left[\frac{m_2 m_3}{(m_2+m_3)(3m_2+m_1)}\right]\frac{v}{\mu g} \quad .$$

On the other hand, if $\mu \to \infty$, then the cylinder rolls without slipping along the box and the frictional force satisfies $|f| \le \mu|N|$. Therefore, the balance equations of linear impulse-momentum and angular-impulse momentum about the center of mass C of the cylinder give

$$\widehat{\boldsymbol{F}}=\hat{f}\boldsymbol{e}_1+\widehat{N}\boldsymbol{e}_2=m_1(v_{C2}-v_{C1})\boldsymbol{e}_1=m_1 v_{C2}\boldsymbol{e}_1 \quad\Rightarrow\quad \widehat{N}=0 \quad , \quad \hat{f}=m_1 v_{C2} \quad ,$$

$$R\hat{f}=\frac{1}{2}m_1R^2(\omega-\omega_0)=\frac{1}{2}m_1R^2\omega \quad\Rightarrow\quad \hat{f}=\frac{1}{2}m_1R\omega \quad .$$

Moreover, the balance of linear impulse-momentum equation of the system consisting the box and the block yields

$$-\widehat{\boldsymbol{F}}=m_2(\boldsymbol{v}_{B2}-\boldsymbol{v}_{B1})+m_3(\boldsymbol{v}_{A2}-\boldsymbol{v}_{A1})=[(m_2+m_3)u+m_3v]\boldsymbol{e}_1 \quad\Rightarrow$$

$$-\hat{f}=(m_2+m_3)u+m_3v \quad .$$

Also, since the cylinder rolls without slipping along the box

$$\boldsymbol{v}_{D2}=\boldsymbol{v}_{D'2} \quad\Rightarrow\quad \boldsymbol{v}_{C2}+\boldsymbol{\omega}\times\boldsymbol{x}_{D/C}=(v_{C2}+\omega R)\boldsymbol{e}_1=u\boldsymbol{e}_1 \quad\Rightarrow\quad u=v_{C2}+\omega R \quad .$$

Next, solving the system of equations

$$\hat{f}=m_1v_{C2} \quad , \quad \hat{f}=\frac{1}{2}m_1R\omega \quad , \quad -\hat{f}=(m_2+m_3)u+m_3v \quad , \quad u=v_{C2}+\omega R \quad ,$$

for $\{v_{C2},u,\omega,\hat{f}\}$ it follows that

$$v_{C2}=-\left[\frac{m_3}{m_1+3(m_2+m_3)}\right]v \quad , \quad u=-\left[\frac{3m_3}{m_1+3(m_2+m_3)}\right]v \quad ,$$

$$\omega=-\left[\frac{2m_3}{m_1+3(m_2+m_3)}\right]\frac{v}{R} \quad , \quad \hat{f}=-\left[\frac{m_3}{m_1+3(m_2+m_3)}\right]m_1v \quad .$$

Problem 3

Figure 10.5 shows a disk of mass m and radius b, which is attached to a frame by an inextensible cord of length $3b$ passing through its center C. The frame rotates with a constant angular acceleration $\ddot{\theta} = p$. The coefficient of friction between the disk and the frame at the point of contact B is μ. Initially, at the time $t = 0$, $\theta(0) = 0$ and both the disk and the frame are at rest. The maximum tension in the cord is given by T_{cr}. Also, gravity is neglected.

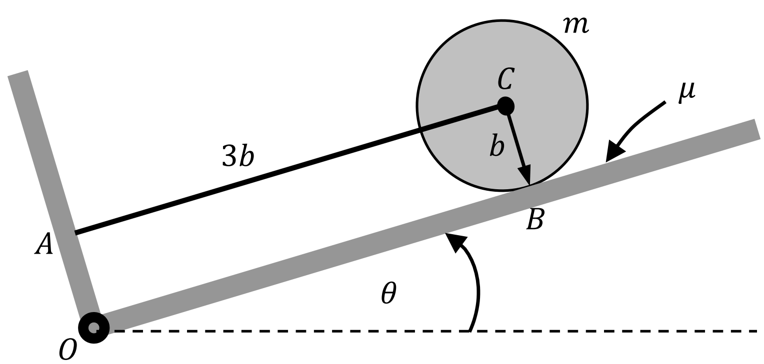

Figure 10.5

1. Assuming that $\mu = 0$, determine:

1.1. the angular velocity of the disk.

1.2. the tension in the cord.

1.3. the angular velocity of the frame when the disk is on the verge of bouncing off.

2. Determine the critical value of μ, denoted by μ_{cr}, for which the disk slips on the frame at the onset of motion.

3. If $\mu > \mu_{cr}$, determine the angular velocity of the frame when the disk is on the verge of slipping.

4. Assuming that the disk does not slip on the frame, determine:

4.1. the kinetic energy of the disk.

4.2. the angular momentum of the disk about the fixed point O.

4.3. the minimum value of the angular acceleration of the disk for which the cord tears at the onset of motion.

Solution:

The free body diagram of the disk is shown in Fig. 10.6.

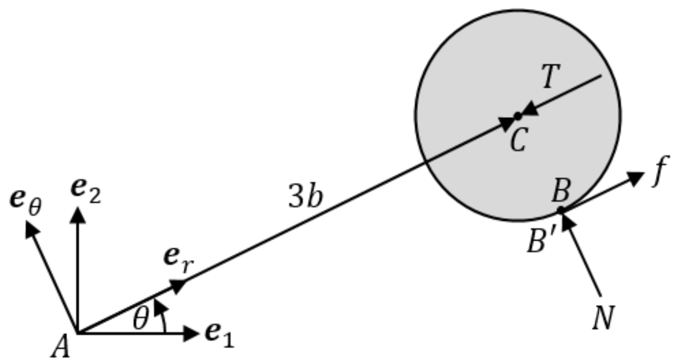

Figure 10.6

In the absence of friction ($f = 0$), the balance of linear momentum and angular momentum about the center of mass of the disk yield

$$\boldsymbol{M}_C = \boldsymbol{0} = I_C \dot{\omega} \boldsymbol{e}_3 \;\;\Rightarrow\;\; \omega = \omega(0) = 0 \;\;,\;\; -T\boldsymbol{e}_r + N\boldsymbol{e}_\theta = m\bar{\boldsymbol{a}} \;\;,$$

$$\bar{\boldsymbol{v}} = \boldsymbol{v}_{C/A} = \frac{d(3b\boldsymbol{e}_r + b\boldsymbol{e}_\theta)}{dt} = b\big(-\dot{\theta}\boldsymbol{e}_r + 3\dot{\theta}\boldsymbol{e}_\theta\big) \;\;\Rightarrow$$

$$\bar{\boldsymbol{a}} = \dot{\bar{\boldsymbol{v}}} = b\big[-\big(3\dot{\theta}^2 - \ddot{\theta}\big)\boldsymbol{e}_r + \big(3\ddot{\theta} - \dot{\theta}^2\big)\boldsymbol{e}_\theta\big] \;\;,$$

Thus,

$$T = mb\big(3\dot{\theta}^2 - \ddot{\theta}\big) \;\;,\;\; N = mb\big(3\ddot{\theta} - \dot{\theta}^2\big) \;\;.$$

Now, using

$$\ddot{\theta} = p \;\;,\;\; \dot{\theta}(0) = 0 \;\;\Rightarrow\;\; \dot{\theta} = pt \;\;,$$

it follows that

$$T = mb(3p^2t^2 - p) \quad , \quad N = mb(3p - p^2t^2).$$

Also, the angular velocity of the frame when the disk is on the verge of bouncing off is given by

$$N(\dot{\theta}_c) = 0 \quad \Rightarrow \quad mb(3p - \dot{\theta}_c^2) = 0 \quad \Rightarrow \quad \dot{\theta}_c = \sqrt{3p} \quad .$$

Next, assuming that the disk rolls without slipping along the frame at the onset of motion, then the equations of motion become

$$f - T = mb(3\dot{\theta}^2 - \ddot{\theta}) \quad , \quad N = mb(3\ddot{\theta} - \dot{\theta}^2) \quad , \quad bf = \frac{1}{2}mb^2\dot{\omega} \quad .$$

Moreover, this no-slip condition yields

$$\boldsymbol{v}_B = \boldsymbol{v}_{B'} = 3b\dot{\theta}\boldsymbol{e}_\theta \quad ,$$

such that

$$\begin{aligned}\bar{\boldsymbol{v}} &= b(-\dot{\theta}\boldsymbol{e}_r + 3\dot{\theta}\boldsymbol{e}_\theta) \\ &= \boldsymbol{v}_B + \boldsymbol{\omega} \times \boldsymbol{x}_{C/B} = 3b\dot{\theta}\boldsymbol{e}_\theta + \omega\boldsymbol{e}_3 \times b\boldsymbol{e}_\theta = b(-\omega\boldsymbol{e}_r + 3\dot{\theta}\boldsymbol{e}_\theta)\end{aligned}$$

which gives

$$\omega = \dot{\theta} \quad \Rightarrow \quad \dot{\omega} = \ddot{\theta} = p \quad .$$

Therefore, the static frictional force takes the form

$$f = \frac{1}{2}mb\dot{\omega} = \frac{1}{2}mbp \quad ,$$

and it must satisfy

$$|f| \le \mu|N| \quad \Rightarrow \quad \mu \ge \mu_{cr} \quad , \quad \mu_{cr} = \frac{|f|}{|N|} \quad .$$

Now, at the onset of motion

$$N(0) = 3mbp \quad .$$

Thus,

$$\mu_{cr} = \frac{\frac{1}{2}mbp}{3mbp} = \frac{1}{6} \ .$$

Also, when the disk is on the verge of slipping, it follows that

$$|f| = \mu|N| \ \Rightarrow \ \dot{\theta} = \sqrt{\left(3 - \frac{1}{2\mu}\right)p} \ .$$

Next, if the disk does not slip along the frame for all times, then its kinetic becomes

$$T = \frac{1}{2}m(\bar{\boldsymbol{v}} \cdot \bar{\boldsymbol{v}}) + \frac{1}{2}\left(\frac{1}{2}mb^2\right)\omega^2 \ ,$$

where (see the previous results),

$$\bar{\boldsymbol{v}} = b\big(-\omega\boldsymbol{e}_r + 3\dot{\theta}\boldsymbol{e}_\theta\big) = b\dot{\theta}(-\boldsymbol{e}_r + 3\boldsymbol{e}_\theta) = bpt(-\boldsymbol{e}_r + 3\boldsymbol{e}_\theta) \ ,$$

such that

$$T = \frac{1}{2}m(10b^2p^2t^2) + \frac{1}{4}mb^2p^2t^2 = \frac{21}{4}mb^2p^2t^2 \ .$$

Moreover, the angular momentum of the disk about the fixed point O is given by

$$\boldsymbol{H}_O = \bar{\boldsymbol{H}} + \boldsymbol{x}_{C/O} \times m\bar{\boldsymbol{v}} = \frac{1}{2}mb^2pt\boldsymbol{e}_3 + \begin{vmatrix} \boldsymbol{e}_r & \boldsymbol{e}_\theta & \boldsymbol{e}_3 \\ 3b & b & 0 \\ -mbpt & 3mbpt & 0 \end{vmatrix} = \frac{21}{4}mb^2pt\boldsymbol{e}_3 \ .$$

Now, recall that the equations of motion in this case (no-slip) take the forms

$$f - T = mb\big(3\dot{\theta}^2 - p\big) \ , \ N = mb\big(3p - \dot{\theta}^2\big) \ , \ bf = \frac{1}{2}mb^2\dot{\omega} = \frac{1}{2}mb^2p \ .$$

Hence, if the cord tears at the onset of motion, then $T = T_{cr}$ and $\dot{\theta} = 0$ so that

$$f - T_{cr} = -mbp_{cr} \ , \ bf = \frac{1}{2}mb^2p_{cr} \ .$$

Solving these two equations for $\{p_{cr}, f\}$ yields

$$p_{cr} = \frac{2}{3}\frac{T_{cr}}{mb} \ , \ f = \frac{1}{3}T_{cr} \ .$$

Problem 4

Consider the assembly shown in Fig. 10.7. The hanging block of mass m_2 is attached to the cylinder of center C, mass m_1 and radius R by an inextensible cord, wrapped around the cylinder and passes over a massless pulley. The coefficient of friction between the cylinder and the ground is μ. Moreover, the system is released from rest with the cylinder being at a distance $4R$ relative to the fixed vertical wall. The coefficient of restitution between the vertical wall and the cylinder is given by $e = 1/2$.

1. Assuming that the cylinder rolls without slipping along the ground, determine:

1.1. the acceleration of its center C at the onset of motion.

1.2. the minimum value of μ for this to happen.

1.3. the velocity of its center C just before impact with the wall.

2. Determine the velocity of the center C of the cylinder just after impact with the wall.
3. Determine the angular velocity of the cylinder just after impact with the wall.
4. Does the cylinder slip along the ground just after impact with the wall? Explain your answer.

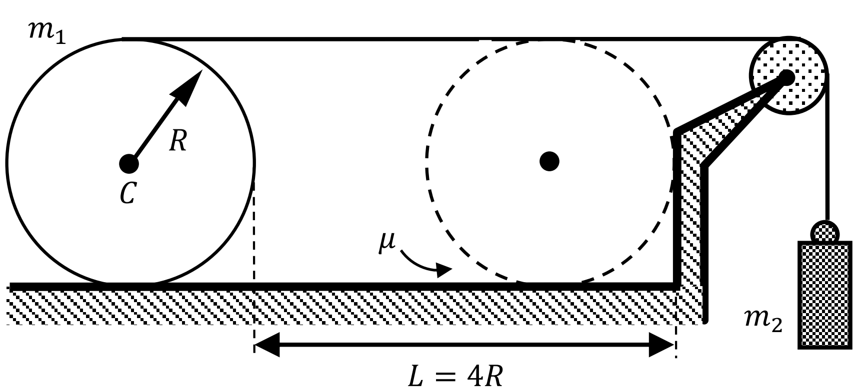

Figure 10.7

Solution:

The free body diagrams of the cylinder and the block before impact are shown in Fig. 10.8.

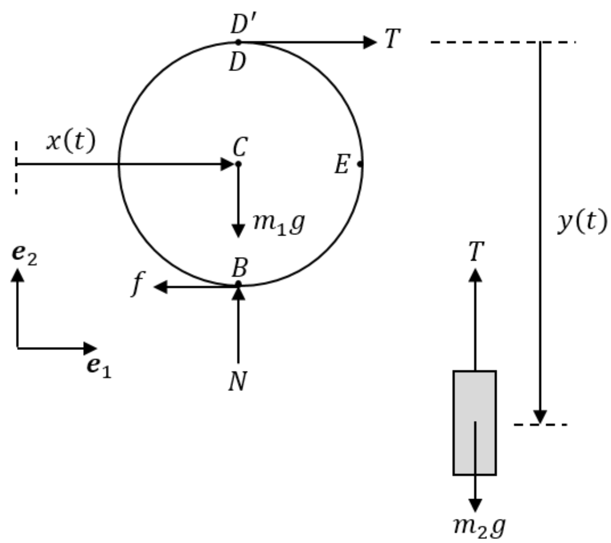

Figure 10.8

If the cylinder rolls without slipping on the ground, then

$$\boldsymbol{v}_B = \mathbf{0} \;\; \Rightarrow \;\; \boldsymbol{v}_C = \dot{x}\boldsymbol{e}_1 = \boldsymbol{v}_B + \boldsymbol{\omega} \times \boldsymbol{x}_{C/B} = \omega \boldsymbol{e}_3 \times R\boldsymbol{e}_2 = -\omega R \boldsymbol{e}_1 \;\; \Rightarrow \;\; \ddot{x} = -\dot{\omega} R \quad .$$

Furthermore, assuming that the cord does not slip along the cylinder, it follows that

$$\boldsymbol{v}_{D/D'} = \mathbf{0} \quad ,$$

where,

$$\boldsymbol{v}_D = \boldsymbol{v}_C + \boldsymbol{\omega} \times \boldsymbol{x}_{C/B} = \ddot{x}\boldsymbol{e}_1 + \omega \boldsymbol{e}_3 \times R\boldsymbol{e}_2 = (\ddot{x} - \omega R)\boldsymbol{e}_1 \quad , \quad \boldsymbol{v}_{D'} = \dot{y}\boldsymbol{e}_1 \quad ,$$

such that

$$\ddot{x} - \dot{\omega} R = \ddot{y} \quad .$$

Next, using that balance equation of linear momentum of the block and the cylinder together with the balance equation of angular momentum about the center of mass C of the cylinder, it follows that

$$T - m_2 g = -m_2 \ddot{y} \quad , \quad T - f = m_1 \ddot{x} \quad , \quad N = m_1 g \quad , \quad -Rf = \frac{1}{2} m R^2 \dot{\omega} \quad .$$

Solving the system of equations

$$\ddot{x} = -\dot{\omega}R \quad , \quad \ddot{x} - \dot{\omega}R = \ddot{y} \quad ,$$

$$T - m_2 g = -m_2\ddot{y} \quad , \quad T - f = m_1\ddot{x} \quad , \quad N = m_1 g \quad , \quad -Rf = \frac{1}{2}mR^2\dot{\omega} \quad ,$$

for $\{\ddot{x}, \ddot{y}, \dot{\omega}, T, f\}$ yield

$$\ddot{x} = \left(\frac{2m_2}{3m_1 + 4m_2}\right)g \quad , \quad \ddot{y} = \left(\frac{4m_2}{3m_1 + 4m_2}\right)g \quad , \quad \dot{\omega} = -\left(\frac{2m_2}{3m_1 + 4m_2}\right)\frac{g}{R} \quad ,$$

$$T = \frac{3m_1 m_2 g}{3m_1 + 4m_2} \quad , \quad f = \frac{m_1 m_2 g}{3m_1 + 4m_2} \quad .$$

Furthermore, since the frictional force is static in this case

$$f \le \mu N \quad \Rightarrow \quad \mu \ge \frac{m_2}{3m_1 + 4m_2} \quad .$$

Now, the velocity $\dot{x}$ of the center of mas C of the cylinder is given by

$$\dot{x} = \dot{x}(0) + \left(\frac{2m_2}{3m_1 + 4m_2}\right)gt \quad , \quad \dot{x}(0) = 0 \quad \Rightarrow \quad \dot{x} = \left(\frac{2m_2}{3m_1 + 4m_2}\right)gt \quad .$$

However,

$$\Delta x = 4R = \dot{x}(0)t + \frac{1}{2}\left(\frac{2m_2}{3m_1 + 4m_2}\right)gt^2 \quad \Rightarrow \quad t = 2\sqrt{\frac{(3m_1 + 4m_2)R}{m_2 g}} \quad .$$

Hence,

$$\dot{x} = 4\sqrt{\frac{m_2 g R}{3m_1 + 4m_2}} \quad .$$

Also, the angular velocity ω of the cylinder becomes

$$\dot{\omega} = -\frac{\dot{x}}{R} \quad \Rightarrow \quad \omega = -4\sqrt{\left(\frac{m_2}{3m_1 + 4m_2}\right)\frac{g}{R}} \quad .$$

Next, the free body diagram of the cylinder at impact is shown in Fig. 10.9.

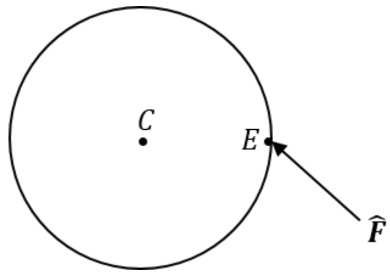

Figure 10.9

Using the definition of the coefficient of restitution, it follows that

$$e = -\frac{\boldsymbol{v}_{E2}\cdot\boldsymbol{e}_1}{\boldsymbol{v}_{E1}\cdot\boldsymbol{e}_1} \quad , \quad \boldsymbol{v}_E(t) = \boldsymbol{v}_C + \boldsymbol{\omega}\times\boldsymbol{x}_{E/C} = \dot{x}(t)\boldsymbol{e}_1 \quad \Rightarrow \quad \dot{x}(t_2) = -e\dot{x}(t_1) \quad \Rightarrow$$

$$\dot{x}(t_2) = -4e\sqrt{\frac{m_2 g R}{3m_1 + 4m_2}} \quad .$$

Moreover, using the balance equations of impulse-linear momentum and impulse-angular momentum about the center of mass of the cylinder at impact yield

$$\hat{\boldsymbol{F}} = \hat{F}_1\boldsymbol{e}_1 + \hat{F}_2\boldsymbol{e}_2 = m_1[\dot{x}(t_2) - \dot{x}(t_1)]\boldsymbol{e}_1 \quad \Rightarrow \quad \hat{F}_2 = 0 \quad ,$$

$$\hat{F}_1 = -4(e+1)m_1\sqrt{\frac{m_2 g R}{3m_1 + 4m_2}} \quad ,$$

$$-R\hat{F}_2 = 0 = \frac{1}{2}m_1 R^2[\omega(t_2) - \omega(t_1)] \quad \Rightarrow \quad \omega(t_2) = \omega(t_1) = -4\sqrt{\left(\frac{m_2}{3m_1 + 4m_2}\right)\frac{g}{R}} \quad .$$

This shows that the cylinder slips along the ground just after impact since

$$|\dot{x}(t_2)| \neq \omega(t_2)R \quad .$$

Problem Set 11 Solutions

Problem 1

A uniform circular disk of mass $m = 23$ [kg] and radius $R = 0.4$ [m] rolls without slipping along a horizontal surface in such a manner that its plane is inclined with the vertical at a constant angle α and its center C moves along a circular path of radius $b = 0.6$ [m] with the speed $v = 2.54$ [m/s], as shown in Fig. 11.1.

1. Determine the value of α.
2. Determine the forces exerted on the disk by the horizontal surface.

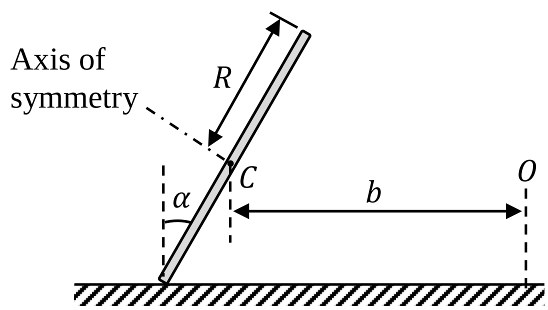

Figure 11.1

Solution:

The free body of the disk is shown in Fig. 11.2.

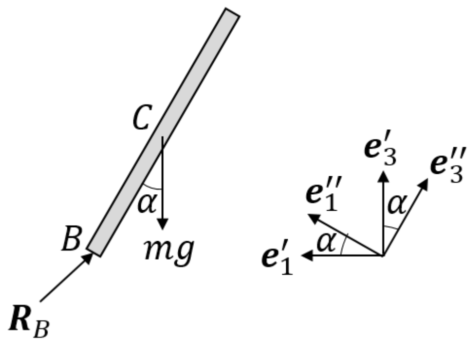

Figure 11.2

The systems $\{\boldsymbol{e}'_i, \boldsymbol{e}''_i\}$ rotate with an angular velocity $\boldsymbol{\Omega}$ and the disk rotates with an angular velocity $\boldsymbol{\omega}$, such that

$$\dot{\boldsymbol{e}}'_i = \boldsymbol{\Omega} \times \boldsymbol{e}'_i \ \ , \ \ \dot{\boldsymbol{e}}'_i = \boldsymbol{\Omega} \times \boldsymbol{e}'_i \ \ , \ \ \boldsymbol{\Omega} = \Omega \boldsymbol{e}'_3 \ \ , \ \ \boldsymbol{\omega} = \boldsymbol{\Omega} + \dot{\varphi} \boldsymbol{e}''_1 \ \ ,$$

$$\boldsymbol{e}'_1 = \cos(\alpha)\, \boldsymbol{e}''_1 - \sin(\alpha)\, \boldsymbol{e}''_3 \ \ , \ \ \boldsymbol{e}'_2 = \boldsymbol{e}''_2 \ \ , \ \ \boldsymbol{e}'_3 = \sin(\alpha)\, \boldsymbol{e}''_1 + \cos(\alpha)\, \boldsymbol{e}''_3 \ \ .$$

Now, the velocity $\boldsymbol{v}_C = v\boldsymbol{e}'_2$ of the disk's center of mass C takes the form

$$\boldsymbol{v}_C = v\boldsymbol{e}'_2 = \boldsymbol{v}_{C/O} = \boldsymbol{\Omega} \times \boldsymbol{x}_{C/O} = \Omega \boldsymbol{e}'_3 \times b\boldsymbol{e}'_1 = \Omega b \boldsymbol{e}'_2 \ \ .$$

Therefore,

$$\Omega = \frac{v}{b} \ \Rightarrow \ \ \boldsymbol{\omega} = \left[\frac{v}{b}\sin(\alpha) + \dot{\varphi}\right] \boldsymbol{e}''_1 + \frac{v}{b}\cos(\alpha)\, \boldsymbol{e}''_3 \ \ .$$

Using the no-slip condition, then $\boldsymbol{v}_C$ can be expressed as

$$\boldsymbol{v}_C = v\boldsymbol{e}'_2 = \boldsymbol{v}_{C/B} = \boldsymbol{\omega} \times \boldsymbol{x}_{C/B} = \boldsymbol{\omega} \times b\boldsymbol{e}''_3 = -[v\sin(\alpha) + b\dot{\varphi}]\boldsymbol{e}''_2 \ \ \Rightarrow$$

$$\dot{\varphi} = -\frac{v}{b}[1 + \sin(\alpha)] \ \ .$$

Hence,

$$\boldsymbol{\omega} = -\frac{v}{b}\boldsymbol{e}''_1 + \frac{v}{b}\cos(\alpha)\, \boldsymbol{e}''_3 \ \ .$$

Next, the balance of linear momentum of the disk yields

$$\boldsymbol{R}_B - mg\boldsymbol{e}'_3 = m\boldsymbol{a}_C \quad , \quad \boldsymbol{R}_B = R''_{Bi}\boldsymbol{e}''_i \quad ,$$

$$\boldsymbol{a}_C = \dot{\boldsymbol{v}}_C = v\dot{\boldsymbol{e}}'_2 = v(\Omega\boldsymbol{e}'_3 \times \boldsymbol{e}'_2) = -\Omega v\boldsymbol{e}'_1 = -\frac{v^2}{b}\boldsymbol{e}'_1 \quad .$$

Thus,

$$R''_{B1} = m\left[g\sin(\alpha) - \frac{v^2\cos(\alpha)}{b}\right] \quad , \quad R''_{B2} = 0 \quad , \quad R''_{B3} = m\left[g\cos(\alpha) + \frac{v^2\sin(\alpha)}{b}\right] \quad .$$

Furthermore, the balance of angular momentum about the center of mass C of the disk gives

$$\boldsymbol{M}_C = \dot{\boldsymbol{H}}_C \quad , \quad \boldsymbol{M}_C = \boldsymbol{x}_{B/C} \times \boldsymbol{R}_B = \begin{vmatrix} \boldsymbol{e}''_1 & \boldsymbol{e}''_2 & \boldsymbol{e}''_3 \\ 0 & 0 & -b \\ R''_{B1} & 0 & R''_{B3} \end{vmatrix} = -bR''_{B1}\boldsymbol{e}''_2 \quad ,$$

$$H''_{Ci} = I''_{Cij}\omega''_j = \frac{mR^2v}{4b}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} -1 \\ 0 \\ \cos(\alpha) \end{Bmatrix} = \frac{mR^2v}{4b}\begin{Bmatrix} -2 \\ 0 \\ \cos(\alpha) \end{Bmatrix} \Rightarrow$$

$$\boldsymbol{H}_C = \frac{mR^2v}{4b}[-2\boldsymbol{e}''_1 + \cos(\alpha)\,\boldsymbol{e}''_3] \ ,$$

$$\dot{\boldsymbol{H}}_C = \frac{\delta\boldsymbol{H}_C}{\delta t} + \boldsymbol{\Omega} \times \boldsymbol{H}_C = \boldsymbol{\Omega} \times \boldsymbol{H}_C \quad \Rightarrow$$

$$\dot{\boldsymbol{H}}_C = \frac{mR^2v^2}{4b^2}\begin{vmatrix} \boldsymbol{e}''_1 & \boldsymbol{e}''_2 & \boldsymbol{e}''_3 \\ \sin(\alpha) & 0 & \cos(\alpha) \\ -2 & 0 & \cos(\alpha) \end{vmatrix} = -\frac{mR^2v^2\cos(\alpha)}{4b^2}[2 + \sin(\alpha)]\boldsymbol{e}''_2 \quad .$$

Therefore,

$$R''_{B1} = \frac{mR^2v^2\cos(\alpha)}{4b^3}[2 + \sin(\alpha)] \quad .$$

Now, equating the expressions of R''_{B1}, it follows that

$$4b^3g\sin(\alpha) = 2v^2(2b^2 + R^2)\cos(\alpha) + R^2v^2\cos(\alpha)\sin(\alpha)$$

Substituting the values of $\{m, b, R, v\}$ together with $g = 9.81$ $[\mathrm{m/s^2}]$ into this equation and solving for α, it follows that the only possible solution is given by

$$\alpha \approx 55.22^{o} \quad .$$

Moreover, the reaction force $\boldsymbol{R}_B$ exerted on the disk by the horizontal surface becomes

$$\boldsymbol{R}_B = m\left[g\sin(\alpha) - \frac{v^2\cos(\alpha)}{b}\right]\boldsymbol{e}_1'' + m\left[g\cos(\alpha) + \frac{v^2\sin(\alpha)}{b}\right]\boldsymbol{e}_3''$$

$$\approx 44.23\boldsymbol{e}_1'' + 331.8\boldsymbol{e}_3'' \ \ [\mathrm{N}] \quad .$$

Problem 2

Consider the assembly shown in Fig. 11.3. The two disks A and B, each having a mass m and radius R, are welded at the two ends of the shaft AB of length $2L$, which coincides with the axis of symmetry of each disk. A third disk, C, of mass m and radius R is welded at the midpoint of the shaft in such a manner that its plane is inclined with the horizontal at a constant angle β. Moreover, the system $\mathbf{e}_i''$ is attached to the shaft and it is assumed that the torques at the bearing A and B are negligible.

1. Determine the angular momentum of the system about C.
2. Determine the normal bearing reactions acting on the shaft AB at A and B.
3. Now, the point masses m_A and m_B are attached at the rim of the disks A and B, respectively (see Fig. 2.1). Determine the values of $\{m_A, m_B\}$ and $\{\varphi_A, \varphi_B\}$ that would eliminate the bearing reactions at A and B.

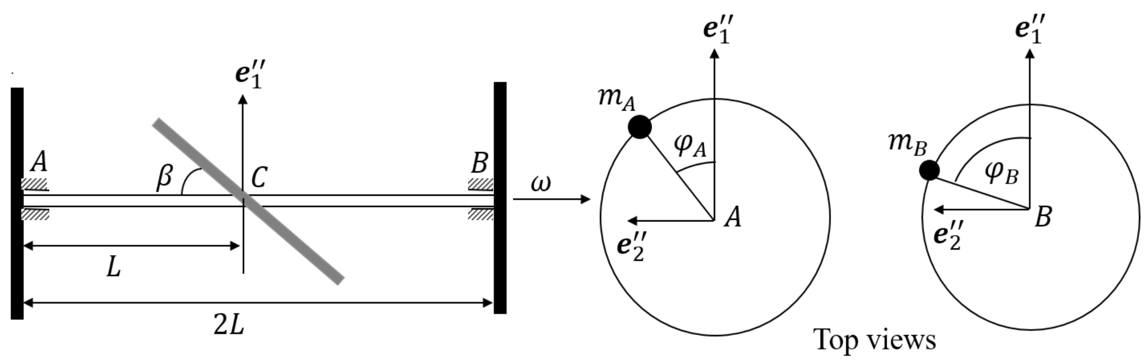

Figure 11.3

Solution:

The free body of the system is shown in Fig. 11.4.

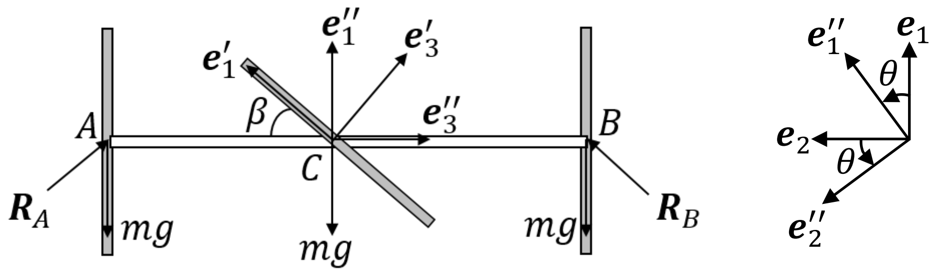

Figure 11.4

The systems $\{\boldsymbol{e}_i'', \boldsymbol{e}_i'\}$ rotate with the same angular velocity $\boldsymbol{\omega}$ such that

$$\dot{\boldsymbol{e}}_i'' = \boldsymbol{\omega} \times \boldsymbol{e}_i'' \quad , \quad \dot{\boldsymbol{e}}_i' = \boldsymbol{\omega} \times \boldsymbol{e}_i' \quad , \quad \boldsymbol{\omega} = \omega \boldsymbol{e}_3'' \quad ;$$

$$\boldsymbol{e}_1' = \sin(\beta)\, \boldsymbol{e}_1'' - \cos(\beta)\, \boldsymbol{e}_3'' \quad , \quad \boldsymbol{e}_2' = \boldsymbol{e}_2'' \quad , \quad \boldsymbol{e}_3' = \cos(\beta)\, \boldsymbol{e}_1'' + \sin(\beta)\, \boldsymbol{e}_3'' \quad .$$

Also, the system $\boldsymbol{e}_i''$ is related to the fixed Cartesian system $\boldsymbol{e}_i$ by

$$\boldsymbol{e}_1 = \cos(\theta)\, \boldsymbol{e}_1'' - \sin(\theta)\, \boldsymbol{e}_2'' \quad , \quad \boldsymbol{e}_2 = \sin(\theta)\, \boldsymbol{e}_1'' + \cos(\theta)\, \boldsymbol{e}_2'' \quad , \quad \boldsymbol{e}_3 = \boldsymbol{e}_3'' \quad .$$

Next, the angular momentum $\boldsymbol{H}_C$ of the system about C can be expressed as

$$\boldsymbol{H}_C = \boldsymbol{H}_C^{(A)} + \boldsymbol{H}_C^{(B)} + \boldsymbol{H}_C^{(C)} \quad ,$$

where the angular momentums $\left\{\boldsymbol{H}_C^{(A)}, \boldsymbol{H}_C^{(B)}, \boldsymbol{H}_C^{(C)}\right\}$ of the disks $\{A, B, C\}$, respectively, are given by

$$\boldsymbol{H}_C^{(A)} = \overline{\boldsymbol{H}}^{(A)} + \boldsymbol{x}_{A/C} \times \boldsymbol{v}_A = \overline{\boldsymbol{H}}^{(A)} = \overline{\boldsymbol{I}}^{(A)} \boldsymbol{\omega} \quad , \quad \boldsymbol{H}_C^{(B)} = \overline{\boldsymbol{H}}^{(B)} = \overline{\boldsymbol{I}}^{(B)} \boldsymbol{\omega} \quad ,$$

$$\overline{\boldsymbol{I}}^{(A)} = \overline{\boldsymbol{I}}^{(B)} = \frac{mR^2}{4} (\boldsymbol{e}_1'' \otimes \boldsymbol{e}_1'' + \boldsymbol{e}_2'' \otimes \boldsymbol{e}_2'') + \frac{mR^2}{2} \boldsymbol{e}_3'' \otimes \boldsymbol{e}_3'' \quad \Rightarrow$$

$$\boldsymbol{H}_C^{(A)} = \boldsymbol{H}_C^{(B)} = \frac{mR^2 \omega}{2} \boldsymbol{e}_3'' \quad ,$$

$$\boldsymbol{H}_C^{(C)} = \overline{\boldsymbol{H}}^{(C)} = \overline{\boldsymbol{I}}^{(C)} \boldsymbol{\omega} \quad , \quad \overline{\boldsymbol{I}}^{(C)} = \frac{mR^2}{4} (\boldsymbol{e}_1' \otimes \boldsymbol{e}_1' + \boldsymbol{e}_2' \otimes \boldsymbol{e}_2') + \frac{mR^2}{2} \boldsymbol{e}_3' \otimes \boldsymbol{e}_3' \quad \Rightarrow$$

$$\boldsymbol{H}_C^{(C)} \cdot \boldsymbol{e}_i' = \begin{bmatrix} \frac{mR^2}{4} & 0 & 0 \\ 0 & \frac{mR^2}{4} & 0 \\ 0 & 0 & \frac{mR^2}{2} \end{bmatrix} \begin{Bmatrix} -\omega\cos(\beta) \\ 0 \\ \omega\sin(\beta) \end{Bmatrix} = \begin{Bmatrix} -\frac{mR^2\omega\cos(\beta)}{4} \\ 0 \\ \frac{mR^2\omega\sin(\beta)}{2} \end{Bmatrix} \Rightarrow$$

$$\boldsymbol{H}_C^{(C)} = \frac{mR^2\omega}{4}[-\cos(\beta)\,\boldsymbol{e}_1' + 2\sin(\beta)\,\boldsymbol{e}_3'] \ .$$

Moreover, using the transformation relations it follows that

$$\boldsymbol{H}_C^{(C)} = \frac{mR^2\omega}{4}[\sin(\beta)\cos(\beta)\,\boldsymbol{e}_1'' + \{\cos^2(\beta) + 2\sin^2(\beta)\}\boldsymbol{e}_3'']$$

$$= \frac{mR^2\omega}{4}\left[\frac{1}{2}\sin(2\beta)\,\boldsymbol{e}_1'' + \{1 + \sin^2(\beta)\}\boldsymbol{e}_3''\right] \ .$$

Next, the balance of linear momentum of the system gives

$$\boldsymbol{R}_A + \boldsymbol{R}_B - 3mg\boldsymbol{e}_1 = m\overline{\boldsymbol{a}} \ , \ \overline{\boldsymbol{a}} = \boldsymbol{0} \ .$$

Therefore,

$$R_{A1}'' + R_{B1}'' = 3mg\cos(\theta) \ , \ R_{A2}'' + R_{B2}'' = -3mg\sin(\theta) \ , \ R_{A3}'' + R_{B3}'' = 0 \ .$$

Furthermore, the balance of angular momentum of the system about C yields

$$\boldsymbol{M}_C = \dot{\boldsymbol{H}}_C \ ,$$

$$\dot{\boldsymbol{H}}_C = \frac{\delta \boldsymbol{H}_C}{\delta t} + \boldsymbol{\omega} \times \boldsymbol{H}_C = \boldsymbol{\omega} \times \boldsymbol{H}_C = \frac{mR^2\omega^2}{8}\sin(2\beta)\,\boldsymbol{e}_2'' \ ,$$

$$\boldsymbol{M}_C = \boldsymbol{x}_{A/C} \times (\boldsymbol{R}_A - mg\boldsymbol{e}_1) + \boldsymbol{x}_{B/C} \times (\boldsymbol{R}_B - mg\boldsymbol{e}_1)$$

$$= \begin{vmatrix} \boldsymbol{e}_1'' & \boldsymbol{e}_2'' & \boldsymbol{e}_3'' \\ 0 & 0 & -L \\ R_{A1}'' - mg\cos(\theta) & R_{A2}'' + mg\sin(\theta) & R_{A3}'' \end{vmatrix}$$

$$+ \begin{vmatrix} \boldsymbol{e}_1'' & \boldsymbol{e}_2'' & \boldsymbol{e}_3'' \\ 0 & 0 & L \\ R_{B1}'' - mg\cos(\theta) & R_{B2}'' + mg\sin(\theta) & R_{B3}'' \end{vmatrix} \Rightarrow$$

$$\boldsymbol{M}_C = L[R''_{A2} + mg\sin(\theta)]\boldsymbol{e}''_1 - L[R''_{A1} - mg\cos(\theta)]\boldsymbol{e}''_2 - L[R''_{B2} + mg\sin(\theta)]\boldsymbol{e}''_1 + L[R''_{B1} - mg\cos(\theta)]\boldsymbol{e}''_2 = L(R''_{A2} - R''_{B2})\boldsymbol{e}''_1 + L(R''_{B1} - R''_{A1})\boldsymbol{e}''_2 \quad .$$

Hence,

$$R''_{B1} - R''_{A1} = \frac{mR^2\omega^2}{8L}\sin(2\beta) \quad , \quad R''_{A2} - R''_{B2} = 0 \quad .$$

Solving the system of equations

$$R''_{A1} + R''_{B1} = 3mg\cos(\theta) \quad , \quad R''_{A2} + R''_{B2} = -3mg\sin(\theta) \quad , \quad R''_{A3} + R''_{B3} = 0 \quad ,$$

$$R''_{B1} - R''_{A1} = \frac{mR^2\omega^2}{8L}\sin(2\beta) \quad , \quad R''_{A2} - R''_{B2} = 0 \quad ,$$

for the reactions forces under the assumption that $R''_{A3} = 0$ it follows that

$$R''_{A1} = \frac{m}{2}\left[g\cos(\theta) - \frac{R^2\omega^2}{8L}\sin(2\beta)\right] \quad , \quad R''_{B1} = \frac{m}{2}\left[3g\cos(\theta) + \frac{R^2\omega^2}{8L}\sin(2\beta)\right] \quad ,$$

$$R''_{A2} = R''_{B2} = -\frac{3mg}{2}\sin(\theta) \quad , \quad R''_{A3} = R''_{B3} = 0 \quad .$$

This shows that the system is dynamically unbalanced.

Next, by attaching the point masses $\{m_A, m_B\}$ to the disks $\{A, B\}$, respectively, then the center of mass of the system moves to the position

$$\overline{\boldsymbol{x}} = \frac{m_A \boldsymbol{x}_{m_A/C} + m_B \boldsymbol{x}_{m_B/C}}{3m + m_A + m_B} \quad ,$$

where,

$$\boldsymbol{x}_{m_A/C} = R\cos(\varphi_A)\,\boldsymbol{e}''_1 + R\sin(\varphi_A)\,\boldsymbol{e}''_2 - L\boldsymbol{e}''_3 \quad ,$$

$$\boldsymbol{x}_{m_B/C} = R\cos(\varphi_B)\,\boldsymbol{e}''_1 + R\sin(\varphi_B)\,\boldsymbol{e}''_2 + L\boldsymbol{e}''_3 \quad ,$$

such that

$$\overline{\boldsymbol{x}} = \frac{[m_A \cos(\varphi_A) + m_B \cos(\varphi_B)]R}{3m + m_A + m_B} \boldsymbol{e}_1'' + \frac{[m_A \sin(\varphi_A) + m_B \sin(\varphi_B)]R}{3m + m_A + m_B} \boldsymbol{e}_2''$$

$$+ \frac{(m_B - m_A)L}{3m + m_A + m_B} \boldsymbol{e}_3'' \quad .$$

Moreover, the contribution of these masses to $\boldsymbol{H}_C$ can be expressed as

$$\boldsymbol{H}_C = \boldsymbol{H}_C^{(A)} + \boldsymbol{H}_C^{(B)} + \boldsymbol{H}_C^{(C)} + \sum_{i=A}^{B} \boldsymbol{H}_C^{(m_i)} \quad ,$$

where,

$$\boldsymbol{H}_C^{(m_i)} = \boldsymbol{x}_{m_i/C} \times m_i \boldsymbol{v}_{m_i} \quad , \quad i = \{A, B\} \quad ,$$

such that

$$\boldsymbol{x}_{m_A/C} = R\cos(\varphi_A)\, \boldsymbol{e}_1'' + R\sin(\varphi_A)\, \boldsymbol{e}_2'' - L\boldsymbol{e}_3'' \quad ,$$

$$\boldsymbol{v}_{m_A} = \frac{\delta \boldsymbol{x}_{m_A/C}}{\delta t} + \boldsymbol{\omega} \times \boldsymbol{x}_{m_A/C} = \boldsymbol{\omega} \times \boldsymbol{x}_{m_A/C} = \omega R[-\sin(\varphi_A)\, \boldsymbol{e}_1'' + \cos(\varphi_A)\, \boldsymbol{e}_2''] \quad \Rightarrow$$

$$\boldsymbol{H}_C^{(m_A)} = m_A \omega R L \left[\cos(\varphi_A)\, \boldsymbol{e}_1'' + \sin(\varphi_A)\, \boldsymbol{e}_2'' + \frac{R}{L} \boldsymbol{e}_3''\right] \quad ;$$

$$\boldsymbol{x}_{m_B/C} = R\cos(\varphi_B)\, \boldsymbol{e}_1'' + R\sin(\varphi_B)\, \boldsymbol{e}_2'' + L\boldsymbol{e}_3'' \quad ,$$

$$\boldsymbol{v}_{m_B} = \frac{\delta \boldsymbol{x}_{m_B/C}}{\delta t} + \boldsymbol{\omega} \times \boldsymbol{x}_{m_B/C} = \boldsymbol{\omega} \times \boldsymbol{x}_{m_A/C} = \omega R[-\sin(\varphi_B)\, \boldsymbol{e}_1'' + \cos(\varphi_B)\, \boldsymbol{e}_2''] \quad \Rightarrow$$

$$\boldsymbol{H}_C^{(m_B)} = m_B \omega R L \left[-\cos(\varphi_B)\, \boldsymbol{e}_1'' - \sin(\varphi_B)\, \boldsymbol{e}_2'' + \frac{R}{L} \boldsymbol{e}_3''\right] \quad .$$

Hence,

$$\boldsymbol{H}_C = \omega R L \left[\frac{mR}{8L} \sin(2\beta) + m_A \cos(\varphi_A) - m_B \cos(\varphi_B)\right] \boldsymbol{e}_1''$$

$$+\omega RL[m_A \sin(\varphi_A) - m_B \sin(\varphi_B)]\boldsymbol{e}_2'' + \omega R^2 \left[\frac{m\{2 + \sin^2(\beta)\}}{4} + m_A + m_B\right] \boldsymbol{e}_3'' \quad .$$

Now, dynamic balancing requires that the center of mass be situated on the axis of rotation $\boldsymbol{e}_3''$ and that the axis of rotation be a principal axis of inertia. Therefore,

$$\bar{\boldsymbol{x}} \cdot \boldsymbol{e}_1'' = 0 \quad \Rightarrow \quad m_A \cos(\varphi_A) + m_B \cos(\varphi_B) = 0 \quad ,$$

$$\bar{\boldsymbol{x}} \cdot \boldsymbol{e}_2'' = 0 \quad \Rightarrow \quad m_A \sin(\varphi_A) + m_B \sin(\varphi_B) = 0 \quad ,$$

$$\boldsymbol{H}_C \cdot \boldsymbol{e}_2'' = 0 \quad \Rightarrow \quad m_A \sin(\varphi_A) - m_B \sin(\varphi_B) = 0 \quad ,$$

$$\boldsymbol{H}_C \cdot \boldsymbol{e}_1'' = 0 \quad \Rightarrow \quad \frac{mR}{8L}\sin(2\beta) + m_A \cos(\varphi_A) - m_B \cos(\varphi_B) = 0 \quad .$$

Solving this system of equations for $\{m_A, m_B, \varphi_A, \varphi_B\}$ yields

$$\sin(\varphi_A) = 0 \quad , \quad \sin(\varphi_B) = 0 \quad ,$$

$$m_A = -\frac{mR\sin(2\beta)}{16L\cos(\varphi_A)} \quad , \quad m_B \cos(\varphi_B) = \frac{mR\sin(2\beta)}{16L\cos(\varphi_B)} \quad .$$

However, since $m_A > 0$ and $m_B > 0$

$$\left\{\varphi_A = \pi \quad , \quad m_A = \frac{mR\sin(2\beta)}{16L}\right\} \quad , \quad \left\{\varphi_B = 0 \quad , \quad m_B = \frac{mR\sin(2\beta)}{16L}\right\} \quad .$$

Problem 3

Fig. 11.5 shows a bar AC of mass m and length L, which is attached at one end to the center C of a disk of mass m and radius R, and the other end is placed on a stationary, frictionless circular plate at the point A. The bar coincides with the axis of symmetry of the disk. Moreover, the disk is constrained to roll without slipping along the rim of the plate in such a manner that its center C moves along a circular path with the speed v_0.

1. Determine the angular velocity and acceleration of the disk.
2. Determine the forces exerted on the disk by the plate.

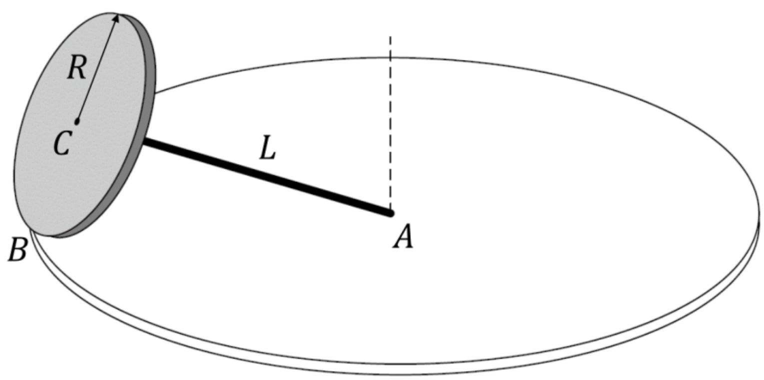

Figure 11.5

Solution:

The free body of the system is shown in Fig. 11.6.

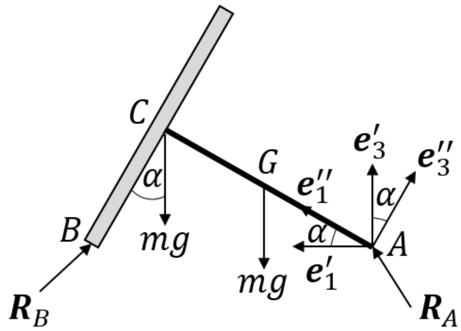

Figure 11.6

The angle α is related to the geometry by the expressions

$$\sin(\alpha) = \frac{R}{\sqrt{L^2+R^2}} \quad , \quad \cos(\alpha) = \frac{L}{\sqrt{L^2+R^2}} \quad .$$

The systems $\{\boldsymbol{e}'_i, \boldsymbol{e}''_i\}$ rotate with an angular velocity $\boldsymbol{\Omega}$ and the disk rotates with an angular velocity $\boldsymbol{\omega}$, such that

$$\dot{\boldsymbol{e}}''_i = \boldsymbol{\Omega} \times \boldsymbol{e}''_i \quad , \quad \dot{\boldsymbol{e}}'_i = \boldsymbol{\Omega} \times \boldsymbol{e}'_i \quad , \quad \boldsymbol{\Omega} = \Omega \boldsymbol{e}'_3 \quad , \quad \boldsymbol{\omega} = \boldsymbol{\Omega} + \dot{\varphi}\boldsymbol{e}''_1 \quad ,$$

$$\boldsymbol{e}'_1 = \cos(\alpha)\,\boldsymbol{e}''_1 - \sin(\alpha)\,\boldsymbol{e}''_3 \quad , \quad \boldsymbol{e}'_2 = \boldsymbol{e}''_2 \quad , \quad \boldsymbol{e}'_3 = \sin(\alpha)\,\boldsymbol{e}''_1 + \cos(\alpha)\,\boldsymbol{e}''_3 \quad .$$

Now, the velocity $\boldsymbol{v}_C = v_0 \boldsymbol{e}'_2$ of the disk's center of mass C takes the form

$$\boldsymbol{v}_C = v_0 \boldsymbol{e}''_2 = \boldsymbol{v}_{C/A} = \boldsymbol{\Omega} \times \boldsymbol{x}_{C/A} = \Omega[\sin(\alpha)\,\boldsymbol{e}''_1 + \cos(\alpha)\,\boldsymbol{e}''_3] \times L\boldsymbol{e}''_1 = \Omega L \cos(\alpha)\,\boldsymbol{e}''_2 \quad .$$

Therefore,

$$\Omega = \frac{v_0}{L\cos(\alpha)} \quad \Rightarrow \quad \boldsymbol{\omega} = \left[\frac{v_0 \sin(\alpha)}{L\cos(\alpha)} + \dot{\varphi}\right]\boldsymbol{e}''_1 + \frac{v_0}{L}\boldsymbol{e}''_3 \quad .$$

Using the no-slip condition, then $\boldsymbol{v}_C$ can be expressed as

$$\boldsymbol{v}_C = v_0 \boldsymbol{e}''_2 = \boldsymbol{v}_{C/B} = \boldsymbol{\omega} \times \boldsymbol{x}_{C/B} = [\{\Omega\sin(\alpha) + \dot{\varphi}\}\boldsymbol{e}''_1 + \Omega\cos(\alpha)\,\boldsymbol{e}''_3] \times R\boldsymbol{e}''_3$$

$$= -R[\Omega\sin(\alpha) + \dot{\varphi}]\boldsymbol{e}''_2 \quad .$$

Hence,

$$\dot{\varphi} = -v_0\left[\frac{1}{R} + \frac{\sin(\alpha)}{L\cos(\alpha)}\right] \;\Rightarrow\; \boldsymbol{\omega} = -\frac{v_0}{R}\boldsymbol{e}_1'' + \frac{v_0}{L}\boldsymbol{e}_3'' \quad .$$

Notice that $\boldsymbol{\omega}$ must lie along the line joining the points A and B since $\boldsymbol{v}_A = \boldsymbol{v}_B = \boldsymbol{0}$. In particular, its direction must satisfy

$$\boldsymbol{e}_\omega = \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|} = -\boldsymbol{e}_1' \quad .$$

Using the result obtained previously, it follows that

$$\boldsymbol{e}_\omega = -\left(\frac{L}{\sqrt{L^2+R^2}}\right)\boldsymbol{e}_1'' + \left(\frac{R}{\sqrt{L^2+R^2}}\right)\boldsymbol{e}_3'' = -\boldsymbol{e}_1' \quad ,$$

as it should be.

Furthermore, the disk's angular acceleration $\dot{\boldsymbol{\omega}}$ takes the form

$$\dot{\boldsymbol{\omega}} = \frac{\delta\boldsymbol{\omega}}{\delta t} + \boldsymbol{\Omega}\times\boldsymbol{\omega} = \boldsymbol{\Omega}\times\boldsymbol{\omega} = \begin{vmatrix} \boldsymbol{e}_1'' & \boldsymbol{e}_2'' & \boldsymbol{e}_3'' \\ \Omega\sin(\alpha) & 0 & \Omega\cos(\alpha) \\ \Omega\sin(\alpha)+\dot{\varphi} & 0 & \Omega\cos(\alpha) \end{vmatrix} = \dot{\varphi}\Omega\cos(\alpha)\,\boldsymbol{e}_2'' \;\Rightarrow$$

$$\dot{\boldsymbol{\omega}} = -\frac{v_0^2(L^2+R^2)}{L^3R}\boldsymbol{e}_2''$$

Next, the balance of linear momentum of the system yields

$$\boldsymbol{R}_A + \boldsymbol{R}_B - 2mg\boldsymbol{e}_3' = m(\boldsymbol{a}_C + \boldsymbol{a}_G) \quad , \quad \boldsymbol{R}_A = R_{Ai}''\boldsymbol{e}_i'' \quad , \quad \boldsymbol{R}_B = R_{Bi}''\boldsymbol{e}_i'' \quad ,$$

$$\boldsymbol{a}_C = \frac{d}{dt}(v_0\boldsymbol{e}_2') = v_0(\boldsymbol{\Omega}\times\boldsymbol{e}_2') = v_0\Omega\boldsymbol{e}_3'\times\boldsymbol{e}_2' = -v_0\Omega\boldsymbol{e}_1' = \frac{v_0^2}{L}\left(-\boldsymbol{e}_1'' + \frac{R}{L}\boldsymbol{e}_3''\right) \quad ,$$

$$\boldsymbol{a}_G = \frac{\boldsymbol{a}_C}{2} = \frac{v_0^2}{2L}\left(-\boldsymbol{e}_1'' + \frac{R}{L}\boldsymbol{e}_3''\right) \quad .$$

Thus,

$$R_{A1}'' + R_{B1}'' - \frac{2mgR}{\sqrt{L^2+R^2}} = -\frac{3mv_0^2}{2L} \quad , \quad R_{A2}'' + R_{B2}'' = 0 \quad ,$$

$$R_{A3}'' + R_{B3}'' - \frac{2mgL}{\sqrt{L^2+R^2}} = \frac{3mv_0^2R}{2L^2} \quad .$$

Also, the balance of angular momentum of the system about the point A gives

$$\boldsymbol{M}_A = \dot{\boldsymbol{H}}_C + \dot{\boldsymbol{H}}_G + \boldsymbol{x}_{C/A} \times m\boldsymbol{a}_C + \boldsymbol{x}_{G/A} \times m\boldsymbol{a}_G = \dot{\boldsymbol{H}}_C + \dot{\boldsymbol{H}}_G - \frac{5mv_0^2R}{4L}\boldsymbol{e}_2'' \quad ,$$

$$H_{Gi}'' = I_{Gij}''\Omega'' = \frac{mL^2\Omega}{12}\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}\sin(\alpha)\\ 0\\ \cos(\alpha)\end{Bmatrix} = \frac{mL^2\Omega}{12}\begin{Bmatrix}0\\ 0\\ \cos(\alpha)\end{Bmatrix} \Rightarrow$$

$$\boldsymbol{H}_G = \frac{mLv_0}{12}\boldsymbol{e}_3'' \quad \Rightarrow \quad \dot{\boldsymbol{H}}_G = \frac{\delta \boldsymbol{H}_G}{\delta t} + \boldsymbol{\Omega} \times \boldsymbol{H}_G = \boldsymbol{\Omega} \times \boldsymbol{H}_G = -\frac{mv_0^2R}{12L}\boldsymbol{e}_2'' \quad ,$$

$$H_{Ci}'' = I_{Cij}''\omega'' = \frac{mR^2v_0}{4}\begin{bmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}-1/R\\ 0\\ 1/L\end{Bmatrix} = \frac{mR^2v_0}{4}\begin{Bmatrix}-2/R\\ 0\\ 1/L\end{Bmatrix} \Rightarrow$$

$$\boldsymbol{H}_C = \frac{mR^2v_0}{4}\left(-\frac{2}{R}\boldsymbol{e}_1'' + \frac{1}{L}\boldsymbol{e}_3''\right) \Rightarrow$$

$$\dot{\boldsymbol{H}}_C = \frac{\delta \boldsymbol{H}_C}{\delta t} + \boldsymbol{\Omega} \times \boldsymbol{H}_C = \boldsymbol{\Omega} \times \boldsymbol{H}_C = \frac{mR^2v_0}{4}\begin{vmatrix}\boldsymbol{e}_1'' & \boldsymbol{e}_2'' & \boldsymbol{e}_3''\\ \Omega\sin(\alpha) & 0 & \Omega\cos(\alpha)\\ -2/R & 0 & 1/L\end{vmatrix}$$

$$= -\frac{mv_0^2R(2L^2+R^2)}{4L^3}\boldsymbol{e}_2'' \quad ,$$

$$\boldsymbol{M}_A = \boldsymbol{x}_{B/A} \times \boldsymbol{R}_B + \boldsymbol{x}_{G/A} \times (-mg\boldsymbol{e}_3') + \boldsymbol{x}_{C/A} \times (-mg\boldsymbol{e}_3')$$

$$= \begin{vmatrix}\boldsymbol{e}_1'' & \boldsymbol{e}_2'' & \boldsymbol{e}_3''\\ L & 0 & -R\\ R_{B1}'' & R_{B2}'' & R_{B3}''\end{vmatrix} - mg\left(\frac{3L}{2}\boldsymbol{e}_1''\right) \times [\sin(\alpha)\,\boldsymbol{e}_1'' + \cos(\alpha)\,\boldsymbol{e}_3'']$$

$$= RR_{B2}''\boldsymbol{e}_1'' + \left[\frac{3mgL^2}{2\sqrt{L^2+R^2}} - RR_{B1}'' - LR_{B3}''\right]\boldsymbol{e}_2'' + LR_{B2}''\boldsymbol{e}_3'' \quad .$$

Consequently,

$$R_{B2}'' = 0 \quad , \quad RR_{B1}'' + LR_{B3}'' - \frac{3mgL^2}{2\sqrt{L^2+R^2}} = \frac{mv_0^2R}{L}\left(\frac{11}{6} + \frac{R^2}{4L^2}\right) \quad .$$

Now, the equations of motion to be solved are given by

$$R_{B2}'' = R_{A2}'' = 0 \quad , \quad R_{A1}'' + R_{B1}'' - \frac{2mgR}{\sqrt{L^2+R^2}} = -\frac{3mv_0^2}{2L} \quad ,$$

$$R''_{A3} + R''_{B3} - \frac{2mgL}{\sqrt{L^2+R^2}} = \frac{3mv_0^2 R}{2L^2} \quad , \quad RR''_{B1} + LR''_{B3} - \frac{3mgL^2}{2\sqrt{L^2+R^2}} = \frac{mv_0^2 R}{L}\left(\frac{11}{6} + \frac{R^2}{4L^2}\right).$$

Assuming that the $\boldsymbol{e}'_1$ component of the reaction force at A vanishes ($\boldsymbol{R}_A \cdot \boldsymbol{e}'_1 = 0$), it follows that

$$\boldsymbol{R}_A \cdot \boldsymbol{e}'_1 = (R''_{A1}\boldsymbol{e}''_1 + R''_{A3}\boldsymbol{e}''_3) \cdot [\cos(\alpha)\,\boldsymbol{e}''_1 - \sin(\alpha)\,\boldsymbol{e}''_3] = 0 \;\;\Rightarrow\;\; LR''_{A1} - RR''_{A3} = 0 \;\; .$$

Solving the system of four equations

$$R''_{A1} + R''_{B1} - \frac{2mgR}{\sqrt{L^2+R^2}} = -\frac{3mv_0^2}{2L} \quad , \quad R''_{A3} + R''_{B3} - \frac{2mgL}{\sqrt{L^2+R^2}} = \frac{3mv_0^2 R}{2L^2} \quad ,$$

$$RR''_{B1} + LR''_{B3} - \frac{3mgL^2}{2\sqrt{L^2+R^2}} = \frac{mv_0^2 R}{L}\left(\frac{11}{6} + \frac{R^2}{4L^2}\right) \quad , \quad LR''_{A1} - RR''_{A3} = 0 \quad ,$$

for the reaction forces $\{R''_{A1}, R''_{A3}, R''_{B1}, R''_{B3}\}$ yields

$$R''_{A1} = \frac{mR}{2(L^2+R^2)}\left[\frac{g(L^2+4R^2)}{\sqrt{L^2+R^2}} - \frac{11v_0^2 R}{3L}\left(1 + \frac{3R^2}{22L^2}\right)\right] \quad ,$$

$$R''_{A3} = \frac{mL}{2(L^2+R^2)}\left[\frac{g(L^2+4R^2)}{\sqrt{L^2+R^2}} - \frac{11v_0^2 R}{3L}\left(1 + \frac{3R^2}{22L^2}\right)\right] \quad ,$$

$$R''_{B1} = \frac{3mL}{2(L^2+R^2)}\left[\frac{gRL}{\sqrt{L^2+R^2}} - v_0^2\left(1 - \frac{2R^2}{9L^2} - \frac{R^4}{6L^4}\right)\right] \quad ,$$

$$R''_{B3} = \frac{3mL}{2(L^2+R^2)}\left[\frac{gL^2}{\sqrt{L^2+R^2}} + \frac{20v_0^2 R}{9L}\left(1 + \frac{21R^4}{40L^4}\right)\right] \quad .$$