DYNAMICS

ME 34010

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1. Introduction

Sir Issac Newton (1642-1727) was the first to discover the correct laws of motion of particles. Since then much work has been done to verify the validity of these laws and to generalize them for deformable media. The study of rigid body dynamics is concerned with developing and analyzing the equations of motion of: a single particle, a system of particles, a rigid body, and a system of rigid bodies.

To analyze the motion of a particle it is necessary to first develop kinematical expressions for the position, velocity and acceleration of the particle. Then, it is necessary to consider the kinetic equations of motions which characterize the influence of the forces applied to the particle on its motion. Thus, the analysis of both kinematics and kinetics are necessary for a complete formulation of a specific problem. Since the analysis of the motion of a single particle is simpler than that of a rigid body, most courses in dynamics develop the material in the following order. The kinematics and kinetics of motion of a single particle are discussed for the simplest case of motion in a straight line. Then, the equations are generalized to motion in a plane followed by motion in three-dimensional space. Next, the kinematics and kinetics of motion of a system of particle is developed. The analysis of rigid body motion starts with analysis in a plane and then is concluded with analysis in three-dimensional space. This approach has the advantage that mathematical complexity increases gradually and that physical concepts are presented in their simplest forms. However, it has the disadvantage that the more complicated mathematical tools required to analyze general three-dimensional motion are presented near the end of the course when there is often not sufficient time to fully absorb the material.

This course in dynamics presents the material in a different order from that in a standard presentation. The course is loosely separated into two parts. Part 1 includes sections 2-15 which develop the analysis of kinematics in three-dimensions and Part 2 includes sections 16-39 which introduce the kinetic equations of motion to analyze forces and energies of systems of rigid bodies. This approach has the advantage that the more complicated mathematical tools of analyzing motion in rotating coordinate systems is developed in Part 1. Since the analysis of the kinetics of particles and rigid bodies necessarily requires the determination of acceleration, the more complicated mathematical tools for analyzing motion in three-dimensions are used in almost all example problems in Part 2. This ensures that these mathematical tools are fully absorbed. Moreover, it ensures that at the end of the course each student can confidently formulate even the most complicated dynamics problems in three-dimensions.

2. Vector Algebra and Indicial Notation

In mechanics it is necessary to use vectors and vector equations to express physical laws. To conveniently express the component forms of these vector equations it is desirable to use a language called indicial notation which develops simple rules governing manipulations of the components of these vector equations. For the purposes of describing this language we introduce a set of right-handed orthonormal base vectors denoted by $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$, such that

$$\mathbf{e}_1 \bullet \mathbf{e}_1 = 1 \ , \ \mathbf{e}_2 \bullet \mathbf{e}_2 = 1 \ , \ \mathbf{e}_3 \bullet \mathbf{e}_3 = 1 \ , \tag{2.1a,b,c}$$

$$\mathbf{e}_1 \bullet \mathbf{e}_2 = 0 \ , \ \mathbf{e}_1 \bullet \mathbf{e}_3 = 0 \ , \ \mathbf{e}_2 \bullet \mathbf{e}_3 = 0 \ , \tag{2.1d,e,f}$$

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3 \ , \ \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2 \ , \ \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1 \ , \tag{2.1g,h,i}$$

where $\mathbf{e}_1 \bullet \mathbf{e}_2$ denotes the dot product between the two vectors, and $\mathbf{e}_1 \times \mathbf{e}_2$ denotes the cross product between the two vectors. In this text we will use a bold faced symbol like **a** to indicate a vector quantity and in the written form we will use a wavy line under the symbol like $\underset{\sim}{a}$ to indicate the same vector quantity **a**.

VECTOR ALGEBRA

Rules of Vector Addition and Multiplication by a Scalar: Let **a**, **b** , and **c** be vectors and α and β be scalars. Then the commutative and associative laws of vector addition are (see Fig. 2.1)

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \ \text{(commutative law)} \ , \tag{2.1a}$$

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c} \ \text{(associative law)} \ , \tag{2.1b}$$

Furthermore, the associative laws of scalar multiplication may be summarized as

$$\alpha\,(\beta\,\mathbf{a}) = (\alpha\beta)\,\mathbf{a} = \beta\,(\alpha\,\mathbf{a}) = \mathbf{a}\,(\alpha\,\beta) \ , \tag{2.2a}$$

$$(\alpha + \beta)\,\mathbf{a} = \alpha\,\mathbf{a} + \beta\,\mathbf{a} \ , \tag{2.2b}$$

$$\alpha\,(\mathbf{a} + \mathbf{b}) = \alpha\,\mathbf{a} + \alpha\,\mathbf{b} \ . \tag{2.2c}$$

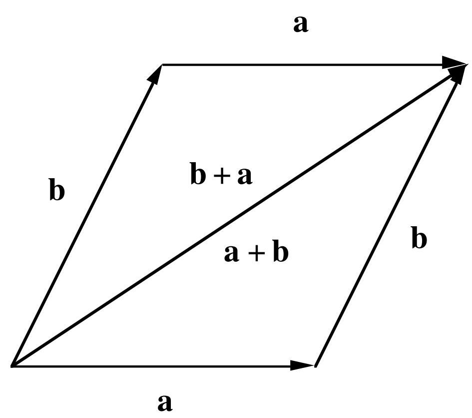

Fig. 2.1

<u>Components of a Vector</u>: An arbitrary vector **a** in Euclidean three-dimensional space may be expressed in terms of its components $\{a_1, a_2, a_3\}$ relative to the fixed base vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ such that

$$\mathbf{a} = a_1\,\mathbf{e}_1 + a_2\,\mathbf{e}_2 + a_3\,\mathbf{e}_3 \quad . \tag{2.3}$$

<u>Scalar (Dot) Product: Magnitude and Direction of a Vector</u>: The scalar (or dot) product between two vectors **a** and **b** is defined by

$$\begin{aligned} \mathbf{a} \bullet \mathbf{b} &= (a_1\,\mathbf{e}_1 + a_2\,\mathbf{e}_2 + a_3\,\mathbf{e}_3) \bullet (b_1\,\mathbf{e}_1 + b_2\,\mathbf{e}_2 + b_3\,\mathbf{e}_3) \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3 \quad , \end{aligned} \tag{2.4}$$

where $\{b_1, b_2, b_3\}$ are the components of **b** relative to the base vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. It follows that the magnitude a and the unit direction $\mathbf{e}_a$ of vector **a** may be defined by

$$a = |\,\mathbf{a}\,| = (\mathbf{a} \bullet \mathbf{a})^{1/2} = (a_1^2 + a_2^2 + a_3^2)^{1/2} \quad , \tag{2.5a}$$

$$\mathbf{e}_a = \frac{\mathbf{a}}{a} = \frac{a_1}{a}\;\mathbf{e}_1 + \frac{a_2}{a}\;\mathbf{e}_2 + \frac{a_2}{a}\;\mathbf{e}_2 \;\;,\;\; \mathbf{e}_a \bullet \mathbf{e}_a = 1 \quad , \tag{2.5b,c}$$

so that **a** may be represented in the form

$$\mathbf{a} = a\,\mathbf{e}_a \quad . \tag{2.6}$$

The scalar product $\mathbf{a} \bullet \mathbf{b}$ may also be written in the more physical form

$$\mathbf{a} \bullet \mathbf{b} = a\,b\cos\theta \quad , \tag{2.7a}$$

$$\mathbf{a} \bullet \mathbf{b} = a\,(\mathbf{b} \bullet \mathbf{e}_a) = a\,(b\cos\theta) \;\;,\;\; \mathbf{a} \bullet \mathbf{b} = b\,(\mathbf{a} \bullet \mathbf{e}_b) = b\,(a\cos\theta) \;, \tag{2.7b,c}$$

where θ is the angle between **a** and **b** (see Fig. 2.2). The representation (2.7b) expresses $\mathbf{a} \bullet \mathbf{b}$ as the magnitude of **a** times the projection of **b** in the direction of **a**, whereas the representation (2.7c) expresses $\mathbf{a} \bullet \mathbf{b}$ as the magnitude of $\mathbf{b}$ times the projection of $\mathbf{a}$ in the direction of $\mathbf{b}$. Furthermore, we note that the scalar product is commutative so that

$$\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a} \ . \tag{2.8}$$

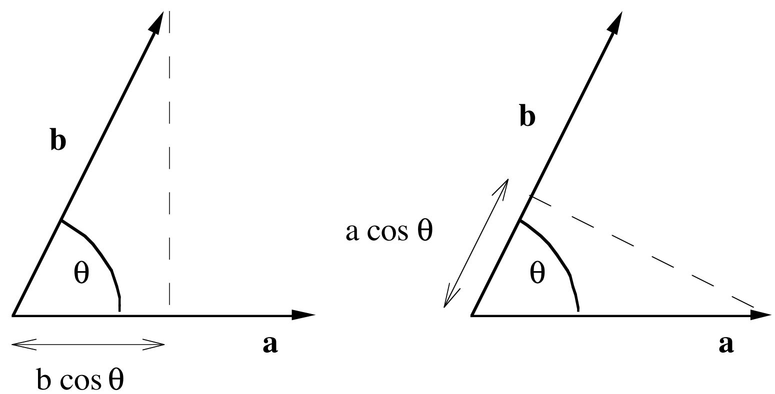

Fig. 2.2

Also, it follows that the components of $\mathbf{a}$ may be calculated by using the scalar product by

$$a_1 = \mathbf{a} \bullet \mathbf{e}_1 \ , \ a_2 = \mathbf{a} \bullet \mathbf{e}_2 \ , \ a_3 = \mathbf{a} \bullet \mathbf{e}_3 \ . \tag{2.9a,b,c}$$

Defining $\{\theta_1,\theta_2,\theta_3\}$ as the angles between the vector $\mathbf{a}$ and the base vectors $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, respectively, the components of the direction vector $\mathbf{e}_a$ in (2.5b) may be represented by (see Fig. 2.3)

$$\frac{a_1}{a} = \cos\theta_1 \ , \ \frac{a_2}{a} = \cos\theta_2 \ , \ \frac{a_3}{a} = \cos\theta_3 \ . \tag{2.10a,b,c}$$

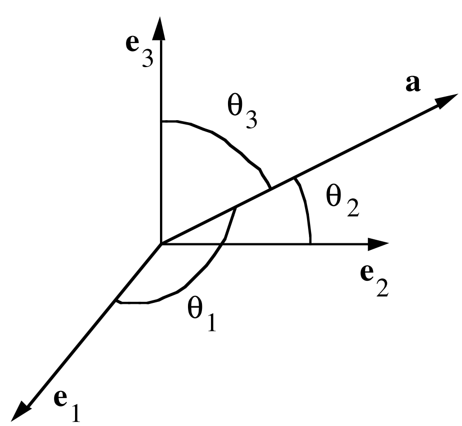

Fig. 2.3

Vector (Cross) Product: The vector product between the two vectors **a** and **b** may be calculated directly using the expressions (2.1g,h,i) or may be calculated using the determinant of a matrix of the form

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad , \tag{2.11a}$$

$$= (a_2 b_3 - a_3 b_2)\, \mathbf{e}_1 - (a_1 b_3 - a_3 b_1)\, \mathbf{e}_2 + (a_1 b_2 - a_2 b_1)\, \mathbf{e}_3 \; . \tag{2.11b}$$

The vector product $\mathbf{a} \times \mathbf{b}$ may also be written in the more physical form

$$\mathbf{a} \times \mathbf{b} = (a\, b\, \sin\theta)\, \mathbf{n} \quad , \quad \mathbf{n} \bullet \mathbf{n} = 1 \quad , \tag{2.12a,b}$$

where θ is the angle between **a** and **b** and **n** is the unit vector that is normal to the plane of the vectors **a** and **b** and is defined by the right-hand rule (see Fig. 2.4). It follows that since **n** is defined by the right-hand rule the vector product is not commutative because

$$\mathbf{a} \times \mathbf{b} = -\, \mathbf{b} \times \mathbf{a} \quad . \tag{2.13}$$

Furthermore, from Fig. 2.4 we realize that the vector product $\mathbf{a} \times \mathbf{b}$ yields the area of the parallelogram formed by the vectors **a** and **b** times the unit normal vector **n**. Thus, the vector product of a vector with itself vanishes ($\mathbf{a} \times \mathbf{a} = 0$).

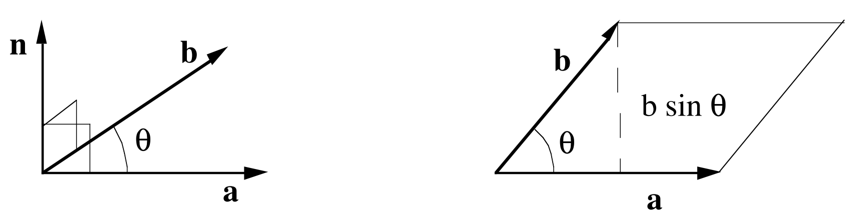

Fig. 2.4

Scalar Triple Product: The scalar triple product between the three vectors **a**,**b**,**c** may be expressed in the following equivalent forms

$$\mathbf{a} \bullet \mathbf{b} \times \mathbf{c} = \mathbf{c} \bullet \mathbf{a} \times \mathbf{b} = \mathbf{b} \bullet \mathbf{c} \times \mathbf{a} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad . \tag{2.14}$$

Physically, the scalar triple product may be interpreted as giving the volume of the parallelepiped formed by the vectors **a**,**b**,**c** (see Fig. 2.5) since

$$\mathbf{a} \times \mathbf{b} \bullet \mathbf{c} = (a\, b\, \sin\theta)\, \mathbf{n} \bullet \mathbf{c} \quad , \tag{2.15}$$

where (a b sin θ) is the area of the base of the parallelepiped and $\mathbf{n} \bullet \mathbf{c}$ is the height of the parallelepiped.

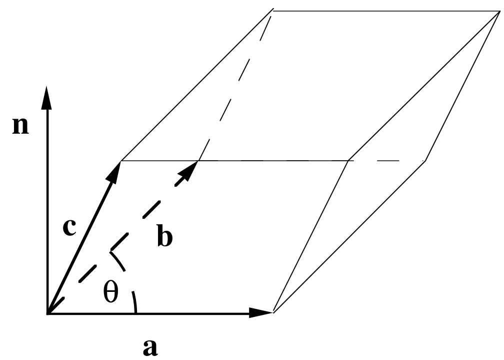

Fig. 2.5

Note that the order of the scalar and vector product may be interchanged without changing the value of the scalar triple product. This can also be seen by realizing that the volume of the parallelepiped in Fig. 2.5 can be obtained by expressing the scalar triple product in terms of the vector product of an two of the three vectors **a**,**b**,**c** whose normal **n** (according to the right-hand rule) points toward the interior of the parallelepiped.

<u>Vector Triple Product</u>: The vector triple product between the three vectors **a**,**b**,**c** may be expanded in the form

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{c})\, \mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\, \mathbf{c} \quad . \tag{2.16}$$

It is important to emphasize that since the vector product between two vectors generates another vector it is essential to include parentheses in the definition (2.16). Also note that the vector $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is perpendicular to the vector $(\mathbf{b} \times \mathbf{c})$. But the vector $(\mathbf{b} \times \mathbf{c})$ is perpendicular to the plane formed by **b** and **c**. This means that the vector $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ must lie in the plane of **b** and **c**, which is consistent with the result (2.16). Furthermore, the vector $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ must also be perpendicular to **a**, which is consistent with the result (2.16).

INDICIAL NOTATION

Quantities written in indicial notation will have a finite number of indices attached to them. Since the number of indices can be zero a quantity with no index can also be considered to be written in index notation. The language of indicial notation is quite simple because only two types of indices may appear in any term. Either the index is a free index or it is a repeated index. Also, we will define a simple summation convention which applies only to repeated indices. These two types of indices and the summation convention are defined as follows.

Free Indices: Indices that appear only once in a given term are known as free indices. For our purposes each of these free indices will take the values (1,2,3). For example, i is a free index in each of the following expressions

$$(x_1 , x_2 , x_3) = x_i \quad (i=1,2,3) \ , \tag{2.17a}$$

$$(\mathbf{e}_1 , \mathbf{e}_2 , \mathbf{e}_3) = \mathbf{e}_i \quad (i=1,2,3) \ . \tag{2.17b}$$

Notice that the free index i in (2.17) refers to the group of three quantities defined by i taking the values 1,2,3.

Repeated Indices: Indices that appear twice in a given term are known as repeated indices. For example i and j are free indices and m and n are repeated indices in the following expressions

$$a_i \, b_j \, c_m \, T_{mn} \, d_n \ , \ A_{immjnn} \ , \ A_{imn} \, B_{jmn} \ . \tag{2.18a,b,c}$$

It is important to emphasize that in the language of indicial notation an index can never appear more than twice in any term.

Einstein Summation Convention: When an index appears as a repeated index in a term, that index is understood to take on the values (1,2,3) and the resulting terms are summed. Thus, for example,

$$x_i \, \mathbf{e}_i = x_1 \, \mathbf{e}_1 + x_2 \, \mathbf{e}_2 + x_3 \, \mathbf{e}_3 \ . \tag{2.19}$$

Because of this summation convention, repeated indices are also known as dummy indices since their replacement by any other letter not appearing as a free index and also not appearing as another repeated index does not change the meaning of the term in which they occur. For examples,

$$x_i \, \mathbf{e}_i = x_j \, \mathbf{e}_j \ , \ a_i \, b_m c_m = a_i \, b_j \, c_j \ . \tag{2.20a,b}$$

It is important to emphasize that the same free indices must appear in each term in an equation so that for example the free index i in (2.20b) must appear on each side of the equation.

Kronecker Delta: The Kronecker delta symbol δ_{ij} is defined by

$$\delta_{ij} = \mathbf{e}_i \bullet \mathbf{e}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad . \tag{2.21}$$

Since the Kronecker delta δ_{ij} vanishes unless i=j it exhibits the following exchange property

$$\delta_{ij}\, x_j = (\, \delta_{1j}\, x_j \,,\, \delta_{2j}\, x_j \,,\, \delta_{3j}\, x_j \,) = (\, x_1 \,,\, x_2 \,,\, x_3 \,) = x_i \quad . \tag{2.22}$$

Notice that the Kronecker symbol may be removed by replacing the repeated index j in (2.22) by the free index i.

Recalling that an arbitrary vector $\mathbf{a}$ in Euclidean 3-Space may be expressed as a linear combination of the base vectors $\mathbf{e}_i$ such that

$$\mathbf{a} = a_i\, \mathbf{e}_i \quad , \tag{2.23}$$

it follows that the components a_i of $\mathbf{a}$ can be calculated using the Kronecker delta

$$a_i = \mathbf{e}_i \bullet \mathbf{a} = \mathbf{e}_i \bullet (a_m\, \mathbf{e}_m) = (\mathbf{e}_i \bullet \mathbf{e}_m)\, a_m = \delta_{im}\, a_m = a_i \quad . \tag{2.24}$$

Notice that when the expression (2.23) for $\mathbf{a}$ was substituted into (2.24) it was necessary to change the repeated index i in (2.23) to another letter (m) because the letter i already appeared in (2.24) as a free index. It also follows that the Kronecker delta may be used to calculate the dot product between two vectors $\mathbf{a}$ and $\mathbf{b}$ with components a_i and b_i, respectively by

$$\mathbf{a} \bullet \mathbf{b} = (a_i\, \mathbf{e}_i) \bullet (b_j\, \mathbf{e}_j) = a_i\, (\mathbf{e}_i \bullet \mathbf{e}_j)\, b_j = a_i\, \delta_{ij}\, b_j = a_i\, b_i \quad . \tag{2.25}$$

Contraction: Contraction is the process of identifying two free indices in a given expression together with the implied summation convention. For example we may contract on the free indices i,j in δ_{ij} to obtain

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3 \quad . \tag{2.26}$$

Note that contraction on the set of $9{=}3^2$ quantities T_{ij} can be performed by multiplying T_{ij} by δ_{ij} to obtain

$$T_{ij}\,\delta_{ij} = T_{ii} = T_{11} + T_{22} + T_{33}. \tag{2.27}$$

Matrix Multiplication: In order to connect the summation convention with standard matrix multiplication consider two vectors with components a_i, b_i, and three square matrices with components A_{ij}, B_{ij}, C_{ij} and define

$$b_i = A_{ij}\,a_j \ , \ C_{ij} = A_{im}B_{mj} \ . \tag{2.28a,b}$$

Since the first index of A_{ij} indicates the row and the second index indicates the column, it can easily be seen that the summation on the index j in (2.28a) yields the same result of the multiplication of the matrix A_{ij} with the column vector a_j to obtain the column vector b_i. Similarly, since the second index of A_{im} in (2.28b) is summed with the first index of B_{mj} it is easy to see that C_{ij} is the matrix that is obtained by multiplying the matrix A_{im} with the matrix B_{mj}.

Transpose: Let A_{ij} be the components of a matrix **A**. Then the components of the transpose $\mathbf{A}^T$ of **A** are given by

$$(\mathbf{A}^T)_{ij} = A^T_{ij} = A_{ji} \ . \tag{2.29}$$

In the above we have considered terms that have no free indices like in equations (2.19), (2.20a), (2.23), (2.25)-(2.27); that have one free index like in equations (2.22), (2.24), (2.28a); and that have two free indices like in equations (2.18), (2.21), (2.28b), (2.29). Obviously, it is possible to write terms that have any number of free indices. In general, a term is said to be of order zero if it has no free index, of order one if it has one free index and of order n if it has no free indices. Usually in mechanics terms with indices are components of quantities called tensors, which are generalizations of vectors. In particular, when **a** and **b** are vectors the quantity **a** • **b** in (2.25) is called a scalar or zero order tensor (or a tensor of order zero). Also, the quantities a_i in (2.24) are the components of the vector **a**, which is also called a first order tensor (or a tensor of order one). In this course we will consider tensors up to order three.

<u>Permutation Symbol</u>: The permutation symbol ε_{ijk} is defined by

$$\varepsilon_{ijk} = \mathbf{e}_i \times \mathbf{e}_j \bullet \mathbf{e}_k = \begin{cases} +1 & \text{for (i,j,k) equal to an even permutation of (1,2,3),} \\ & \text{(i,j,k) = (1,2,3), (3,1,2), (2,3,1)} \\ -1 & \text{for (i,j,k) equal to an odd permutation of (1,2,3),} \\ & \text{(i,j,k) = (1,3,2), (2,1,3), (3,2,1)} \\ 0 & \text{whenever any of the indices (i,j,k) are, repeated more than} \\ & \text{once (i.e i=j, or i=k, or j=k, or i=j=k)} \end{cases} \tag{2.30}$$

It can be shown that the nine vectors $\mathbf{e}_i \times \mathbf{e}_j$ can be expressed in terms of the permutation symbol using the expression

$$\mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk}\, \mathbf{e}_k \ . \tag{2.31}$$

Thus, the vector product between the two vectors **a** and **b** may be expressed in the form

$$\mathbf{a} \times \mathbf{b} = a_i\, \mathbf{e}_i \times b_j\, \mathbf{e}_j = \varepsilon_{ijk}\, a_i\, b_j\, \mathbf{e}_k \ . \tag{2.32}$$

For convenience we summarize the expanded and short forms of a number of vector quantities in Table 2.1.

Quantity	Expanded Form	Short Form
Rectangular Cartesian Coordinates	x_1, x_2, x_3	x_i (i=1,2,3)
Rectangular Cartesian Base Vectors	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	$\mathbf{e}_i$
Position Vector	$\mathbf{x} = x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2 + x_3\,\mathbf{e}_3$	$x_i\,\mathbf{e}_i$
Components of Vector **a**	$a_1 = \mathbf{a} \bullet \mathbf{e}_1$, $a_2 = \mathbf{a} \bullet \mathbf{e}_2$ $a_3 = \mathbf{a} \bullet \mathbf{e}_3$	$a_i = \mathbf{a} \bullet \mathbf{e}_i$
Vector **a**	$\mathbf{a} = a_1\,\mathbf{e}_1 + a_2\,\mathbf{e}_2 + a_3\,\mathbf{e}_3$	$\mathbf{a} = a_i\,\mathbf{e}_i$
Scalar Product	$\mathbf{a} \bullet \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$	$\mathbf{a} \bullet \mathbf{b} = a_i\, b_i$
Vector Product	$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2)\,\mathbf{e}_1$ $+ (a_3 b_1 - a_1 b_3)\,\mathbf{e}_2$ $+ (a_1 b_2 - a_2 b_1)\,\mathbf{e}_3$	$\mathbf{a} \times \mathbf{b} = \varepsilon_{ijk}\, a_i\, b_j\, \mathbf{e}_k$
Gradient of a Scalar ϕ	$\nabla\,\phi = \frac{\partial \phi}{\partial x_1}\,\mathbf{e}_1 + \frac{\partial \phi}{\partial x_2}\,\mathbf{e}_2$ $+ \frac{\partial \phi}{\partial x_3}\,\mathbf{e}_3$	$\nabla\,\phi = \frac{\partial \phi}{\partial x_i}\,\mathbf{e}_i$
Divergence of a Vector **a**	$\nabla \bullet \mathbf{a} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}$	$\nabla \bullet \mathbf{a} = \frac{\partial a_i}{\partial x_i}$
Curl of a Vector **a**	$\nabla \times \mathbf{a} = \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right)\mathbf{e}_1$ $+ \left(\frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}\right)\mathbf{e}_2$ $+ \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)\mathbf{e}_3$	$\nabla \times \mathbf{a} = \varepsilon_{ijk}\, \frac{\partial a_j}{\partial x_i}\, \mathbf{e}_k$

Table 2.1

3. Vector Calculus

In dynamics most vectors will be considered to be functions of time t and many of the equations in dynamics will be differential equations that need to be integrated. Consequently, it is important to learn how to differentiate and integrate vector functions.

Differentiation: To this end, we define the time derivative of the vector function $\mathbf{a}(t)$ by the same limiting process that derivatives of scalar functions are defined

$$\frac{d\mathbf{a}}{dt} = \dot{\mathbf{a}} = \lim_{\Delta t \to 0} \frac{\mathbf{a}(t+\Delta t) - \mathbf{a}(t)}{\Delta t} \quad . \tag{3.1}$$

In (3.1) and throughout the text we use the notation $d\mathbf{a}/dt$ and a superposed (•) to denote time differentiation. It is important to note that both the magnitude and direction of a vector can change with time (see Fig. 3.1).

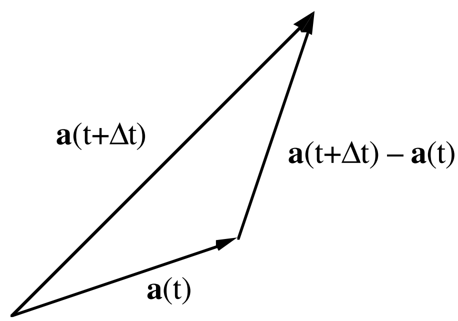

Fig.3.1

The standard rules of differentiation of scalar functions apply to vector functions except that the commutative law does not apply to the vector product between two vectors. It follows that if $\mathbf{a}$ and $\mathbf{b}$ are vector functions of time and α is a scalar function of time that

$$\frac{d}{dt}(\mathbf{a} + \mathbf{b}) = \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{b}}{dt} \quad , \tag{3.2a}$$

$$\frac{d}{dt}(\alpha\, \mathbf{a}) = \frac{d\alpha}{dt}\, \mathbf{a} + \alpha \frac{d\mathbf{a}}{dt} \quad , \tag{3.2b}$$

$$\frac{d}{dt}(\mathbf{a} \bullet \mathbf{b}) = \frac{d\mathbf{a}}{dt} \bullet \mathbf{b} + \mathbf{a} \bullet \frac{d\mathbf{b}}{dt} \quad , \tag{3.2c}$$

$$\frac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \frac{d\mathbf{a}}{dt} \times \mathbf{b} + \mathbf{a} \times \frac{d\mathbf{b}}{dt} \quad . \tag{3.2d}$$

Vector of Constant Magnitude: Since we usually refer vectors to base vectors that have unit length it is desirable to consider the derivative of a general vector $\mathbf{a}$ of constant magnitude. Thus, let

$$\mathbf{a} \bullet \mathbf{a} = \text{constant} \quad . \tag{3.3}$$

Taking the derivative of (3.3) we have

$$\dot{\mathbf{a}} \bullet \mathbf{a} + \mathbf{a} \bullet \dot{\mathbf{a}} = 2\, \mathbf{a} \bullet \dot{\mathbf{a}} = 0 \quad , \tag{3.4}$$

which means that $\dot{\mathbf{a}}$ is perpendicular to $\mathbf{a}$ (see Fig. 3.2) so that the vector $\mathbf{a}$ can only rotate.

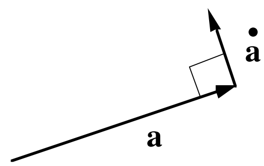

Fig. 3.2

Indefinite Integral of a Vector: The indefinite integral of the vector vector function $\mathbf{f}(t)$ is denoted by

$$\int^{t} \mathbf{f}(\tau)\, d\tau \,, \tag{3.5}$$

where the lower limit of integration is understood to be any arbitrary fixed value and τ is merely a variable of integration. It follows that (3.5) denotes all functions whose derivative is $\mathbf{f}$, so that

$$\frac{d}{dt}\int^{t} \mathbf{f}(\tau)\, d\tau = \mathbf{f}(t) \quad . \tag{3.6}$$

Definite Integral of a Vector: The definite integral of a vector function $\mathbf{f}(t)$ from time t_0 to time t is denoted by

$$\int_{t_0}^{t} \mathbf{f}(\tau)\, d\tau \quad . \tag{3.7}$$

Integral of a Vector Differential Equation: Consider the vector differential equation given by

$$\frac{d\mathbf{F}}{dt} = \mathbf{f}(t) \quad . \tag{3.8}$$

Integrating (3.8) using the indefinite integral (3.5) we deduce that

$$\mathbf{F}(t) = \int^{t} \mathbf{f}(\tau)\, d\tau + \mathbf{C} \quad , \tag{3.9}$$

where $\mathbf{C}$ is an arbitrary constant vector. Now in order to determine the function $\mathbf{F}(t)$ uniquely we need to specify the initial value of $\mathbf{F}$ [say $\mathbf{F}(t_0)$]. Thus, with the help of this initial condition we may determine the value of $\mathbf{C}$ in (3.9) by the equation

$$\mathbf{F}(t_0) = \int^{t_0} \mathbf{f}(\tau)\, d\tau + \mathbf{C} \ . \tag{3.10}$$

Now substituting (3.10) into (3.9) we deduce that

$$\mathbf{F}(t) = \mathbf{F}(t_0) + \int_{t_0}^{t} \mathbf{f}(\tau)\, d\tau \ , \tag{3.11}$$

where we have used the fact that

$$\int_{t_0}^{t} \mathbf{f}(\tau)\, d\tau = \int^{t} \mathbf{f}(\tau)\, d\tau - \int^{t_0} \mathbf{f}(\tau)\, d\tau \ . \tag{3.12}$$

Finally, we note that the integral of the sum of two vector functions $\mathbf{a}$,$\mathbf{b}$ is equal to the sum of the integrals of the functions

$$\int (\mathbf{a} + \mathbf{b})\, dt = \int \mathbf{a}\, dt + \int \mathbf{b}\, dt \ . \tag{3.13}$$

4. Position, Velocity, Acceleration

Let $\mathbf{x}(t)$ be the position vector relative to a fixed origin of a point that moves in space along a curve C (see Fig. 4.1). The average velocity $\mathbf{v}_{avg}$ over the time period $[t,t+\Delta t]$ is defined by

$$\mathbf{v}_{avg} = \frac{\mathbf{x}(t+\Delta t) - \mathbf{x}(t)}{\Delta t} \quad , \tag{4.1}$$

and the velocity (instantaneous velocity) $\mathbf{v}$ is defined as the derivative of the position vector

$$\mathbf{v}(t) = \dot{\mathbf{x}}(t) = \underset{\Delta t \varnothing 0}{\text{limit}} \frac{\mathbf{x}(t+\Delta t) - \mathbf{x}(t)}{\Delta t} \quad . \tag{4.2}$$

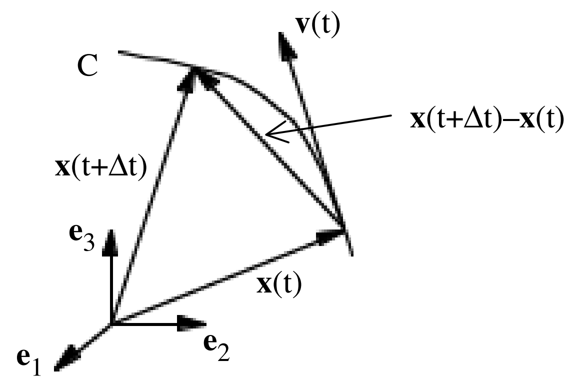

Fig. 4.1

Note from Fig. 4.1 that in the limit that Δt approaches zero the vector $\mathbf{x}(t+\Delta t)-\mathbf{x}(t)$ becomes tangent to the curve C so the instantaneous velocity $\mathbf{v}$ is always tangent to the path traversed by the point. Furthermore, the acceleration $\mathbf{a}$ of a point is defined as the derivative of the velocity so that

$$\mathbf{a} = \dot{\mathbf{v}} = \underset{\Delta t \to 0}{\text{limit}} \frac{\mathbf{v}(t+\Delta t) - \mathbf{v}(t)}{\Delta t} = \ddot{\mathbf{x}} \quad . \tag{4.3}$$

The displacement of a point from the position $\mathbf{x}(t_1)$ at time t_1 to its position $\mathbf{x}(t_2)$ at time t_2 may be obtained by integrating the velocity $\mathbf{v}$

$$\mathbf{x}(t_2) - \mathbf{x}(t_1) = \int_{t_1}^{t_2} \mathbf{v}(\tau)\, d\tau \quad . \tag{4.4}$$

Also, the distance traveled along the path traversed by the point is an increasing function of t so the distance traveled from time t_1 to time t_2 is calculated by integrating the magnitude of the velocity **v** and is denoted by $D_{2/1}$

$$D_{2/1} = \int_{t_1}^{t_2} |\mathbf{v}(\tau)|\, d\tau \quad . \tag{4.5}$$

5. Tangential and Normal Coordinates

Consider a space curve that is parameterized by the arclength s so that the position vector $\mathbf{x}(s)$ may be expressed in terms of its Rectangular Cartesian coordinates

$$\mathbf{x}(s) = x_1(s)\,\mathbf{e}_1 + x_2(s)\,\mathbf{e}_2 + x_3(s)\,\mathbf{e}_3 = x_i(s)\,\mathbf{e}_i \ , \tag{5.1}$$

where the arclength s is determined by integrating the element of arclength ds which is given by

$$(ds)^2 = d\mathbf{x} \bullet d\mathbf{x} \ , \tag{5.2}$$

Recalling that $d\mathbf{x}$ is tangent to the space curve (see Fig. 5.1) we may define the unit tangent vector $\mathbf{e}_t$ by

$$\mathbf{e}_t = \frac{d\mathbf{x}}{ds} \ . \tag{5.3}$$

The vector $\mathbf{e}_t$ can easily be shown to be a unit vector by using (5.2)

$$\mathbf{e}_t \bullet \mathbf{e}_t = \frac{d\mathbf{x}}{ds} \bullet \frac{d\mathbf{x}}{ds} = \frac{d\mathbf{x} \bullet d\mathbf{x}}{(ds)^2} = 1 \ . \tag{5.4}$$

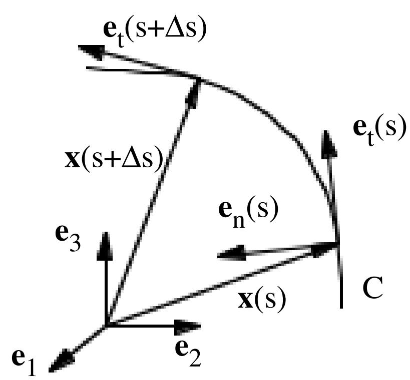

Fig. 5.1

Notice from Fig. 5.1 that the tangent vector $\mathbf{e}_t$ may change its direction as the curve is traversed. Recalling that $\mathbf{e}_t$ is a unit vector of constant length its derivative with respect to s must be perpendicular to it so that we may write

$$\frac{d\mathbf{e}_t}{ds} = \kappa\,\mathbf{e}_n = \frac{1}{\rho}\,\mathbf{e}_n \ , \quad \kappa = \frac{1}{\rho} = \left| \frac{d\mathbf{e}_t}{ds} \right| \geq 0 \ , \tag{5.5a,b}$$

where κ is the curvature and ρ is the radius of curvature of the space curve. Note that since κ is nonnegative the vector $\mathbf{e}_n$ points towards the inside of the space curve (see Fig. 5.1).

To help understand the curvature consider the special case of a planar curve for which

$$\mathbf{x}(s) = x_1(s)\,\mathbf{e}_1 + x_2(s)\,\mathbf{e}_2 \ .$$

Furthermore, let $\Delta\psi$ be the angle between the tangent vector $\mathbf{e}_t(s)$ at s and the tangent vector $\mathbf{e}_t(s+\Delta s)$ at $s+\Delta s$ (see Fig. 5.2). It follows from geometry that the vector $\mathbf{e}_t(s+\Delta s)$ may be expressed in terms of the base vectors $\mathbf{e}_t(s)$ and $\mathbf{e}_n(s)$ at s such that

$$\mathbf{e}_t(s+\Delta s) = \cos\Delta\psi\,\mathbf{e}_t(s) + \sin\Delta\psi\,\mathbf{e}_n(s) \ . \tag{5.7}$$

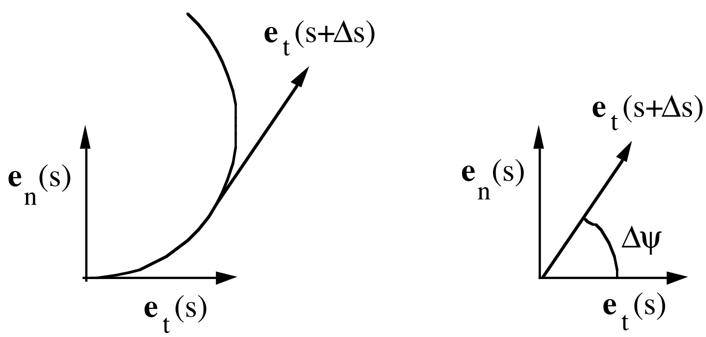

Fig. 5.2

Recalling the definition of the derivative we have

$$\frac{d\mathbf{e}_t}{ds} = \lim_{\Delta s \to 0} \frac{\mathbf{e}_t(s+\Delta s) - \mathbf{e}_t(s)}{\Delta s} \ ,$$

$$= \lim_{\Delta s \to 0} \left[\left(\frac{\cos\Delta\psi - 1}{\Delta s}\right)\mathbf{e}_t(s) + \left(\frac{\sin\Delta\psi}{\Delta s}\right)\mathbf{e}_n(s) \right] \ . \tag{5.8b}$$

But using the Taylor series expansions of $\cos\Delta\psi$ and $\sin\Delta\psi$

$$\cos\Delta\psi \approx 1 - \frac{(\Delta\psi)^2}{2} + \ldots \ , \quad \sin\Delta\psi \approx \Delta\psi - \frac{(\Delta\psi)^3}{6} + \ldots \ , \tag{5.9a,b}$$

we may rewrite (5.8b) in the form

$$\frac{d\mathbf{e}_t}{ds} = \lim_{\Delta s \to 0} \left[-\frac{(\Delta\psi)^2}{2\Delta s}\,\mathbf{e}_t(s) + \left(\frac{\Delta\psi}{\Delta s}\right)\mathbf{e}_n(s) \right] = \frac{d\psi}{ds}\,\mathbf{e}_n(s) \ . \tag{5.10}$$

Thus, comparing the result (5.10) with the definition (5.5) we have

$$\frac{1}{\rho} = \frac{d\psi}{ds} \quad , \quad ds = \rho \, d\psi \ . \tag{5.11a,b}$$

Note that the relationship (5.11b) is consistent with the relationship that connects the arclength of a circle with the radius of the circle and the angular displacement (see Fig. 5.3).

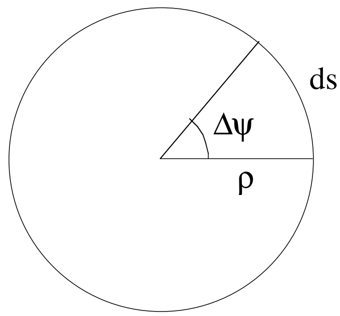

Fig. 5.3

Note also from Fig. 5.4 that for planar curves it is easy to find $\mathbf{e}_n$ (to within a plus or minus sign) by the formula

$$\mathbf{e}_n = \pm\, \mathbf{e}_3 \times \mathbf{e}_t \ , \tag{5.12}$$

since $\mathbf{e}_3 \times \mathbf{e}_t$ is a vector that lies in the $\mathbf{e}_1$–$\mathbf{e}_2$ plane and is normal to the space curve.

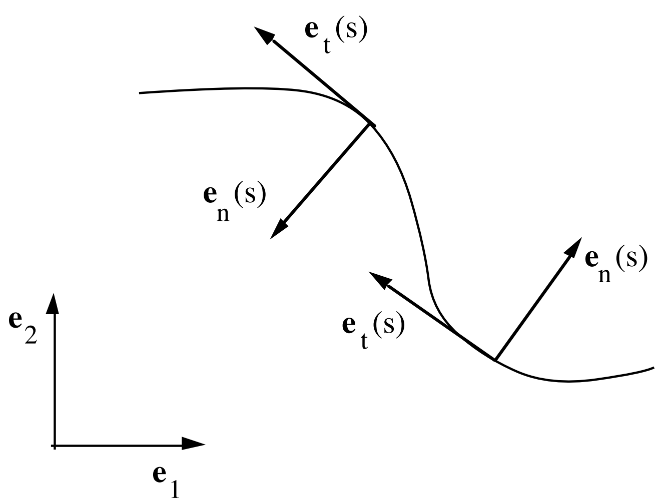

Fig. 5.4

For some problems it is convenient to use tangential and normal coordinates to describe motion of a particle moving in space. Within this context, the position vector **x** depends on time parametrically through the specification of s(t) so that

$$\mathbf{x} = \mathbf{x}(s(t)) \quad . \tag{5.13}$$

Thus, with the help of (5.3) and (5.5a) we may use the chain rule of differentiation to calculate the velocity **v** and acceleration **a** in the forms

$$\mathbf{v} = \dot{\mathbf{x}} = \frac{d\mathbf{x}}{ds}\,\dot{s} = \dot{s}\,\mathbf{e}_t \ ,$$

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{s}\,\mathbf{e}_t + \dot{s}\,\dot{\mathbf{e}}_t = \ddot{s}\,\mathbf{e}_t + \dot{s}^{\,2}\frac{d\mathbf{e}_t}{ds} = \ddot{s}\,\mathbf{e}_t + \frac{\dot{s}^2}{\rho}\,\mathbf{e}_n \ . \tag{5.14b}$$

It follows that the tangential and normal components of the velocity and acceleration become

$v_t = \mathbf{v} \bullet \mathbf{e}_t = \dot{s}$	the component of the velocity tangent to the curve defining the path of motion
$v_n = \mathbf{v} \bullet \mathbf{e}_n = 0$	the velocity is always tangent to the path of motion
$a_t = \mathbf{a} \bullet \mathbf{e}_t = \ddot{s}$	the component of acceleration tangent to the curve
$a_n = \mathbf{a} \bullet \mathbf{e}_n = \dfrac{\dot{s}^2}{\rho}$	the component of the acceleration normal to the path of motion and directed towards the center of curvature of the path

Note that even if the speed is constant [$\dot{s}$ = constant, $\ddot{s}$ = 0] the acceleration does not vanish when the path is curved ($\rho \neq \infty$) because the velocity changes direction.

Summary of Tangential and Normal Coordinates

Position vector	$\mathbf{x}(s(t))$
arclength	$(ds)^2 = d\mathbf{x} \bullet d\mathbf{x}$
tangent vector	$\mathbf{e}_t = \dfrac{d\mathbf{x}}{ds}$
curvature	$\kappa = \left\lvert \dfrac{d\mathbf{e}_t}{ds} \right\rvert$
radius of curvature	$\rho = \dfrac{1}{\kappa}$
normal vector	$\mathbf{e}_n = \rho \dfrac{d\mathbf{e}_t}{ds}$
velocity	$\mathbf{v} = \dot{s}\, \mathbf{e}_t$
acceleration	$\mathbf{a} = \ddot{s}\, \mathbf{e}_t + \dfrac{\dot{s}^2}{\rho}\, \mathbf{e}_n$

6. Rectilinear Motion

For rectilinear motion (motion in a straight line) the radius of curvature ρ becomes infinite so that the tangent vector $\mathbf{e}_t$ becomes constant.

$$\rho \to \infty \ , \ \frac{d\mathbf{e}_t}{ds} = 0 \ . \tag{6.1a,b}$$

Thus, we may chose our coordinate system so that $\mathbf{e}_t$ is in the positive $\mathbf{e}_1$ direction and deduce that the position, velocity and acceleration are characterized by the scalars s,v,a, respectively, such that

$$\mathbf{x} = s\ \mathbf{e}_1 \ , \tag{6.2a}$$

$$\mathbf{v} = v\ \mathbf{e}_1 \ , \ v = \dot{s} \ , \tag{6.2b,c}$$

$$\mathbf{a} = a\ \mathbf{e}_1 \ , \ a = \dot{v} = \ddot{s} \ . \tag{6.2d,e}$$

In what follows we consider four cases where the acceleration a is specified by different functional forms. For each case we develop equations that express the velocity v and the position s in terms of the initial position s_1 and velocity v_1 at the initial time t_1

$$s(t_1) = s_1 \ , \ v(t_1) = v_1 \ . \tag{6.3a,b}$$

<u>Case 1</u>: a = constant

For the case when the acceleration a is constant we may integrate the differential equations (6.2c,e) to obtain

$$\frac{dv}{dt} = a \ \Rightarrow \ v(t) = C_1 + \int_{t_1}^{t} a\ d\tau = C_1 + a\,(t - t_1) \ , \tag{6.4a}$$

$$\frac{ds}{dt} = v \ \Rightarrow \ s(t) = C_2 + \int_{t_1}^{t} v(\tau)\ d\tau = C_2 + C_1(t - t_1) + \frac{1}{2}\ a\,(t - t_1)^2 \ . \tag{6.4b}$$

Now using the initial conditions (6.3) we deduce that

$$v(t) = v_1 + a(t - t_1) \ , \tag{6.5a}$$

$$s(t) = s_1 + v_1(t - t_1) + \frac{1}{2}\ a\,(t - t_1)^2 \ . \tag{6.5b}$$

<u>Case 2</u>: a = a(t)

For the case when the acceleration is a general function of time we can only express the velocity v and position s in terms of integrals that need to be evaluated

$$v(t) = v_1 + \int_{t_1}^{t} a(\tau)\, d\tau \ , \tag{6.6a}$$

$$s(t) = s_1 + \int_{t_1}^{t} v(\tau)\, d\tau \ . \tag{6.6b}$$

Case 3: $a = a(v)$

The case when the acceleration is a function of velocity occurs often when damping or air drag are modeled. For this case the differential equation (6.2e) yields

$$\frac{dv}{dt} = a(v) \ \Rightarrow \ \frac{dv}{a(v)} = dt \ . \tag{6.7a,b}$$

Thus, with the help of the initial condition (6.3b) we may obtain an equation for v(t) of the form

$$\int_{v_1}^{v(t)} \frac{dV}{a(V)} = \int_{t_1}^{t} d\tau = (t - t_1) \ . \tag{6.8}$$

Then s(t) can be determined by the solution (6.6b).

Alternatively, sometimes it is of interest to find v(s) directly. For this case v is thought of as a function of s and the chain rule of differentiation is used to deduce that

$$a(v) = \frac{d}{dt}\left[v(s(t))\right] = \frac{dv(s)}{ds}\, \dot{s} = v\, \frac{dv(s)}{ds} \ . \tag{6.9}$$

Thus, the velocity v(s) can be determined by evaluating the integral

$$\int_{v_1}^{v(s)} \frac{VdV}{a(V)} = \int_{s_1}^{s} dS = (s - s_1) \ . \tag{6.10}$$

Finally, using (6.10) the differential equation (6.2c) yields

$$\frac{ds}{dt} = v(s) \ \Rightarrow \ \frac{ds}{v(s)} = dt \ , \tag{6.11a,b}$$

which may be integrated to obtain an equation for s(t) of the form

$$\int_{s_1}^{s(t)} \frac{dS}{v(S)} = \int_{t_1}^{t} d\tau = (t - t_1) \ . \tag{6.12}$$

Case 4: $a = a(s)$

The case when the acceleration is a function of position occurs often when springs are modeled. For this case we may multiply the differential equation (6.2e) by $\dot{s} = v$ to obtain

$$\dot{s}\,\frac{dv}{dt} = v\,\frac{dv}{dt} = a(s)\,\dot{s}\ , \tag{6.13}$$

which may be integrated using the initial conditions (6.3) to obtain

$$\int_{v_1}^{v(s)} V\, dV = \frac{1}{2}\ [v(s)^2 - v_1^2] = \int_{s_1}^{s} a(S)\, dS\ \ . \tag{6.14}$$

Thus, v(s) becomes

$$v(s) = \pm \left[v_1^2 + 2\int_{s_1}^{s} a(S)\, dS\ \right]^{1/2}. \tag{6.15}$$

Then, using (6.12) it is possible to determine s(t).

7. Polar Coordinates

By way of introduction to the description of general planar motion in terms of polar coordinates let us first consider circular planar motion. For this case the position vector may be expressed in the form

$$\mathbf{x} = x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2 \;, \tag{7.1}$$

where the rectangular Cartesian coordinates x_1, x_2 may be expressed in terms of the radius r of the circle and the angle θ by (see Fig. 7.1)

$$x_1 = r\cos\theta \;, \quad x_2 = r\sin\theta \;. \tag{7.2a,b}$$

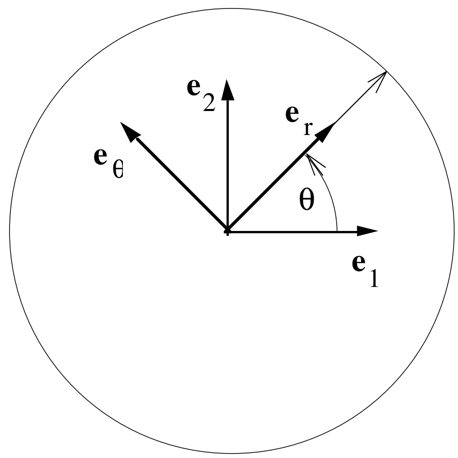

Fig. 7.1

Thus, the position vector may be expressed in the form

$$\mathbf{x} = r\,(\cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2) = r\,\mathbf{e}_r \;, \tag{7.3}$$

where $\mathbf{e}_r$ is the unit vector in the direction of the point of interest

$$\mathbf{e}_r = \mathbf{e}_r(\theta) = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2 \;, \quad \mathbf{e}_r \bullet \mathbf{e}_r = 1 \;. \tag{7.4a,b}$$

Furthermore, we define the unit vector $\mathbf{e}_\theta$ by

$$\mathbf{e}_\theta(\theta) = \frac{d\mathbf{e}_r}{d\theta} = -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2 \;, \quad \mathbf{e}_\theta \bullet \mathbf{e}_\theta = 1 \;, \tag{7.5a,b}$$

which points in the direction of increasing θ (see Fig. 7.1). Also, note that $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are orthogonal vectors

$$\mathbf{e}_r \bullet \mathbf{e}_\theta = 0 \;, \tag{7.6}$$

and that the derivative of $\mathbf{e}_\theta$ with respect to θ is related to $\mathbf{e}_r$ by the expression

$$\frac{d\mathbf{e}_\theta}{d\theta} = - \mathbf{e}_r \ . \tag{7.7}$$

It follows from (5.2) and the above definitions that for constant radius r the increment of arclength ds is related to the angle increment $d\theta$ by

$$(ds)^2 = d\mathbf{x} \bullet d\mathbf{x} = \frac{d\mathbf{x}}{d\theta} \bullet \frac{d\mathbf{x}}{d\theta} (d\theta)^2 = (r\,\mathbf{e}_\theta) \bullet (r\,\mathbf{e}_\theta)\,(d\theta)^2 \ , \tag{7.8a}$$

$$(ds)^2 = r^2\,(d\theta)^2 \ , \ \ ds = r\,d\theta \ . \tag{7.8b,c}$$

Furthermore, recall from (5.3) that the tangent vector is given by

$$\mathbf{e}_t = \frac{d\mathbf{x}}{ds} = \frac{d\mathbf{x}}{d\theta}\,\frac{d\theta}{ds} = (r\,\mathbf{e}_\theta)\,\frac{1}{r} = \mathbf{e}_\theta \ . \tag{7.9}$$

Thus, for circular motion $\mathbf{e}_\theta$ is tangent to the path. Also, using (5.5) we may determine that the normal vector $\mathbf{e}_n$ and the radius of curvature ρ are given by

$$\frac{d\mathbf{e}_t}{ds} = \frac{d\mathbf{e}_\theta}{d\theta}\,\frac{d\theta}{ds} = -\frac{1}{r}\,\mathbf{e}_r = \frac{1}{\rho}\,\mathbf{e}_n \ , \tag{7.10a}$$

$$\rho = r \ , \ \ \mathbf{e}_n = - \mathbf{e}_r \ . \tag{7.10b,c}$$

For general planar motion the position vector is a function of time parametrically through the polar coordinates $\{r,\theta\}$ and may be expressed in terms of the polar base vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ by

$$\mathbf{x} = \mathbf{x}(t) = r\,\mathbf{e}_r(\theta) \ , \ \ r = r(t) \ , \ \ \theta = \theta(t) \ . \tag{7.11a,b,c}$$

It is important to emphasize that unlike for rectangular Cartesian coordinates the position vector in polar coordinates is not equal to the sum of the coordinates times their associated base vectors. This is because the base vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are functions of the angular coordinate θ, so that $\mathbf{e}_r$ already is directed towards the point of interest. Now, differentiation of (7.1a) yields

$$\mathbf{v} = \dot{\mathbf{x}} = \dot{r}\,\mathbf{e}_r + r\,\dot{\mathbf{e}}_r \ . \tag{7.12}$$

Using the definition (7.5a) we may deduce that

$$\dot{\mathbf{e}}_r = \frac{d\mathbf{e}_r}{d\theta}\,\dot{\theta} = \dot{\theta}\,\mathbf{e}_\theta \;, \tag{7.13}$$

so the velocity becomes

$$\mathbf{v} = \dot{r}\;\mathbf{e}_r + r\,\dot{\theta}\,\mathbf{e}_\theta = v_r\,\mathbf{e}_r + v_\theta\,\mathbf{e}_\theta \;. \tag{7.14}$$

Recalling from (5.14a) that the velocity is always tangent to the particle path the tangent vector $\mathbf{e}_t$ may be determined by the equation

$$\mathbf{e}_t = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{v_r\,\mathbf{e}_r + v_\theta\,\mathbf{e}_\theta}{(v_r^2 + v_\theta^2)^{1/2}} \;. \tag{7.15}$$

This shows that for the general planar case $\mathbf{e}_\theta$ is not tangent to the path of motion. Next, differentiation of (7.14) yields the acceleration in the form

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{r}\;\mathbf{e}_r + \dot{r}\,\dot{\mathbf{e}}_r + \dot{r}\,\dot{\theta}\,\mathbf{e}_\theta + r\,\ddot{\theta}\;\mathbf{e}_\theta + + r\,\dot{\theta}\,\dot{\mathbf{e}}_\theta \,, \tag{7.16a}$$

$$= \ddot{r}\;\mathbf{e}_r + \dot{r}\,\dot{\theta}\,\mathbf{e}_\theta + \dot{r}\,\dot{\theta}\,\mathbf{e}_\theta + r\,\ddot{\theta}\;\mathbf{e}_\theta - r\,\dot{\theta}^2\,\mathbf{e}_r \;, \tag{7.16b}$$

$$= [\ddot{r} - r\,\dot{\theta}^2]\;\mathbf{e}_r + [r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}]\;\mathbf{e}_\theta \;, \tag{7.16c}$$

$$= [\ddot{r} - r\,\dot{\theta}^2]\;\mathbf{e}_r + \frac{1}{r}\,\frac{d(r^2\dot{\theta})}{dt}\;\mathbf{e}_\theta = a_r\,\mathbf{e}_r + a_\theta\,\mathbf{e}_\theta \;, \tag{7.16d}$$

The physical interpretation of the velocity components in (7.14) and the acceleration components in (7.16c) may be summarized as follows

$\dot{r}\,\mathbf{e}_r$	velocity due to changing length of the position vector
$r\,\dot{\theta}\,\mathbf{e}_\theta$	velocity due to changing direction of the position vector
$\ddot{r}\;\mathbf{e}_r$	acceleration due to changing radial velocity
$-\,r\,\dot{\theta}^2$	Centripetal acceleration: acceleration due to changing $\mathbf{e}_\theta$ direction
$r\,\ddot{\theta}\;\mathbf{e}_\theta$	acceleration due to changing angular speed $\dot{\theta}$
$2\,\dot{r}\,\dot{\theta}\,\mathbf{e}_\theta$	Corriolis acceleration due to motion in a rotating coordinate system

8. Cylindrical Polar Coordinates

For cylindrical polar coordinates the position vector $\mathbf{x}$ of a point is defined in terms of the three coordinates $\{r,\theta,x_3\}$ (see Fig. 8.1)

$$\mathbf{x} = r\,\mathbf{e}_r(\theta) + x_3\,\mathbf{e}_3 \ , \tag{8.1}$$

and all vectors are expressed in terms of the right-handed orthonormal set of base vectors $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_3\}$ defined by

$$\mathbf{e}_r(\theta) = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2 \ , \quad \frac{d\mathbf{e}_r}{d\theta} = \mathbf{e}_\theta \ , \tag{8.2a,b}$$

$$\mathbf{e}_\theta(\theta) = -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2 \ , \quad \frac{d\mathbf{e}_\theta}{d\theta} = -\mathbf{e}_r \ , \tag{8.2c,d}$$

$$\mathbf{e}_3 = \mathbf{e}_3 \ , \quad \frac{d\mathbf{e}_3}{d\theta} = 0 \ . \tag{8.2e,f}$$

Since $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_3\}$ are a right-handed orthonormal set of base vectors they satisfy the relations

$$\mathbf{e}_r \bullet \mathbf{e}_r = 1 \ , \ \mathbf{e}_\theta \bullet \mathbf{e}_\theta = 1 \ , \ \mathbf{e}_3 \bullet \mathbf{e}_3 = 1 \ , \tag{8.3a,b,c}$$

$$\mathbf{e}_r \bullet \mathbf{e}_\theta = 0 \ , \ \mathbf{e}_r \bullet \mathbf{e}_3 = 0 \ , \ \mathbf{e}_\theta \bullet \mathbf{e}_3 = 0 \ , \tag{8.3d,e,f}$$

$$\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{e}_3 \ , \ \mathbf{e}_3 \times \mathbf{e}_r = \mathbf{e}_\theta \ , \ \mathbf{e}_\theta \times \mathbf{e}_3 = \mathbf{e}_r \ . \tag{8.3g,h,i}$$

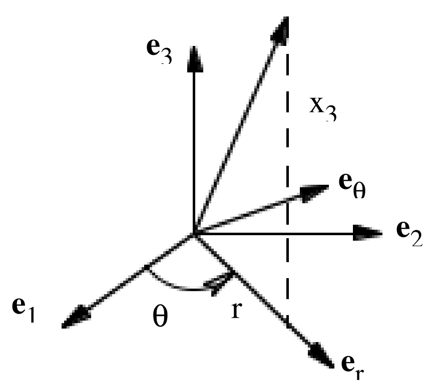

Fig. 8.1

Using the results of the previous section and the fact that $\mathbf{e}_3$ is a constant vector the velocity and acceleration may be calculated and the results are summarized below.

Summary of Cylindrical Polar Coordinates

coordinates	r, θ, x_3
base vectors	$\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_3$
derivatives of base vectors	$\dfrac{d\mathbf{e}_r}{d\theta} = \mathbf{e}_\theta \quad , \quad \dfrac{d\mathbf{e}_\theta}{d\theta} = -\,\mathbf{e}_r$
position vector	$\mathbf{x} = r\,\mathbf{e}_r(\theta) + x_3\,\mathbf{e}_3$
velocity	$\mathbf{v} = \dot{r}\,\mathbf{e}_r + r\,\dot{\theta}\,\mathbf{e}_\theta + \dot{x}_3\,\mathbf{e}_3$
acceleration	$[\ddot{r} - r\,\dot{\theta}^2]\,\mathbf{e}_r + \dfrac{1}{r}\,\dfrac{d(r^2\dot{\theta})}{dt}\,\mathbf{e}_\theta + \ddot{x}_3\,\mathbf{e}_3$

9. Relative Motion

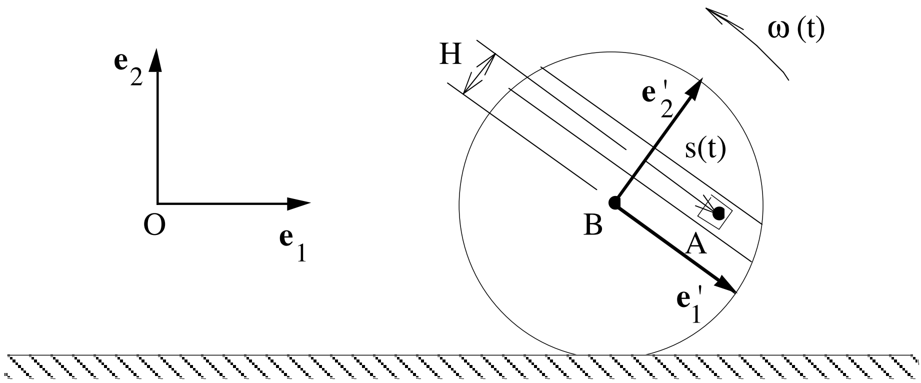

Fig. 9.1

By way of introduction to the topic of relative motion consider the example shown in Fig. 9.1 of a cylinder rolling on a flat surface with point A moving in a slot that rotates with the cylinder. It is quite difficult to describe the motion of point A relative to the fixed origin O directly by writing the position vector $\mathbf{x}(t)$ relative to $\mathbf{e}_1$ and $\mathbf{e}_2$. However, it is possible to describe this complicated motion by separating the description into smaller simpler parts. In separating the description of motion it is often convenient to use moving and rotating coordinate axes. It is important to emphasize that depending on how we separate the motion we can either simplify or complicate the kinematic description. Since this separation is not unique we will have to develop experience solving many problems in order to learn the advantages and disadvantages of different separations of motion.

With reference to Fig. 9.2 we can describe the general motion of point A in terms of its position $\mathbf{x}$, velocity $\mathbf{v}$, and acceleration $\mathbf{a}$, relative to the fixed origin O by separating the motion into the sum of the motion of point B, with position vector $\mathbf{X}$ and the motion of A relative to B, with the position vector $\mathbf{p}$. Thus, we have

$$\mathbf{x} = \mathbf{X} + \mathbf{p} \quad , \quad \mathbf{x} = \mathbf{x}_A \quad , \quad \mathbf{X} = \mathbf{x}_B \quad , \tag{9.1a,b,c}$$

$$\mathbf{v} = \dot{\mathbf{X}} + \dot{\mathbf{p}} \quad , \quad \mathbf{v} = \mathbf{v}_A \quad , \quad \dot{\mathbf{X}} = \mathbf{v}_B \quad , \tag{9.1d,e,f}$$

$$\mathbf{a} = \ddot{\mathbf{X}} + \ddot{\mathbf{p}} \quad , \quad \mathbf{a} = \mathbf{a}_A \quad , \quad \ddot{\mathbf{X}} = \mathbf{a}_B \quad . \tag{9.1g,h,i}$$

In (9.1) the subscripts A or B are used to denote quantities characterizing the motion of the points A and B. Also, the position vector $\mathbf{p}$, velocity $\dot{\mathbf{p}}$, and acceleration $\ddot{\mathbf{p}}$, describe the relative motion of point A relative to point B. Sometimes it is convenient to express these vectors in alternative forms that emphasize their relative nature

$$\mathbf{p} = \mathbf{x}_{A/B} = \mathbf{x}_A - \mathbf{x}_B \quad , \tag{9.2a}$$

$$\dot{\mathbf{p}} = \mathbf{v}_{A/B} = \mathbf{v}_A - \mathbf{v}_B \quad , \tag{9.2b}$$

$$\ddot{\mathbf{p}} = \mathbf{a}_{A/B} = \mathbf{a}_A - \mathbf{a}_B \quad , \tag{9.2c}$$

where the notation $\mathbf{x}_{A/B}$ denotes the position of A relative to B.

ABSOLUTE AND RELATIVE MOTION

Motion relative to a fixed point in space is called absolute motion whereas motion relative to a moving point is called relative motion. Thus, with reference to Fig. 9.2, the quantities $\mathbf{x}_A$, $\mathbf{v}_A$, $\mathbf{a}_A$ are called the absolute position, velocity, and acceleration, respectively, and the quantities $\mathbf{x}_{A/B}$, $\mathbf{v}_{A/B}$, $\mathbf{a}_{A/B}$ are called the relative position, velocity, and acceleration, respectively.

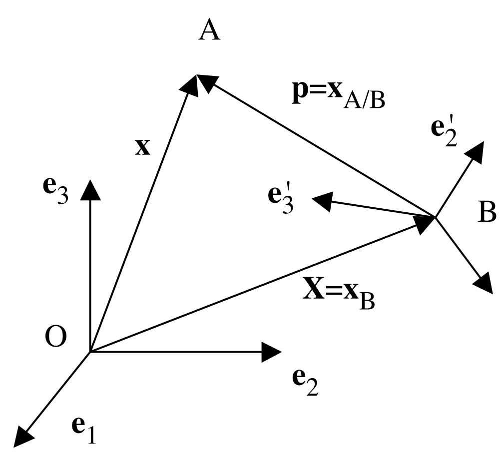

Fig. 9.2

10. Rotating Coordinate Axes and Angular Velocity

In the description of relative motion it is often convenient to use a rotating coordinate system like the one shown in Fig. 9.1. Here and throughout the text we let $\mathbf{e}_i$ be a fixed right-handed orthonormal set of coordinate axes and let $\mathbf{e}'_i$ be another right-handed orthonormal set of coordinate axes that is allowed to rotate in space. By way of introduction let us first consider the simple case where $\mathbf{e}'_i$ rotates about a fixed axis $\mathbf{e}'_3$, which for convenience is identified with $\mathbf{e}_3$ (see Fig. 10.1). Thus we take

$$\mathbf{e}'_1 = \cos\theta\, \mathbf{e}_1 + \sin\theta\, \mathbf{e}_2 \ , \tag{10.1a}$$

$$\mathbf{e}'_2 = -\sin\theta\, \mathbf{e}_1 + \cos\theta\, \mathbf{e}_2 \ , \tag{10.1b}$$

$$\mathbf{e}'_3 = \mathbf{e}_3 \ . \tag{10.1c}$$

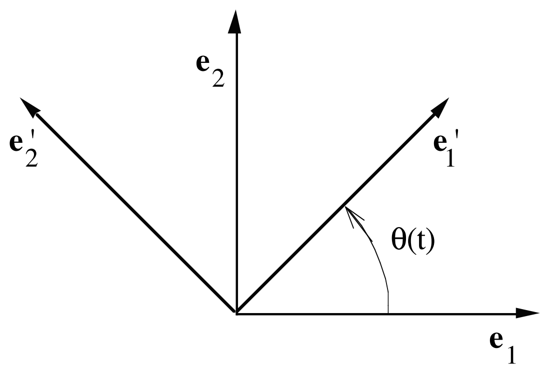

Fig. 10.1

Since $\mathbf{e}_i$ are fixed their derivatives vanish ($\dot{\mathbf{e}}_i$ =0) so differentiation of (10.1) yields

$$\dot{\mathbf{e}}'_1 = \dot{\theta}\ (-\sin\theta\, \mathbf{e}_1 + \cos\theta\, \mathbf{e}_2) = \dot{\theta}\, \mathbf{e}'_2 \ , \tag{10.2a}$$

$$\dot{\mathbf{e}}'_2 = \dot{\theta}\ (-\cos\theta\, \mathbf{e}_1 - \sin\theta\, \mathbf{e}_2) = -\dot{\theta}\, \mathbf{e}'_1 \ , \tag{10.2b}$$

$$\dot{\mathbf{e}}'_3 = 0 \ . \tag{10.2c}$$

Now from Fig. 10.1 we observe that the angle θ characterizes the rotation of the $\mathbf{e}'_i$ axes about the fixed $\mathbf{e}'_3$ axis so that $\dot{\theta}$ characterizes the angular velocity. In this regard it is important to emphasize that the origin of the rotating coordinate axes $\mathbf{e}'_i$ can move without changing the description (10.2). Furthermore, by introducing the angular velocity vector $\boldsymbol{\omega}$ defined by

$$\boldsymbol{\omega} = \dot{\theta}\, \mathbf{e}'_3 \ , \tag{10.3}$$

we can conveniently rewrite equations (10.2) in the compact form

$$\dot{\mathbf{e}}'_i = \boldsymbol{\omega} \times \mathbf{e}'_i \ , \tag{10.4}$$

which explicitly states that $\boldsymbol{\omega}$ is the angular velocity of the $\mathbf{e}'_i$ coordinate axes.

In the above we have proved equation (10.4) for the special case of rotation about a fixed axis in space. However, it can be shown that (10.4) holds even if the angular velocity $\boldsymbol{\omega}(t)$ is a function of times whose magnitude and direction change. To prove this we recall that since $\mathbf{e}'_i$ form an orthonormal set of axes they satisfy the conditions

$$\mathbf{e}'_i \bullet \mathbf{e}'_j = \delta_{ij} \ . \tag{10.5}$$

In view of the fact that (10.5) is symmetric in the indices (i,j) these equations represent six constraints on the nine scalar quantities that characterize the three vectors $\mathbf{e}'_i$. Thus, the coordinate axes $\mathbf{e}'_i$ have only three degrees of freedom, which corespond to three independent rotations. To show that (10.4) is consistent with the constraints (10.5) we differentiate (10.5) to obtain

$$\dot{\mathbf{e}}'_i \bullet \mathbf{e}'_j + \mathbf{e}'_i \bullet \dot{\mathbf{e}}'_j = 0 \ . \tag{10.6}$$

Next, substitution of (10.4) into (10.6) yields

$$\begin{aligned} \dot{\mathbf{e}}'_i \bullet \mathbf{e}'_j + \mathbf{e}'_i \bullet \dot{\mathbf{e}}'_j &= (\boldsymbol{\omega} \times \mathbf{e}'_i) \bullet \mathbf{e}'_j + \mathbf{e}'_i \bullet (\boldsymbol{\omega} \times \mathbf{e}'_j) = \boldsymbol{\omega} \bullet (\mathbf{e}'_i \times \mathbf{e}'_j) + (\mathbf{e}'_j \times \mathbf{e}'_i) \bullet \boldsymbol{\omega} \ , \\ &= \boldsymbol{\omega} \bullet [\mathbf{e}'_i \times \mathbf{e}'_j + \mathbf{e}'_j \times \mathbf{e}'_i] = 0 \ . \end{aligned} \tag{10.7}$$

This means that the differential equations (10.4) satisfy the differential form (10.6) of the constraint (10.5) so that the vectors $\mathbf{e}'_i$ calculated by integrating (10.4) for arbitrary $\boldsymbol{\omega}$ will remain an orthonormal set of vectors. Thus, the components $\omega'_i = \boldsymbol{\omega} \bullet \mathbf{e}'_i$ of the angular velocity $\boldsymbol{\omega}$ represent the rates of rotation of the rotating coordinate axes about the each of the axes $\mathbf{e}'_i$, respectively. Finally, we represent $\boldsymbol{\omega}$ in terms of its magnitude ω and direction $\mathbf{e}_\omega$

$$\boldsymbol{\omega} = \omega\, \mathbf{e}_\omega \ , \ \mathbf{e}_\omega \bullet \mathbf{e}_\omega = 1 \ , \tag{10.8a,b}$$

and note that the sign convention is chosen so that positive values of ω indicate counter-clockwise rotation about the positive $\mathbf{e}_\omega$ axes.

As an example we can reconsider the cylindrical polar coordinate axes shown in Fig. 10.2 and take

$$\mathbf{e}'_1 = \mathbf{e}_r \ , \ \mathbf{e}'_2 = \mathbf{e}_\theta \ , \mathbf{e}'_3 = \mathbf{e}_3 \ . \tag{10.9a,b,c}$$

Noting that the angular velocity $\boldsymbol{\omega}$ is given by

$$\boldsymbol{\omega} = \dot{\theta}\, \mathbf{e}_3 \ , \tag{10.10}$$

the derivatives of the vectors $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_3$ may be calculated using the formula (10.4) to obtain

$$\dot{\mathbf{e}}_r = \boldsymbol{\omega} \times \mathbf{e}_r = \dot{\theta}\, \mathbf{e}_\theta \ , \tag{10.11a}$$

$$\dot{\mathbf{e}}_\theta = \boldsymbol{\omega} \times \mathbf{e}_\theta = -\dot{\theta}\, \mathbf{e}_r \ , \tag{10.11b}$$

$$\dot{\mathbf{e}}_3 = \boldsymbol{\omega} \times \mathbf{e}_3 = 0 \ . \tag{10.11c}$$

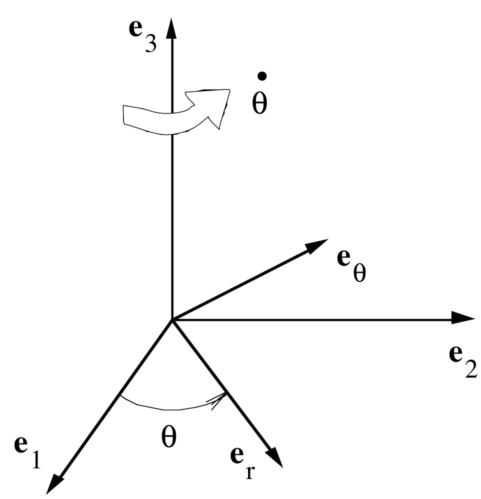

Fig. 10.2

11. General Differential Operator

Returning to our discussion of relative motion we note that vectors can be referred to any complete set of base vectors. Thus, with reference to Fig. 9.2 the vector **p**, which describes the position of point A relative to point B, may be represented in terms of the base vectors $\mathbf{e}'_i$ and its components p'_i relative to $\mathbf{e}'_i$ such that

$$\mathbf{p} = p'_i \, \mathbf{e}'_i \ . \tag{11.1}$$

Since in dynamics we are interested in the time rate of change of vectors it is important to emphasize that whenever we introduce a set of base vectors like $\mathbf{e}'_i$ we must also define the angular velocity $\boldsymbol{\omega}$ with which the base vectors are rotating. To emphasize this we write

$$\dot{\mathbf{e}}'_i = \boldsymbol{\omega} \times \mathbf{e}'_i \ , \tag{11.2}$$

which explicitly indicates that $\boldsymbol{\omega}$ is the angular velocity of the base vectors $\mathbf{e}'_i$. This is particularly important when we use more than one set of rotating base vectors so that more than one angular velocity is used.

Now, differentiation of (11.1) and use of (11.2) yields

$$\dot{\mathbf{p}} = \dot{p}'_i \, \mathbf{e}'_i + p'_i \, \dot{\mathbf{e}}'_i \ , \tag{11.3a}$$

$$\dot{\mathbf{p}} = \dot{p}'_i \, \mathbf{e}'_i + \boldsymbol{\omega} \times (p'_i \, \mathbf{e}'_i) \ . \tag{11.3b}$$

Thus, the derivative of **p** naturally separates into two parts

$$\dot{\mathbf{p}} = \frac{\delta \mathbf{p}}{\delta t} + \boldsymbol{\omega} \times \mathbf{p} \ , \tag{11.4}$$

where the operator $\delta(\ )/\delta t$ is defined as the frame derivative by

$$\frac{\delta \mathbf{p}}{\delta t} = \dot{p}'_i \, \mathbf{e}'_i \ . \tag{11.5}$$

The physical interpretation of these terms may be explained as follows:

$\dfrac{\delta \mathbf{p}}{\delta t} = \dot{p}'_i \, \mathbf{e}'_i$ The frame derivative of $\mathbf{p}$ is the rate of change of the vector $\mathbf{p}$ measured relative to $\mathbf{e}'_i$ assuming that $\mathbf{e}'_i$ do not rotate.

$\boldsymbol{\omega} \times \mathbf{p}$ The rate of change of the vector $\mathbf{p}$ due to the rotation of the coordinate axes $\mathbf{e}'_i$.

The differential operator (11.4) is sometimes called the general operator because it is valid even if the coordinate system is rotating. It is important to emphasize that the angular velocity $\boldsymbol{\omega}$ that appears in (11.4) characterizes the rate of rotation of the same coordinate system in which $\mathbf{p}$ is represented. For example, if we were to consider a second coordinate system with base vectors $\mathbf{e}''_i$ which rotate with angular velocity $\boldsymbol{\Omega}$

$$\dot{\mathbf{e}}''_i = \boldsymbol{\Omega} \times \mathbf{e}''_i \ , \tag{11.6}$$

Then the general operator (11.4) would take the form

$$\dot{\mathbf{p}} = \frac{\delta \mathbf{p}}{\delta t} + \boldsymbol{\Omega} \times \mathbf{p} \ , \quad \frac{\delta \mathbf{p}}{\delta t} = \dot{p}''_i \, \mathbf{e}''_i \ . \tag{11.7a,b}$$

Recalling that the vector product $\boldsymbol{\omega} \times \mathbf{p}$ may be calculated using the determinant

$$\boldsymbol{\omega} \times \mathbf{p} = \begin{vmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3 \\ \omega'_1 & \omega'_2 & \omega'_3 \\ p'_1 & p'_2 & p'_3 \end{vmatrix} \ , \tag{11.8}$$

it is very convenient to calculate the derivative of a vector referred to a rotating coordinate system by writing the following table.

	$\mathbf{e}_1'$	$\mathbf{e}_2'$	$\mathbf{e}_3'$
$\boldsymbol{\omega}$	ω_1'	ω_2'	ω_3'
$\mathbf{p}$	p_1'	p_2'	p_3'
$\dfrac{\delta \mathbf{p}}{\delta t}$	$\dot{p}_1'$	$\dot{p}_2'$	$\dot{p}_3'$
$\boldsymbol{\omega} \times \mathbf{p}$	$\omega_2' p_3' - \omega_3' p_2'$	$-\omega_1' p_3' + \omega_3' p_1'$	$\omega_1' p_2' - \omega_2' p_1'$
$\dot{\mathbf{p}}$	$\dot{p}_1' + \omega_2' p_3' - \omega_3' p_2'$	$\dot{p}_2' - \omega_1' p_3' + \omega_3' p_1'$	$\dot{p}_3' + \omega_1' p_2' - \omega_2' p_1'$

Using the general operator (11.4) we can calculate the derivative of any vector. For example we can calculate the relative acceleration $\ddot{\mathbf{p}}$ in the form

$$\ddot{\mathbf{p}} = \frac{\delta \dot{\mathbf{p}}}{\delta t} + \boldsymbol{\omega} \times \dot{\mathbf{p}} \ , \tag{11.9a}$$

$$\ddot{\mathbf{p}} = \frac{\delta}{\delta t}\left[\frac{\delta \mathbf{p}}{\delta t} + \boldsymbol{\omega} \times \mathbf{p}\right] + \boldsymbol{\omega} \times \left[\frac{\delta \mathbf{p}}{\delta t} + \boldsymbol{\omega} \times \mathbf{p}\right] \ , \tag{11.9b}$$

$$\ddot{\mathbf{p}} = \frac{\delta^2 \mathbf{p}}{\delta t^2} + \frac{\delta \boldsymbol{\omega}}{\delta t} \times \mathbf{p} + 2\,\boldsymbol{\omega} \times \frac{\delta \mathbf{p}}{\delta t} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{p}) \ . \tag{11.9c}$$

However, using the general operator (11.4) the angular acceleration $\dot{\boldsymbol{\omega}}$ is given by

$$\dot{\boldsymbol{\omega}} = \frac{\delta \boldsymbol{\omega}}{\delta t} + \boldsymbol{\omega} \times \boldsymbol{\omega} = \frac{\delta \boldsymbol{\omega}}{\delta t} \ , \tag{11.10}$$

so the relative acceleration (11.9c) becomes

$$\ddot{\mathbf{p}} = \frac{\delta^2 \mathbf{p}}{\delta t^2} + \dot{\boldsymbol{\omega}} \times \mathbf{p} + 2\,\boldsymbol{\omega} \times \frac{\delta \mathbf{p}}{\delta t} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{p}) \ . \tag{11.11}$$

The physical interpretation of these terms may be explained as follows:

$\dfrac{\delta^2 \mathbf{p}}{\delta t^2}$	The acceleration as measured relative to $\mathbf{e}_i^{\prime}$ assuming that $\mathbf{e}_i^{\prime}$ <u>do not</u> rotate.
$\dot{\boldsymbol{\omega}} \times \mathbf{p}$	The acceleration due to the angular acceleration $\dot{\boldsymbol{\omega}}$ of the coordinate axes $\mathbf{e}_i^{\prime}$.
$2\,\boldsymbol{\omega} \times \dfrac{\delta \mathbf{p}}{\delta t}$	The Corriolis acceleration
$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{p})$	The Centripetal acceleration

12. Spherical Polar Coordinates

For spherical polar coordinates the position vector $\mathbf{x}$ of a point is defined in terms of the three coordinates $\{R,\theta,\phi\}$ (see Fig. 12.1)

$$\mathbf{x} = R\,\mathbf{e}_R(\theta,\phi) \quad , \tag{12.1}$$

and all vectors are expressed in terms of the right-handed orthonormal set of base vectors $\{\mathbf{e}_R,\mathbf{e}_\theta,\mathbf{e}_\phi\}$ defined in terms of the cylindrical polar base vectors $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_3\}$by

$$\mathbf{e}_R(\theta,\phi) = \cos\phi\;\mathbf{e}_r + \sin\phi\;\mathbf{e}_3 \;\; , \tag{12.2a}$$

$$\mathbf{e}_\theta(\theta) = -\sin\theta\;\mathbf{e}_1 + \cos\theta\;\mathbf{e}_2 \;\; , \tag{12.2b}$$

$$\mathbf{e}_\phi(\theta,\phi) = -\sin\phi\;\mathbf{e}_r + \cos\phi\;\mathbf{e}_3 \;\; , \tag{12.2c}$$

Since $\{\mathbf{e}_R,\mathbf{e}_\theta,\mathbf{e}_\phi\}$ are a right-handed orthonormal set of base vectors they satisfy the relations

$$\mathbf{e}_R \bullet \mathbf{e}_R = 1 \;\; , \;\; \mathbf{e}_\theta \bullet \mathbf{e}_\theta = 1 \;\; , \;\; \mathbf{e}_\phi \bullet \mathbf{e}_\phi = 1 \;\; , \tag{12.3a,b,c}$$

$$\mathbf{e}_R \bullet \mathbf{e}_\theta = 0 \;\; , \;\; \mathbf{e}_R \bullet \mathbf{e}_\phi = 0 \;\; , \;\; \mathbf{e}_\theta \bullet \mathbf{e}_\phi = 0 \;\; , \tag{12.3d,e,f}$$

$$\mathbf{e}_R \times \mathbf{e}_\theta = \mathbf{e}_\phi \;\; , \;\; \mathbf{e}_\phi \times \mathbf{e}_R = \mathbf{e}_\theta \;\; , \quad \mathbf{e}_\theta \times \mathbf{e}_\phi = \mathbf{e}_R \;\; . \tag{12.3g,h,i}$$

In order to calculate derivatives of vectors expressed in spherical coordinates it is necessary to calculate derivatives of the base vectors. This can be done directly by deriving the formulas

$$\frac{\partial \mathbf{e}_R}{\partial \theta} = \cos\phi\;\mathbf{e}_\theta \;\; , \;\; \frac{\partial \mathbf{e}_R}{\partial \phi} = \mathbf{e}_\phi \;\; , \tag{12.4a,b}$$

$$\frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\cos\phi\;\mathbf{e}_R + \sin\phi\;\mathbf{e}_\phi \;\; , \;\; \frac{\partial \mathbf{e}_\theta}{\partial \phi} = 0 \;\; , \tag{12.4c,d}$$

$$\frac{\partial \mathbf{e}_\phi}{\partial \theta} = -\sin\phi\;\mathbf{e}_\theta \;\; , \;\; \frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\mathbf{e}_R \;\; , \tag{12.4e,f}$$

and using the chain rule of differentiation. Alternatively we may take

$$\mathbf{e}_1' = \mathbf{e}_R \;\; , \;\; \mathbf{e}_2' = \mathbf{e}_\theta \;\; , \mathbf{e}_3' = \mathbf{e}_\phi \;\; , \tag{12.5a,b,c}$$

and write the angular velocity $\boldsymbol{\omega}$ in the form (see Fig. 12.1)

$$\boldsymbol{\omega} = \dot{\theta}\;\mathbf{e}_3 - \dot{\phi}\;\mathbf{e}_\theta \;\; . \tag{12.6}$$

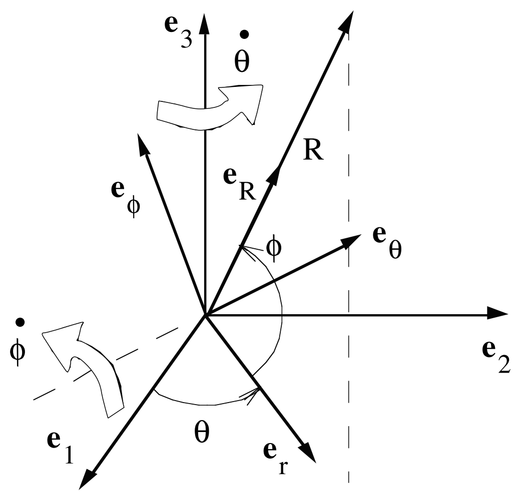

Fig. 12.1

Notice that in (12.6) $\dot{\theta}$ represents the angular velocity about the $\mathbf{e}_3$ axis and $\dot{\phi}$ represents the angular velocity about the $(-\mathbf{e}_\theta)$ axis. Furthermore, using (12.2) we can express the vector $\mathbf{e}_3$ in terms of the spherical base vectors such that

$$\mathbf{e}_3 = \sin\phi\ \mathbf{e}_R + \cos\phi\ \mathbf{e}_\phi\ , \tag{12.7}$$

so the angular velocity $\boldsymbol{\omega}$ becomes

$$\boldsymbol{\omega} = \dot{\theta}\ \sin\phi\ \mathbf{e}_R - \dot{\phi}\ \mathbf{e}_\theta + \dot{\theta}\ \cos\phi\ \mathbf{e}_\phi\ \ . \tag{12.8}$$

Now the rate of change of the base vectors may be calculated by

$$\dot{\mathbf{e}}_R = \boldsymbol{\omega} \times \mathbf{e}_R = \dot{\theta}\ \cos\phi\ \mathbf{e}_\theta + \dot{\phi}\ \mathbf{e}_\phi\ \ , \tag{12.9a}$$

$$\dot{\mathbf{e}}_\theta = \boldsymbol{\omega} \times \mathbf{e}_\theta = -\ \dot{\theta}\ \cos\phi\ \mathbf{e}_R + \dot{\theta}\ \sin\phi\ \mathbf{e}_\phi\ \ , \tag{12.9b}$$

$$\dot{\mathbf{e}}_\phi = \boldsymbol{\omega} \times \mathbf{e}_\phi = -\ \dot{\phi}\ \mathbf{e}_R - \dot{\theta}\ \sin\phi\ \mathbf{e}_\theta\ \ . \tag{12.9c}$$

Using the procedure described in the last section we can calculate the velocity and acceleration in spherical coordinates and the results are summarized in the following table.

Spherical Polar Coordinates

	$\mathbf{e}_R$	$\mathbf{e}_\theta$	$\mathbf{e}_\phi$
$\boldsymbol{\omega}$	$\dot\theta \sin\phi$	$-\dot\phi$	$\dot\theta \cos\phi$
$\mathbf{x}$	R	0	0
$\frac{\delta \mathbf{x}}{\delta t}$	$\dot R$	0	0
$\boldsymbol{\omega} \times \mathbf{x}$	0	$R\,\dot\theta \cos\phi$	$R\,\dot\phi$
$\mathbf{v}$	$\dot R$	$R\,\dot\theta \cos\phi$	$R\,\dot\phi$
$\frac{\delta \mathbf{v}}{\delta t}$	$\ddot R$	$\dot R\,\dot\theta \cos\phi$ $+ R\,\ddot\theta \cos\phi$ $- R\,\dot\theta\,\dot\phi \sin\phi$	$\dot R\,\dot\phi$ $+ R\,\ddot\phi$
$\boldsymbol{\omega} \times \mathbf{v}$	$-R\,\dot\phi^2$ $- R\,\dot\theta^2 \cos^2\phi$	$- R\,\dot\phi\,\dot\theta \sin\phi$ $+\dot R\,\dot\theta \cos\phi$	$R\,\dot\theta^2 \cos\phi \sin\phi$ $+ \dot R\,\dot\phi$
$\mathbf{a}$	$\ddot R$ $-R\,\dot\phi^2$ $- R\,\dot\theta^2 \cos^2\phi$	$R\,\ddot\theta \cos\phi$ $+ 2\,\dot R\,\dot\theta \cos\phi$ $- 2R\,\dot\theta\,\dot\phi \sin\phi$	$R\,\ddot\phi$ $+ R\,\dot\theta^2 \sin\phi \cos\phi$ $+ 2\,\dot R\,\dot\phi$

13. General Rigid Body Motion

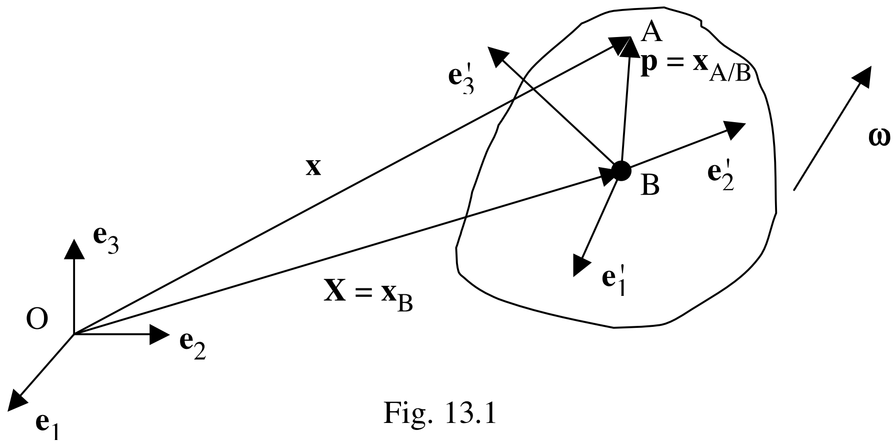

Fig. 13.1

A body is said to be rigid if the distance between <u>any</u> two points remains constant. Letting A and B be two points on the rigid body (see Fig. 13.1) we have

$$|\,\mathbf{x}_{A/B}\,|^2 = \mathbf{x}_{A/B} \bullet \mathbf{x}_{A/B} = \text{constant} \quad . \tag{13.1}$$

It follows that the angle θ between any two material lines on the rigid body remains constant (see Fig. 13.2) so that we can attach a coordinate system $\mathbf{e}_i'$ to the body that will remain orthonormal. This coordinate system is called a <u>body coordinate system</u>. Since the coordinate system $\mathbf{e}_i'$ is attached to the body its angular velocity $\boldsymbol{\omega}$

$$\dot{\mathbf{e}}_i' = \boldsymbol{\omega} \times \mathbf{e}_i' \,, \tag{13.2}$$

is the same as the angular velocity of the rigid body. Now with reference to Fig. 13.1 it is apparent that a rigid body has <u>6 degrees of freedom</u>: 3 translational degrees of freedom characterized by $\mathbf{x}_B$; and 3 rotational degrees of freedom characterized by $\boldsymbol{\omega}$.

The relative velocity between two points A and B on a rigid body may be determined by referring the relative position vector to the body coordinate system

$$\mathbf{x}_{A/B} = \mathbf{p} = p_i' \, \mathbf{e}_i' \quad . \tag{13.3}$$

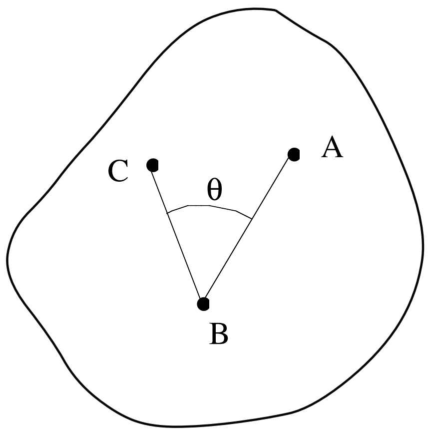

Fig. 13.2

Then, use of the general operator (11.4) and the expression (13.2) we have

$$\mathbf{v}_{A/B} = \dot{\mathbf{p}} = \frac{\delta \mathbf{p}}{\delta t} + \boldsymbol{\omega} \times \mathbf{p} \ . \tag{13.4}$$

However, since A and B lie on the rigid body and $\mathbf{e}_i'$ is a body coordinate system the coordinates p_i' are constant so that $\delta\mathbf{p}/\delta t$ vanishes

$$\frac{\delta \mathbf{p}}{\delta t} = \frac{\delta \mathbf{x}_{A/B}}{\delta t} = \dot{p}_i' \, \mathbf{e}_i' = 0 \ , \tag{13.5}$$

and (13.4) reduces to

$$\mathbf{v}_{A/B} = \boldsymbol{\omega} \times \mathbf{x}_{A/B} \ . \tag{13.6}$$

Note that this means that the relative velocity $\mathbf{v}_{A/B}$ is perpendicular to the relative position vector $\mathbf{x}_{A/B}$ so that

$$\mathbf{v}_{A/B} \bullet \mathbf{x}_{A/B} = 0 \ . \tag{13.7}$$

Also, note that the result (13.7) is consistent with the basic definition of a rigid body because it can be obtained by differentiating (13.1).

Furthermore, it follows from (13.6) that the velocity of a general point A may be expressed in the form

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{x}_{A/B} \ . \tag{13.8}$$

Then, the acceleration of point A can be determined by differentiating (13.8) and using (13.6) to obtain

$$\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\omega}} \times \mathbf{x}_{A/B} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x}_{A/B}) \ . \tag{13.9}$$

Next, we develop a formula to determine the angular velocity $\boldsymbol{\omega}$ of a rigid body using the velocity of three points A,B,C on the rigid body. Specifically, using (13.6) the relative velocity $\mathbf{v}_{C/B}$ becomes

$$\mathbf{v}_{C/B} = \boldsymbol{\omega} \times \mathbf{x}_{C/B} \ , \tag{13.10}$$

so that $\mathbf{v}_{A/B}$ and $\mathbf{v}_{C/B}$ are both perpendicular to the angular velocity $\boldsymbol{\omega}$. This means that the vector $\mathbf{v}_{A/B} \times \mathbf{v}_{C/B}$ is parallel to $\boldsymbol{\omega}$. Consequently, with the help of (13.10) we have

$$\mathbf{v}_{A/B} \times \mathbf{v}_{C/B} = \mathbf{v}_{A/B} \times (\boldsymbol{\omega} \times \mathbf{x}_{C/B}) \ . \tag{13.11}$$

However, the vector triple product may be expanded with the help of (2.16) to obtain

$$\mathbf{v}_{A/B} \times \mathbf{v}_{C/B} = (\mathbf{v}_{A/B} \bullet \mathbf{x}_{C/B}) \, \boldsymbol{\omega} - (\mathbf{v}_{A/B} \bullet \boldsymbol{\omega}) \, \mathbf{x}_{C/B} \ . \tag{13.12}$$

Since $\mathbf{v}_{A/B}$ is perpendicular to $\boldsymbol{\omega}$ we can solve (13.12) for $\boldsymbol{\omega}$ whenever $(\mathbf{v}_{A/B} \bullet \mathbf{x}_{C/B})$ does not vanish

$$\boldsymbol{\omega} = \frac{\mathbf{v}_{A/B} \times \mathbf{v}_{C/B}}{\mathbf{v}_{A/B} \bullet \mathbf{x}_{C/B}} \ . \tag{13.13}$$

Finally, we note that

$$\mathbf{v}_{A/B} \bullet \mathbf{x}_{C/B} = (\boldsymbol{\omega} \times \mathbf{x}_{A/B}) \bullet \mathbf{x}_{C/B} = \boldsymbol{\omega} \bullet (\mathbf{x}_{A/B} \times \mathbf{x}_{C/B}) \ . \tag{13.14}$$

Thus, in order for $(\mathbf{v}_{A/B} \bullet \mathbf{x}_{C/B})$ to be nonzero $\boldsymbol{\omega}$ cannot lie in the plane of $\mathbf{x}_{A/B}$ and $\mathbf{x}_{C/B}$, and $(\mathbf{x}_{A/B} \times \mathbf{x}_{C/B})$ cannot vanish, which means that the points A,B,C cannot lie on the same line.

If $\boldsymbol{\omega}$ lies in the plane of $\mathbf{x}_{A/B}$ and $\mathbf{x}_{C/B}$ then

$$\boldsymbol{\omega} = A \, \mathbf{x}_{A/B} + B \, \mathbf{x}_{C/B} \ , \tag{13.15}$$

However,

$$\mathbf{v}_{A/B} = \boldsymbol{\omega} \times \mathbf{x}_{A/B} = -B \, \mathbf{x}_{A/B} \times \mathbf{x}_{C/B} \ , \ \mathbf{v}_{C/B} = \boldsymbol{\omega} \times \mathbf{x}_{C/B} = A \, \mathbf{x}_{A/B} \times \mathbf{x}_{C/B} \ , \tag{13.16}$$

which yields

$$\boldsymbol{\omega} = \left[\frac{\mathbf{v}_{C/B} \bullet \mathbf{v}_{C/B}}{\mathbf{v}_{C/B} \bullet (\mathbf{x}_{A/B} \times \mathbf{x}_{C/B})}\right] \mathbf{x}_{A/B} - \left[\frac{\mathbf{v}_{A/B} \bullet \mathbf{v}_{A/B}}{\mathbf{v}_{A/B} \bullet (\mathbf{x}_{A/B} \times \mathbf{x}_{C/B})}\right] \mathbf{x}_{C/B} \ . \tag{13.17}$$

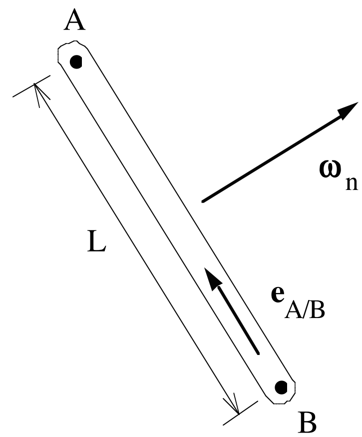

Fig. 13.3

If we only have knowledge of the velocity of two points A and B on a rigid body (such as the velocity of the end points of a rigid bar; Fig. 13.3) then we cannot determine all components of the angular velocity. In particular we cannot determine the component of the angular velocity in the direction of the relative position vector $\mathbf{x}_{A/B}$. Letting L be the length of the vector $\mathbf{x}_{A/B}$ and $\mathbf{e}_{A/B}$ be the unit vector directed from B to A

$$\mathbf{x}_{A/B} = L\, \mathbf{e}_{A/B} \ , \ \mathbf{e}_{A/B} \bullet \mathbf{e}_{A/B} = 1 \ , \tag{13.18a,b}$$

and taking the vector product of (13.6) with $\mathbf{x}_{A/B}$ we may deduce that

$$\begin{aligned} \mathbf{x}_{A/B} \times \mathbf{v}_{A/B} = \mathbf{x}_{A/B} \times (\boldsymbol{\omega} \times \mathbf{x}_{A/B}) &= (\mathbf{x}_{A/B} \bullet \mathbf{x}_{A/B})\, \boldsymbol{\omega} - (\mathbf{x}_{A/B} \bullet \boldsymbol{\omega})\, \mathbf{x}_{A/B} \\ &= L^2 \, [\boldsymbol{\omega} - (\mathbf{e}_{A/B} \bullet \boldsymbol{\omega})\, \mathbf{e}_{A/B}] \ . \end{aligned} \tag{13.19}$$

Thus, the normal component $\boldsymbol{\omega}_n$ of the angular momentum $\boldsymbol{\omega}$ of the bar is given by

$$\boldsymbol{\omega}_n = \boldsymbol{\omega} - (\boldsymbol{\omega} \bullet \mathbf{e}_{A/B})\, \mathbf{e}_{A/B} = \frac{\mathbf{x}_{A/B} \times \mathbf{v}_{A/B}}{L^2} \ . \tag{13.20}$$

14. Instantaneous Screw Motion Of A Rigid Body

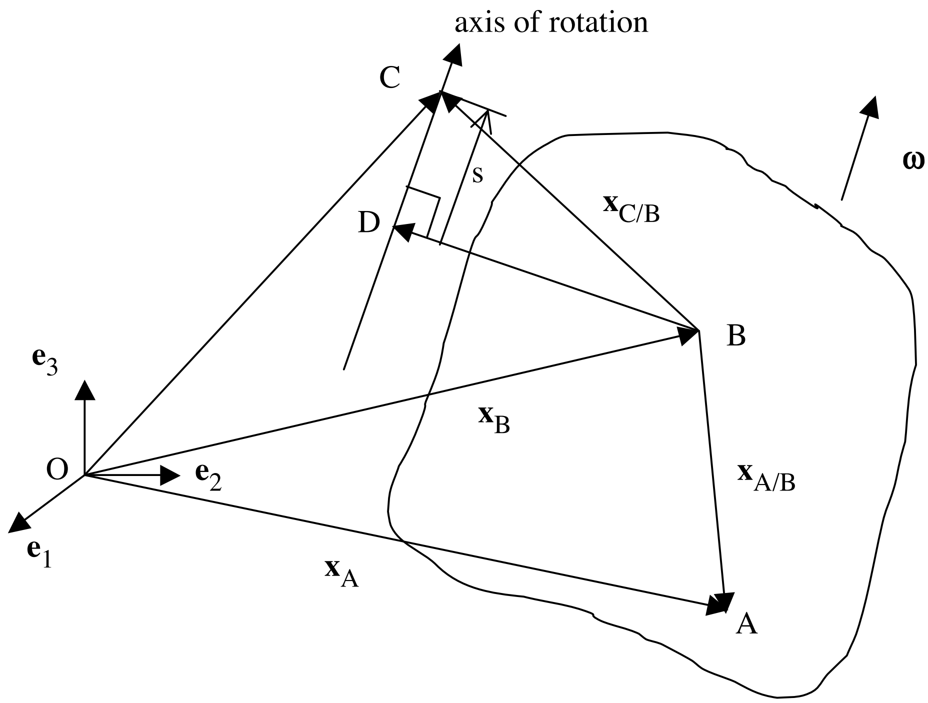

Fig. 14.1

In this section we show that the general motion of a rigid body can be described by the motion of a screw with the rigid body rotating about an axis in space and translating parallel to this axis. This screw motion is considered to be instantaneous in the sense that the axis of rotation and velocity of translation can change with time.

For convenience we express the angular velocity $\boldsymbol{\omega}$ of the rigid body in terms of its magnitude ω and direction $\mathbf{e}_\omega$ by

$$\boldsymbol{\omega} = \omega\, \mathbf{e}_\omega \quad , \quad \mathbf{e}_\omega \bullet \mathbf{e}_\omega = 1 \quad , \tag{14.1a,b}$$

and recall from (13.8) that the velocity of a general point A on a rigid body may be expressed in terms of the velocity $\mathbf{v}_B$ of another point on the rigid body by the formula

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{x}_{A/B} \quad . \tag{14.2}$$

Taking the inner product of (14.2) with the vector $\mathbf{e}_\omega$ we may deduce that

$$\mathbf{v}_A \bullet \mathbf{e}_\omega = \mathbf{v}_B \bullet \mathbf{e}_\omega \quad . \tag{14.3}$$

This means that all points have the same component of velocity in the $\mathbf{e}_\omega$ direction. In other words, the body is advancing in the $\mathbf{e}_\omega$ direction with uniform velocity. Furthermore, since $\mathbf{v}_A$ is not necessarily parallel to $\mathbf{e}_\omega$ we realize that the body is also rotating about some axis parallel to $\mathbf{e}_\omega$. To find this axis of rotation, let $\mathbf{x}_C$ locate an arbitrary point on this axis of rotation and note from (14.3) that all points on this axis have the same absolute velocity which is parallel to $\mathbf{e}_\omega$ so that

$$\mathbf{v}_C = (\mathbf{v}_B \bullet \mathbf{e}_\omega)\,\mathbf{e}_\omega \;. \tag{14.4}$$

However, since all points C are attached to the same rigid body we may write

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{x}_{C/B} \;. \tag{14.5}$$

Using the fact that $\boldsymbol{\omega}$ is parallel to $\mathbf{v}_C$ it follows that

$$0 = \boldsymbol{\omega} \times \mathbf{v}_C = \boldsymbol{\omega} \times \mathbf{v}_B + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x}_{C/B})$$

$$0 = \boldsymbol{\omega} \times \mathbf{v}_B + (\boldsymbol{\omega} \bullet \mathbf{x}_{C/B})\,\boldsymbol{\omega} - (\boldsymbol{\omega} \bullet \boldsymbol{\omega})\,\mathbf{x}_{C/B} \;, \tag{14.6}$$

so that the relative position vector $\mathbf{x}_{C/B}$ may be written in the form

$$\mathbf{x}_{C/B} = \frac{\boldsymbol{\omega} \times \mathbf{v}_B}{\omega^2} + \frac{(\boldsymbol{\omega} \bullet \mathbf{x}_{C/B})\,\boldsymbol{\omega}}{\omega^2} \;, \tag{14.7a}$$

$$\mathbf{x}_{C/B} = \mathbf{x}_{D/C} + s\,\mathbf{e}_\omega \;,\; \mathbf{x}_{D/C} = \frac{\boldsymbol{\omega} \times \mathbf{v}_B}{\omega^2} \;,\; s = \mathbf{x}_{C/B} \bullet \mathbf{e}_\omega \;. \tag{14.7b,c}$$

Notice that the vector $\mathbf{x}_{D/C}$ is perpendicular to $\mathbf{e}_\omega$ so it locates the point D on the axis of rotation that is closest to the point B (see Fig. 14.1). Also, the scalar s in (14.7c) determines the location of an arbitrary point on the axis of rotation as measured from the point D.

In summary, the rigid body is instantaneously rotating with angular velocity ω about the axis DC at the same time that it is advancing in the direction $\mathbf{e}_\omega$ with uniform velocity $(\mathbf{v}_B \bullet \mathbf{e}_\omega)$ so the motion can be described as motion of a screw.

For the simpler case of planar motion in the $\mathbf{e}_1$–$\mathbf{e}_2$ plane the angular velocity $\boldsymbol{\omega}$ is in the constant $\mathbf{e}_3$ direction

$$\boldsymbol{\omega} = \omega\,\mathbf{e}_3 \;, \tag{14.8}$$

the velocity $\mathbf{v}_B$ is in the $\mathbf{e}_1$–$\mathbf{e}_2$ plane so the velocity $\mathbf{v}_C$ of points on the axis of rotation vanishes. Thus, the intersection of the axis of rotation with the x_3=0 plane is the <u>instantaneous center of rotation</u> and is given by (14.7b) with s=0

$$\mathbf{x}_{C/B} = \frac{\boldsymbol{\omega} \times \mathbf{v}_B}{\omega^2} = \frac{|\,\mathbf{v}_B\,|}{\omega} \; \frac{\mathbf{e}_3 \times \mathbf{v}_B}{|\,\mathbf{e}_3 \times \mathbf{v}_B\,|} \quad . \tag{14.9}$$

The formula (14.9) indicates that the instantaneous center of rotation is located along a line perpendicular to the velocity $\mathbf{v}_B$ and a distance $|\,\mathbf{v}_B\,| \,/\, |\,\boldsymbol{\omega}\,|$ from point B (see Fig. 14.2). It is important to note that ω in (14.8) can be positive or negative so the sign of ω controls the direction of $\mathbf{r}_{C/B}$ in (14.9). Furthermore, since $\mathbf{v}_C$ vanishes the velocity of an arbitrary point B on the rigid body is perpendicular to the relative position vector $\mathbf{x}_{B/C}$ because

$$\mathbf{v}_B = \mathbf{v}_C + \boldsymbol{\omega} \times \mathbf{x}_{B/C} = \boldsymbol{\omega} \times \mathbf{x}_{B/C} \quad . \tag{14.10}$$

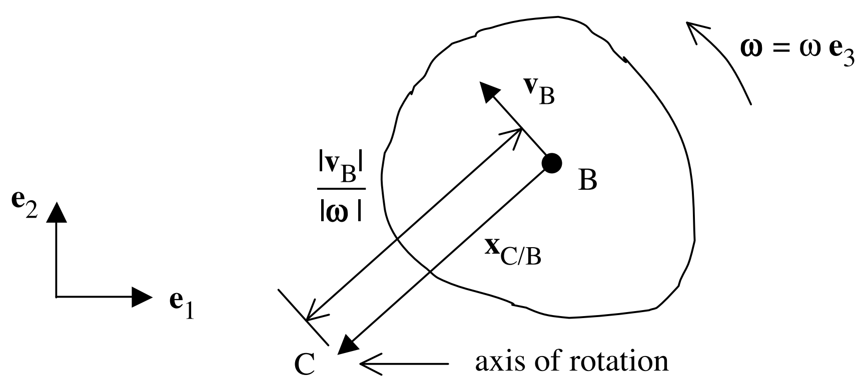

Fig. 14.2

15. Contact of Bodies

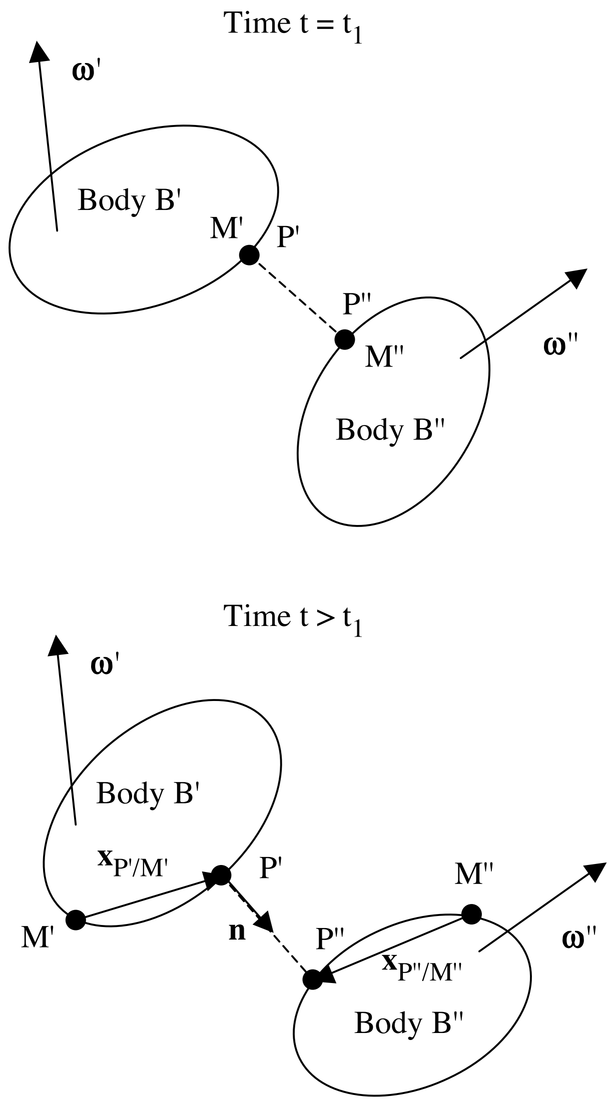

Fig. 15.1

In this section we study the conditions that describe contact and sliding of two bodies. To this end, consider two bodies B' and B'' that are both translating and rotating in space. Let $\mathbf{e}'_i$ be a body coordinate system attached to B' which rotates with angular velocity $\boldsymbol{\omega}'$ so that

$$\dot{\mathbf{e}}'_i = \boldsymbol{\omega}' \times \mathbf{e}'_i \ . \tag{15.1}$$

In what follows it is necessary to distinguish between the locations and velocities of various points. For example, let M' be the material point on body B' that at time t_1 was closest to body B'' and let M'' be the material point on body B'' that at time t_1 was closest to body B'. Also, let P' be the point in space that lies on the surface of body B' (but is not a material point) that is always closest to B''. Similarly, let P'' be the point in space that lies on the surface of body B'' (but is not a material point) that is always closest to B'. In general, as bodies B' and B'' move the points P' and P'' traverse different material points on the surfaces of bodies B' and B'', respectively. Furthermore, since P' and P'' are the points of closest contact between the two bodies, the tangent planes to the bodies B' and B'' are parallel at the points P' and P'', and the vector **n** that is directed from P' towards P'' is normal to these tangent planes (see Fig. 15.1).

Here we are interested in conditions that determine whether the bodies are in contact and whether they will remain in contact or tend to separate. To this end, we note that if the vector $\mathbf{x}_{P'/P''}$ vanishes then the two bodies are instantaneously in contact, whereas if the vector $\mathbf{x}_{P'/P''}$ is nonzero then the bodies are separated. In any case, the normal component $\mathbf{v}_{P'/P''} \bullet \mathbf{n}$ of the relative velocity between the two points P' and P'' determines whether the two bodies maintain contact ($\mathbf{v}_{P'/P''} \bullet \mathbf{n} = 0$), are separating ($\mathbf{v}_{P'/P''} \bullet \mathbf{n} < 0$), or are approaching each other when they are separated ($\mathbf{v}_{P'/P''} \bullet \mathbf{n} > 0$). In general, since P' and P'' move on the surface of the bodies B' and B'', respectively, it is rather difficult to determine their velocities. However, we will show presently that the normal component $\mathbf{v}_{P'} \bullet \mathbf{n}$ of the velocity of point P' is equal to the normal component $\mathbf{v}_{M'} \bullet \mathbf{n}$ of the velocity of the material point M' on body B' that instantaneously coincides with P'. Thus, the conditions of contact can be reformulated in terms of the velocities of the material points M' and M'', which are easily calculated.

The motion of P' relative to the material point M' can be described by the vector

$$\mathbf{x}_{P'/M'} = (x_{P'/M'})_i' \, \mathbf{e}_i' \ , \tag{15.2}$$

so the velocity of P' relative to M' may be expressed in the form

$$\mathbf{v}_{P'/M'} = \frac{\delta \mathbf{x}_{P'/M'}}{\delta t} + \boldsymbol{\omega}' \times \mathbf{x}_{P'/M'} \ , \tag{15.3a}$$

$$\frac{\delta \mathbf{x}_{P'/M'}}{\delta t} = (\dot{x}_{P'/M'})_i' \, \mathbf{e}_i' \ . \tag{15.3b}$$

Notice that since P' moves on the surface of body B' it follows that the vector (15.3b) is instantaneously tangent to the surface of B' so that

$$\frac{\delta \mathbf{x}_{P'/M'}}{\delta t} \bullet \mathbf{n} = 0 \ , \tag{15.4a}$$

$$\mathbf{v}_{P'/M'} \bullet \mathbf{n} = (\boldsymbol{\omega}' \times \mathbf{x}_{P'/M'}) \bullet \mathbf{n} \ . \tag{15.4b}$$

This means that in the limit that t approaches t_1 (and $\mathbf{x}_{P'/M'}$ approaches zero), the normal component of the velocity of the point P' of closest contact with the body B'' is the same as the normal component of the material point M' which instantaneously coincides with P' so that

$$\mathbf{v}_{P'/M'} \bullet \mathbf{n} = 0 \ , \quad \mathbf{v}_{P'} \bullet \mathbf{n} = \mathbf{v}_{M'} \bullet \mathbf{n} \quad \text{for} \ \ \mathbf{x}_{P'/M'} = 0 \ . \tag{15.6a,b}$$

Consequently, using a similar result for the velocity of P'' relative to M''

$$\mathbf{v}_{P''/M''} \bullet \mathbf{n} = 0 \ , \quad \mathbf{v}_{P''} \bullet \mathbf{n} = \mathbf{v}_{M''} \bullet \mathbf{n} \quad \text{for} \ \ \mathbf{x}_{P''/M''} = 0 \ . \tag{15.7a,b}$$

it may be seen that

$$\mathbf{v}_{P'/P''} \bullet \mathbf{n} = \mathbf{v}_{M'/M''} \bullet \mathbf{n} \quad \text{for} \ \ \mathbf{x}_{P'/M'} = \mathbf{x}_{P''/M''} = 0 \ . \tag{15.8}$$

Thus, we can use the instantaneous velocities $\mathbf{v}_{M'}$ and $\mathbf{v}_{M''}$ to make the following physical interpretation of the relative velocity $\mathbf{v}_{M'/M''}$:

$\mathbf{v}_{M'/M''} \bullet \mathbf{n} =$	Normal component of the relative velocity which measures the rate of approach (positive value) or separation (negative value) of the bodies B' and B''.
$\mathbf{v}_{M'/M''} - (\mathbf{v}_{M'/M''} \bullet \mathbf{n})\, \mathbf{n} =$	Magnitude and direction of the slip velocity of material points M' and M'' on the bodies B' and B'', respectively.

It follows that if the two bodies B' and B'' are in contact at some point in time they will remain in contact if the relative velocity $\mathbf{v}_{M'/M''}$ of the contact points M' and M'' has vanishing normal component. Furthermore, if $\mathbf{v}_{M'/M''}$ vanishes then the bodies remain in contact and do not slip.

As an example consider a cylinder of radius r which rotates with angular velocity ω about the $\mathbf{e}_3$ axis and whose center B translates with velocity v in the negative $\mathbf{e}_1$ direction (see Fig. 15.2). At time t_1 the material point M' attached to the cylinder is in contact with a stationary horizontal plane at the material point M''. It follows that the velocity $\mathbf{v}_{M''}$ vanishes so that the velocity of point M' relative to point M'' is given by

$$\mathbf{v}_{M'/M''} = \mathbf{v}_{M'} = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{x}_{M'/B} = -\,\mathrm{v}\, \mathbf{e}_1 + (\omega\, \mathbf{e}_3) \times (-\, \mathrm{r}\, \mathbf{e}_2) \quad , \tag{15.9a}$$

$$\mathbf{v}_{M'/M''} = (\omega \mathrm{r} - \mathrm{v})\, \mathbf{e}_1 \quad . \tag{15.9b}$$

Since the relative velocity $\mathbf{v}_{M'/M''}$ has vanishing normal component ($\mathbf{v}_{M'/M''} \bullet \mathbf{e}_2 = 0$) the cylinder remains in contact with the horizontal plane. Furthermore, if $\omega \mathrm{r} > \mathrm{v}$ then point M' is sliding in the positive $\mathbf{e}_1$ direction relative to the fixed point M'' and if $\omega \mathrm{r} < \mathrm{v}$ then M' is sliding in the negative $\mathbf{e}_1$ direction relative to M''. Finally, if $\omega \mathrm{r} = \mathrm{v}$ then the relative velocity $\mathbf{v}_{M'/M''}$ vanishes and the cylinder rolls without slipping.

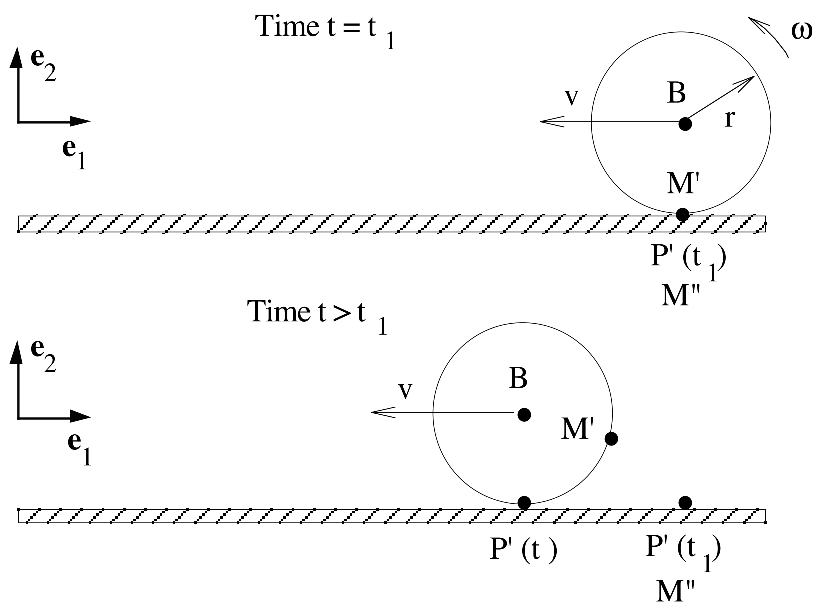

Fig. 15.2

16. Kinetics of a Particle

In the previous sections we have devoted most of our attention to the study of kinematics of particles and rigid bodies by learning how to analyze position, velocity and acceleration. Such kinematical quantities are considered to be primitive quantities because they can be measured directly. In this section we introduce the notion of force which is a kinetical vector quantity that usually is measured indirectly. In particular, the magnitude of a force is often measured by comparing it to the weight of an object or by using the displacement of a spring, which itself has been calibrated by measuring the weights of standard objects.

Obviously, the direction of a force matters because when we push an object in one direction it tends to move in that direction, whereas when we push it in the opposite direction it tends to move in the opposite direction. In fact, for rectilinear motion it can be shown that the acceleration is in the same direction as the force. To see if this observation remains true for more general motions we can consider the motion of a ball on a smooth (frictionless) horizontal plane that is confined to move in a circle by a string that is attached to a weight (Fig. 16.1). The first thing that we can observe is that the acceleration vector rotates and is always pointed towards the center of the circle. Specifically, the acceleration vector **a** always points in the same direction as the <u>force</u> vector **F** which is applied to the ball by the string. Mathematically this means that **F** is parallel to **a**

$$\mathbf{F} \parallel \mathbf{a} \,. \tag{16.1}$$

By keeping the weight constant and changing the radius of the circular path we can observe that the angular velocity of the ball changes in such a way that it preserves the magnitude of the acceleration. Furthermore, by using the same ball but taking different weights we can determine that the magnitude of the force **F** applied by the string on the ball (which is equal to the weight applied) always remains proportional to the magnitude of the acceleration. Thus, for different forces $\{\mathbf{F}_1 , \mathbf{F}_2 , \mathbf{F}_3 \}$ and associated accelerations $\{ \mathbf{a}_1 , \mathbf{a}_2 , \mathbf{a}_3 \}$ we have

$$\frac{|\mathbf{F}_1|}{|\mathbf{a}_1|} = \frac{|\mathbf{F}_2|}{|\mathbf{a}_2|} = \frac{|\mathbf{F}_3|}{|\mathbf{a}_3|} = \mathrm{m} \;, \tag{16.2}$$

where the constant of proportionality m is a property of the ball which is called the <u>mass</u> of the ball.

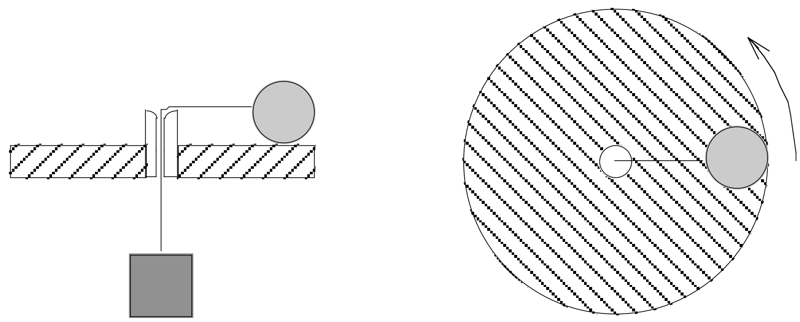

Fig. 16.1

NEWTONS LAWS OF MOTION

Sir Issac Newton (1642-1727) was the first to discover the correct laws of motion of particles which are summarized as the following three laws of motion:

Law I: A particle remains at rest or continues to move in a straight line with uniform velocity if there is no unbalanced force acting on it.

$$\mathbf{F} = 0 \;\Rightarrow\; \mathbf{v} = \text{constant} \;. \tag{16.3}$$

Law II: The resultant force **F** acting on a particle is equal to the rate of change of linear momentum.

$$\mathbf{F} = \frac{d(m\mathbf{v})}{dt} \;. \tag{16.4}$$

Law III: The forces of action and reaction between interacting bodies are equal in magnitude and opposite in direction (Fig. 16.2).

$$\mathbf{F}_{A/B} + \mathbf{F}_{B/A} = 0 \;,\; \mathbf{F}_{A/B} = -\,\mathbf{F}_{B/A} \;. \tag{16.5a,b}$$

($\mathbf{F}_{A/B}$ is the force applied by body B on body A)

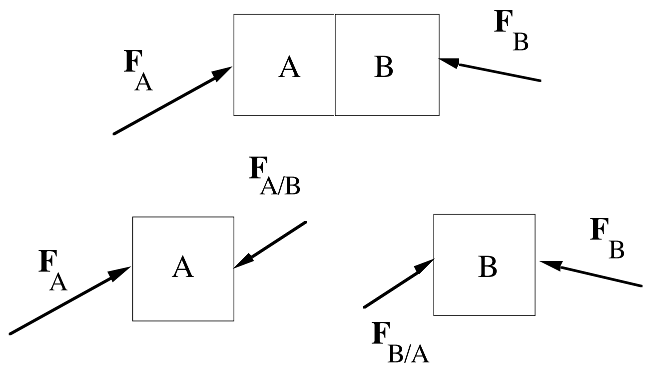

Fig. 16.2

CONSERVATION OF MASS

For our purposes we will only consider bodies that have constant mass so we may write the law of conservation of mass in the form

$$\frac{dm}{dt} = \dot{m} = 0 \ . \tag{16.6}$$

BALANCE OF LINEAR MOMENTUM

Newton's second law of motion is referred to in modern terms as the balance of linear momentum. In words it states that the rate of change of linear momentum ($m\mathbf{v}$) is equal to the total external force $\mathbf{F}$ applied to the body. In view of the conservation of mass (16.6) the balance of linear momentum can be written in the form

$$\frac{d(m\mathbf{v})}{dt} = m\frac{d\mathbf{v}}{dt} = m\,\mathbf{a} = \mathbf{F} \ . \tag{16.7}$$

It is important to emphasize that the velocity $\mathbf{v}$ and the acceleration $\mathbf{a}$ in the balance of linear momentum must be <u>absolute</u> not relative quantities so they must be measured relative to a fixed point.

Since the balance of linear momentum is a vector equation it may be expressed with respect to <u>any</u> convenient set of base vectors. For example, if we consider two sets of rectangular Cartesian base vectors $\mathbf{e}_i$ and $\mathbf{e}_i'$ we may write the following scalar equations

$$F_i = m\,a_i \qquad (\mathbf{e}_i \bullet \mathbf{F} = m\,\mathbf{e}_i \bullet \mathbf{a}) \ , \tag{16.8a}$$

$$F_i' = m\,a_i' \qquad (\mathbf{e}_i' \bullet \mathbf{F} = m\,\mathbf{e}_i' \bullet \mathbf{a}) \ . \tag{16.8b}$$

Similarly, if we refer the vectors to the cylindrical polar base vectors $\{\mathbf{e}_r , \mathbf{e}_\theta , \mathbf{e}_3\}$ or the spherical polar base vectors $\{\mathbf{e}_R , \mathbf{e}_\theta , \mathbf{e}_\phi\}$ we may write

$$F_r = m\, a_r \ , \ F_\theta = m\, a_\theta \ , \ F_3 = m\, a_3 \ , \tag{16.9}$$

or

$$F_R = m\, a_R \ , \ F_\theta = m\, a_\theta \ , \ F_\phi = m\, a_\phi \ . \tag{16.10}$$

FREE BODY DIAGRAM

In order to correctly translate a physical problem into a mathematical problem in dynamics it is essential to carefully draw a free-body diagram which isolates the body of interest and includes all external forces acting on the body (Fig. 16.3). In this regard we should emphasize that the force **F** appearing in the balance of linear momentum is the resultant force which is the sum of all external forces acting on the body.

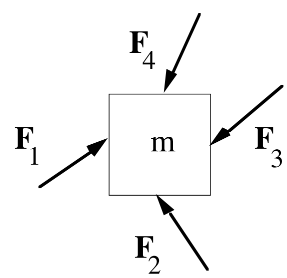

Fig. 16.3

In the above we have focused attention on Newton's second law of motion which we call the balance of linear momentum. This is because the balance of linear momentum actually contains Newton's first and third laws as special cases. It is obvious from the balance of linear momentum (16.7) that when the resultant force **F** vanishes the velocity of the particle is constant because

$$\mathbf{F} = 0 \ \Rightarrow \ \mathbf{a} = \dot{\mathbf{v}} = 0 \ \Rightarrow \ \mathbf{v} = \text{constant} \ . \tag{16.11}$$

Thus, we have proved Newton's first law.

To prove Newton's third law from the balance of linear momentum we return to Fig. 16.2 and note that $\mathbf{F}_A$ and $\mathbf{F}_B$ are the external forces applied to the macroparticle which is composed of the two particles A and B. Assuming that the particles are small enough and remain in contact it follows from continuity that they both move with the same acceleration **a**

$$\mathbf{a}_A = \mathbf{a}_B = \mathbf{a} \ . \tag{16.12}$$

Letting m_A be the mass of particle A and m_B be the mass of particle B, the balance of linear momentum applied to the macroparticle of total mass $m_A + m_B$ yields the equation

$$\mathbf{F}_A + \mathbf{F}_B = (m_A + m_B)\, \mathbf{a} \ . \tag{16.13}$$

Alternatively, we can consider the free body diagrams of the particles A and B separately and denote $\mathbf{F}_{A/B}$ as the internal force (contact force) applied by particle B on particle A, and denote $\mathbf{F}_{B/A}$ as the internal force applied by particle A on particle B. Then the balances of linear momentum of each of the particles may be written in the forms

$$\mathbf{F}_A + \mathbf{F}_{A/B} = m_A\, \mathbf{a} \ , \ \mathbf{F}_B + \mathbf{F}_{B/A} = m_B\, \mathbf{a} \ . \tag{16.14a,b}$$

Next, we add equations (16.14a,b) and subtract (16.13) from the result to deduce that

$$\mathbf{F}_{A/B} + \mathbf{F}_{B/A} = 0 \ , \tag{16.15}$$

which proves Newton's third law. The main theoretical point associated with this proof is the basic assumption that the conservation of mass and the balance of linear momentum are valid for any arbitrary part of a body (which in this case includes both the macroparticle and the two particles, separately).

D'ALEMBERT'S PRINCIPLE

In order to extend the principle of virtual work for static problems to dynamical problems d'Alembert (1717-1783) introduced the notion of the "force of inertia" $\boldsymbol{I}$, defined in terms of the mass of a particle and its absolute acceleration by

$$\boldsymbol{I} = -\, m\, \mathbf{a} \ . \tag{16.16}$$

Using this definition the balance of linear momentum (16.7) can be rewritten in the form

$$\mathbf{F} + \boldsymbol{I} = 0 \ . \tag{16.17}$$

This changes the balance of linear momentum into a principle that states that every body is in a state of "dynamical equilibrium". In our opinion, the introduction of the concept of an inertial force confuses the concept of force since acceleration is a kinematical quantity that can be measured independently of the concept of force and mass is an intrinsic property of the body that is independent of the particular motion of the body.

Furthermore, the introduction of the concept of "dynamical equilibrium" does not simplify the formulation of the balance of linear momentum because it still requires the calculation of the absolute acceleration $\mathbf{a}$. Also, since the balance of linear momentum and the principle of "dynamical equilibrium" (16.17) are mathematically identical, any mathematical operation performed on equation (16.17) can be performed on equation (16.7) to obtain the same result.

TWO MAIN PROBLEMS IN DYNAMICS

There are two main problems in the study of dynamics which can be summarized as follows:

> Problem I: Given the motion and mass of a particle, determine the resultant force necessary to create this motion. This problem is relatively simple because we just need to differentiate the motion to determine the absolute acceleration and then use the balance of linear momentum to determine the resultant force.
>
> Problem II: Given the resultant force acting on a particle and its mass, determine the motion of the particle. This problem is much more difficult than problem I because we need to integrate the equations of motion.

Because of the analytical difficulty with integrating the set of nonlinear equations of motion that usually result in dynamical problems we will focus most of our attention to either problem I or to the formulation (but not solution) of problem II.

17. Vibrations

In this section we consider the physically important problem of both free and forced vibration of a damped spring-mass system. To this end, let us first consider the simple problem of free vibrations of the undamped spring-mass system shown in Fig. 17.1. The spring has a free length L and spring constant k and the body has mass m. Friction is neglected but gravity is included.

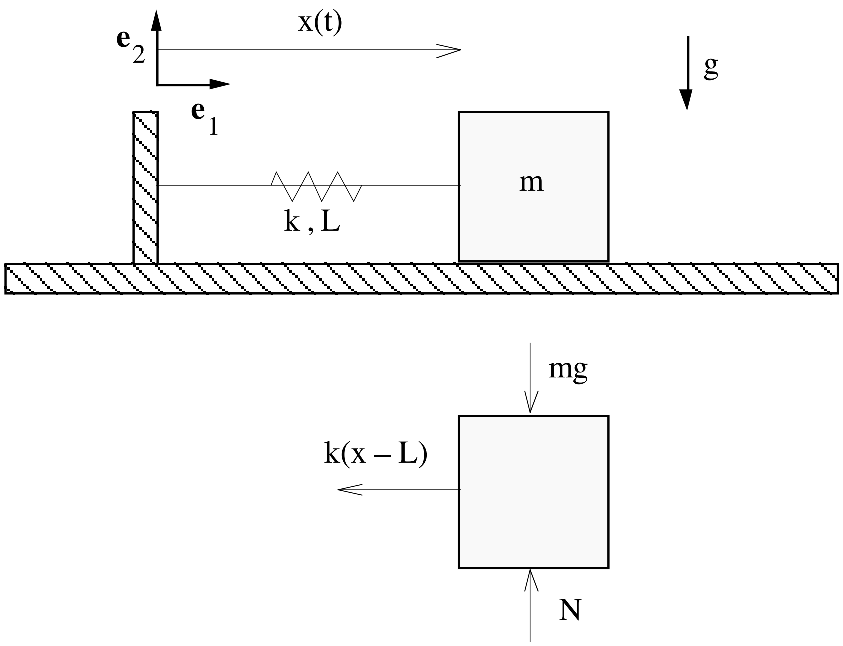

Fig. 17.1

Recalling that the force in a spring is equal to the spring constant k times the change in its length relative to its force-free length L, we may use the free body diagram of the mass m in Fig. 17.1 to write the resultant external force $\mathbf{F}$ in the form

$$\mathbf{F} = -\,k(x - L)\,\mathbf{e}_1 + [\,N - mg\,]\,\mathbf{e}_2 \;, \tag{17.1}$$

where N is the contact force that the horizontal plate applies to the mass and mg is the force of gravity. Notice that the origin of the coordinate system has been chosen conveniently so that x is the current length of the spring and the position of the mass. Furthermore, since the mass is only allowed to move in the horizontal direction its acceleration is given by

$$\mathbf{a} = \ddot{x}\;\mathbf{e}_1 \;. \tag{17.2}$$

Thus, the balance of linear momentum yields the following vector equation

$$-\,k(x-L)\,\mathbf{e}_1 + [\,N - mg\,]\,\mathbf{e}_2 = m\,\ddot{x}\;\mathbf{e}_1 \ , \tag{17.3}$$

which gives the following scalar equations

$$m\,\ddot{x} \;+ k\,(x-L) = 0 \ , \ N = mg \ . \tag{17.4a,b}$$

The first of these equations is a differential equation to determine the position x of the mass and the second merely states that the contact force is equal to the force of gravity.

Noting that L is constant, equation (17.4a) may be rewritten in the form

$$\frac{d^2}{dt^2}(x-L) \;+ \omega_n{}^2\,(x-L) = 0 \ , \ \omega_n{}^2 = \frac{k}{m} \ , \tag{17.5a,b}$$

where the constant ω_n is called the natural frequency and it has the units of inverse time. The general solution of (17.5a) may be written in terms of sines and cosines and takes the form

$$x - L = A_1\,\sin(\omega_n t) + A_2\,\cos(\omega_n t) \ , \tag{17.6}$$

where A_1 and A_2 are constants of integration to be determined by the initial conditions

$$x(0) = x_0 \ , \ \dot{x}(0) \; = v_0 \ . \tag{17.7a,b}$$

However, for ease of interpretation it is more convenient to rewrite (17.6) as

$$x - L = A\,\Big[\,\frac{A_1}{A}\;\sin(\omega_n t) + \frac{A_2}{A}\;\cos(\omega_n t)\Big] \ , \ \ A = (A_1^2\; + A_2^2\,)^{1/2} , \tag{17.8a,b}$$

and recall the trigonometric relation

$$\sin(\omega_n t + \phi) = \sin(\omega_n t)\cos\phi + \cos(\omega_n t)\sin\phi \ , \tag{17.9}$$

to deduce that the general solution of (17.5) may be expressed in the alternative form

$$x - L = A\,\sin(\omega_n t + \phi) \ . \tag{17.10}$$

In (17.10) it is easy to see that A is the <u>amplitude</u> of the vibration and ϕ is the <u>phase angle</u>. These constants (A,ϕ) are determined by the initial conditions (17.7) which yield the equations

$$x_0 - L = A\,\sin\phi \ , \ v_0 = A\,\omega_n\cos\phi \ . \tag{17.11a,b}$$

Furthermore, using the fact that $\sin^2\phi + \cos^2\phi = 1$ we can determine the value of the amplitude A by

$$A = \left[(x_0 - L)^2 + \left(\frac{v_0}{\omega_n}\right)^2 \right]^{1/2} \quad . \tag{17.12}$$

Then the phase angle ϕ, which is restricted to the range $[0 \le \phi < 2\pi]$, is uniquely determined by the two equations (17.11a,b). The general solution (17.10)-(17.12) indicates that the spring-mass system vibrates freely with the natural frequency ω_n and the amplitude of the vibration is determined by the initial conditions. Notice from (17.5b) that ω_n decreases as the body becomes more "massive" and thus has more inertia. Also, ω_n increases as the spring becomes stiffer with a larger spring constant.

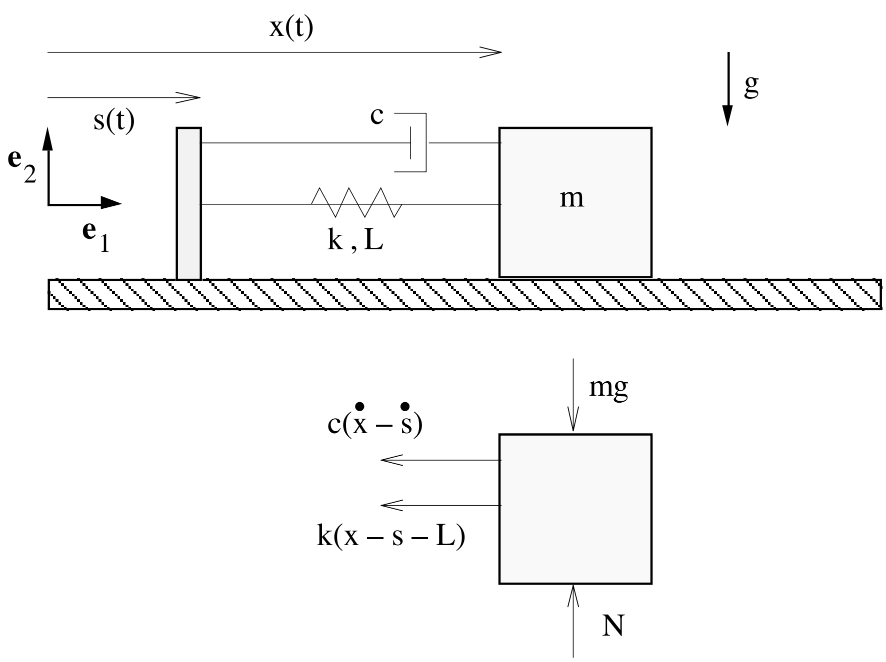

Fig. 17.2

Next we consider the more complicated but also more realistic case of forced vibration of a damped spring-mass system (Fig. 17.2). For this case we include a dashpot damper that creates a force $c(\dot{x} - \dot{s})$ that resists the relative motion between the two ends of the dashpot. This resisting force is modeled as a linear function of the relative velocity, with the damping coefficient c being a property of the dashpot. Furthermore, for this case the base of the spring and dashpot is forced to move with the motion described by s(t). Also, since the current length of the spring is given by (x − s) the resultant force applied to the mass becomes

$$\mathbf{F} = [\,-k(x-s-L) - c(\dot{x} - \dot{s})\,]\,\mathbf{e}_1 + [\,N - mg\,]\,\mathbf{e}_2 \quad . \tag{17.13}$$

Since x is the position of the mass m measured relative to a fixed wall, the absolute acceleration of the mass is still given by (17.2) so the vector equation representing the balance of linear momentum is given by

$$[\,-k(x-s-L) - c(\dot{x} - \dot{s})\,]\,\mathbf{e}_1 + [\,N - mg\,]\,\mathbf{e}_2 = m\,\ddot{x}\,\mathbf{e}_1 \quad , \tag{17.14}$$

and the two scalar equation become

$$m\,\ddot{x} + k(x-s-L) + c(\dot{x} - \dot{s}) = 0 \quad , \quad N = mg \quad . \tag{17.15a,b}$$

By dividing (17.15a) by the mass m and using the definition (17.5b) for the natural frequency ω_n we may rewrite (17.15a) in the form

$$\ddot{w} + \frac{c}{m}\dot{w} + \omega_n^2\, w = -\,\ddot{s} \quad , \quad w = x - s - L \quad , \tag{17.16a,b}$$

where the quantity w denotes the extension of the spring from its stress-free length L. Integration of (17.16a) requires the specification of the function s(t) and the initial conditions (17.7).

In order to understand the influence of damping we first consider the case of free damped vibrations for which the base is held constant so that s(t) vanishes and the balance of linear momentum (17.16a) reduces to

$$\ddot{w} + \frac{c}{m}\dot{w} + \omega_n^2\, w = 0 \quad . \tag{17.17}$$

By using the transformation

$$w(t) = \exp\!\left(-\frac{ct}{2m}\right)\xi(t) \quad , \tag{17.18}$$

together with the results

$$\dot{w} = \exp\!\left(-\frac{ct}{2m}\right)\left[\,\dot{\xi} - \frac{c}{2m}\,\xi\,\right] \quad , \tag{17.19a}$$

$$\ddot{w} = \exp\!\left(-\frac{ct}{2m}\right)\left[\,\ddot{\xi} - \frac{c}{m}\dot{\xi} + \left(\frac{c}{2m}\right)^2 \xi\,\right] \quad . \tag{17.19b}$$

we can transform (17.17) into the simpler form

$$\ddot{\xi} + \omega_n^2 (1-\zeta^2)\, \xi = 0 \ , \ c = 2m\zeta\omega_n \ , \tag{17.20}$$

where ζ is a non-dimensional damping parameter. Now, the initial conditions (17.7) with $\dot{s}(0) = 0$ may be used with the definitions (17.16b) and (17.18) to deduce that

$$w(0) = x_0 - s_0 - L \ , \ \dot{w}(0) = v_0 \ , \tag{17.21a,b}$$

$$\xi(0) = x_0 - s_0 - L \ , \ \dot{\xi}(0) = v_0 + \zeta\omega_n (x_0 - s_0 - L) \ , \tag{17.21c,d}$$

Noting that (17.20) is similar to (17.5a) we obtain three types of solutions which are differentiated by the relative value of the damping coefficient ζ.

Underdamped ($\zeta < 1$)

$$w(t) = \exp(-\zeta\omega_n t)\ A \sin(\omega^* t + \phi) \ , \tag{17.22a}$$

$$\omega^* = \omega_n \sqrt{1-\zeta^2} \ , \tag{17.22b}$$

$$A = \left[(x_0 - s_0 - L)^2 + \left\{ \frac{v_0}{\omega^*} + \frac{\zeta\omega_n}{\omega^*} (x_0 - s_0 - L) \right\}^2 \right]^{1/2} \ , \tag{17.22c}$$

$$\sin\phi = \frac{1}{A} (x_0 - s_0 - L) \ , \ \cos\phi = \frac{1}{A\omega^*} \left[v_0 + \zeta\omega_n (x_0 - s_0 - L) \right] . \tag{17.22d,e}$$

Critically Damped ($\zeta = 1$)

$$w(t) = \exp(-\omega_n t) \left[(x_0 - s_0 - L) + \left\{ v_0 + \omega_n (x_0 - s_0 - L) \right\} t \right] , \tag{17.23}$$

Overdamped ($\zeta > 1$)

$$w(t) = \exp(-\zeta\omega_n t)\ A \sinh(\omega^* t + \phi) \ , \tag{17.24a}$$

$$\omega^* = \omega_n \sqrt{\zeta^2 - 1} \ , \tag{17.24b}$$

$$A = \left[\left\{ \frac{v_0}{\omega^*} + \frac{\zeta\omega_n}{\omega^*} (x_0 - s_0 - L) \right\}^2 - (x_0 - s_0 - L)^2 \right]^{1/2} \ , \tag{17.24c}$$

$$\sinh\phi = \frac{1}{A} (x_0 - s_0 - L) \ , \ \cosh\phi = \frac{1}{A\omega^*} \left[v_0 + \zeta\omega_n (x_0 - s_0 - L) \right] . \tag{17.24d,e}$$

In deriving the solution (17.24a) we have followed similar procedures used to derive the solution (17.10) and have used the hyperbolic identities

$$\sinh(\omega^* t + \phi) = \sinh(\omega^* t) \cosh\phi + \cosh(\omega^* t) \sinh\phi \ , \tag{17.25a}$$

$$\cosh^2\phi - \sinh^2\phi = 1 \ . \tag{17.25b}$$

Notice that because of the presence of the exponential terms in (17.22a),(17.23a) and (17.23a) each of these solutions yields the same equilibrium solution of vanishing w for long times (large values of t). More specifically, (17.22a) is called an underdamped solution because it exhibits oscillations that damp to zero; solution (17.23a) is called critically damped because the damping coefficient ζ attains its critical value (ζ=1, the minimum value necessary to damp to zero without any oscillation); and solution (17.24a) is called overdamped because ζ attains a value larger than its critical value. This response for $v_0 = 0$ is shown in Fig. 17.3 for different values of the damping coefficient ζ.

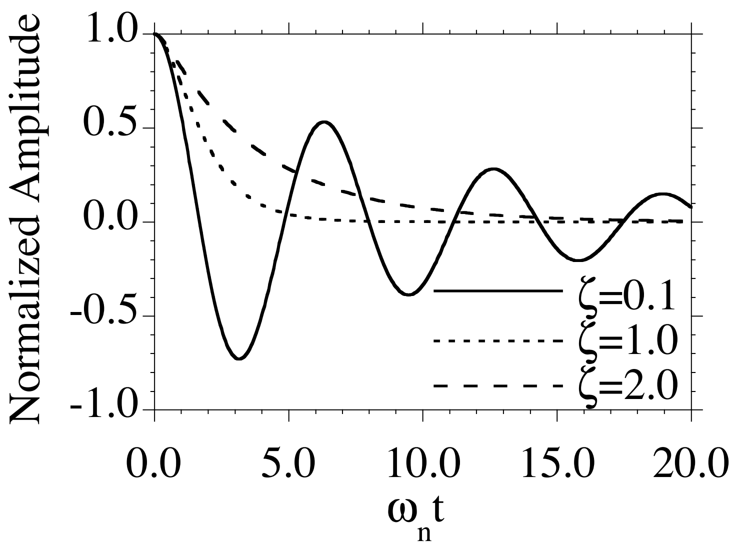

Fig. 17.4

The above solution indicates that when damping is present (ζ >0) the homogeneous solutions of (17.16a) damp out so that after a reasonably long time the only solution that remains is the particular solution. As a special case let us consider the problem where the base vibrates with frequency ω and amplitude s_0 such that

$$s(t) = s_0 \sin(\omega t) \ , \tag{17.26}$$

so that (17.16a) reduces to

$$\ddot{w} + \frac{c}{m}\dot{w} + \omega_n^2 w = s_0\, \omega^2 \sin(\omega t) \ . \tag{17.27}$$

Now, the particular solution w_p of (17.27) may be written in the form

$$w_p(t) = A \sin(\omega t + \phi) \ , \tag{17.28}$$

where the amplitude A and phase angle ϕ are to be determined. Substituting (17.28) into (17.27) we obtain

$$A \left[\left\{ \omega_n^2 - \omega^2 \right\} \sin(\omega t + \phi) + 2\zeta\omega_n\omega \cos(\omega t + \phi) \right] = s_0\omega^2 \sin(\omega t) \ , \tag{17.29a}$$

$$A \left[(\omega_n^2 - \omega^2) \left\{ \sin(\omega t) \cos(\phi) + \cos(\omega t) \sin(\phi) \right\} \right.$$

$$\left. + \ 2\zeta\omega_n\omega \left\{ \cos(\omega t) \cos(\phi) - \sin(\omega t) \sin(\phi) \right\} \right] = s_0 \ \omega^2 \sin(\omega t) \ , \tag{17.29b}$$

$$A \left[(\omega_n^2 - \omega^2) \left\{ \sin(\omega t) \cos(\phi) + \cos(\omega t) \sin(\phi) \right\} \right.$$

$$\left. + \ 2\zeta\omega_n\omega \left\{ \cos(\omega t) \cos(\phi) - \sin(\omega t) \sin(\phi) \right\} \right] = s_0 \ \omega^2 \sin(\omega t) \ , \qquad (17.29b)$$

$$A \left[\left\{ (\frac{\omega_n^2}{\omega^2} - 1) \cos(\phi) - \frac{2\zeta\omega_n}{\omega} \sin(\phi) \right\} \sin(\omega t) \right.$$

$$\left. + \left\{ (\frac{\omega_n^2}{\omega^2} - 1) \sin(\phi) + \frac{2\zeta\omega_n}{\omega} \cos(\phi) \right\} \cos(\omega t) \right] = s_0 \sin(\omega t) \ . \tag{17.29c}$$

It follows that the term $\cos(\omega t)$ can be eliminated by appropriately specifying the phase angle ϕ so the solution becomes

$$A = s_0 \, G(\omega_n,\zeta,\omega) \ , \tag{17.30a}$$

$$G(\omega_n,\zeta,\omega) = \frac{1}{\left[\left(\frac{\omega_n^2}{\omega^2} - 1 \right)^2 + \left(\frac{2\zeta\omega_n}{\omega} \right)^2 \right]^{1/2}} \ , \tag{17.30b}$$

$$\sin\phi = - \frac{2\zeta\omega_n}{\omega} G(\omega_n,\zeta,\omega) \ , \ \cos\phi = \left(\frac{\omega_n^2}{\omega^2} - 1 \right) G(\omega_n,\zeta,\omega) \ , \tag{17.30c,d}$$

where $G(\omega_n,\zeta,\omega)$ is called the amplification function because it determines how much the amplitude A of the response w is amplified relative to the forcing amplitude s_0.

From a physical point of view we are often more interested in the absolute motion of the mass instead of its motion relative to the base which is described by w. Thus, using (17.16b), (17.28) and (17.30) the solution for the position x of the mass becomes

$$x(t) = L + s_0 \sin(\omega t) + G(\omega_n,\zeta,\omega)\, s_0 \sin(\omega t + \phi) \ . \tag{17.31}$$

This solution can be written in the simpler form

$$x(t) = L + s_0\, g(\omega_n,\zeta,\omega) \sin(\omega t + \Phi) \ , \tag{17.32}$$

where the amplification function g and the phase angle Φ associated with the absolute motion of the mass are determined by the equations

$$g \cos\Phi = 1 + G \cos\phi \ , \ g \sin\Phi = G \sin\phi \ . \tag{17.33}$$

Now using the results (17.30b,c,d) we may deduce that

$$g^2 = (1 + G \cos\phi)^2 + (G \sin\phi)^2 = 1 + 2G \cos\phi + G^2 \ , \tag{17.34a}$$

$$g = \left[\frac{1 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2} \right]^{1/2} \ . \tag{17.34b}$$

The main physical implications of this solution can be observed by considering the following limiting cases:

Limit $\omega << \omega_n$

$$g \approx 1 \ , \ \Phi \approx 0 \ , \ x(t) \approx L + s(t) \ , \tag{17.35a}$$

Limit $\omega = \omega_n$

$$g \approx \frac{1}{2\zeta}\sqrt{1+4\zeta^2} \ , \ \Phi \approx -\tan^{-1}\left(\frac{1}{2\zeta}\right), \tag{17.35b}$$

Limit $\omega >> \omega_n$

$$g \approx 0 \ , \ \Phi \approx -\frac{\pi}{2} \ , \ x(t) \approx L \ , \tag{17.35c}$$

This means that for very low forcing frequencies ($\omega << \omega_n$) the mass moves with the base; for very high forcing frequencies ($\omega >> \omega_n$) the mass is isolated from the base vibration because it does not move at all. Furthermore, for very low and very high forcing frequency the limiting solution is independent of the value of damping. In contrast, when the forcing frequency ω equals the natural frequency ω_n, the amplitude of the vibration is totally controlled by the value of damping and increases as the damping coefficient decreases. More specifically, if the damping coefficient vanishes ($\zeta = 0$) then the response amplitude becomes infinite. For this reason it is essential to design a mechanical system to have natural frequencies different from any expected forcing frequency. This response can be seen graphically in Fig. 17.4 which plots the amplification function g.

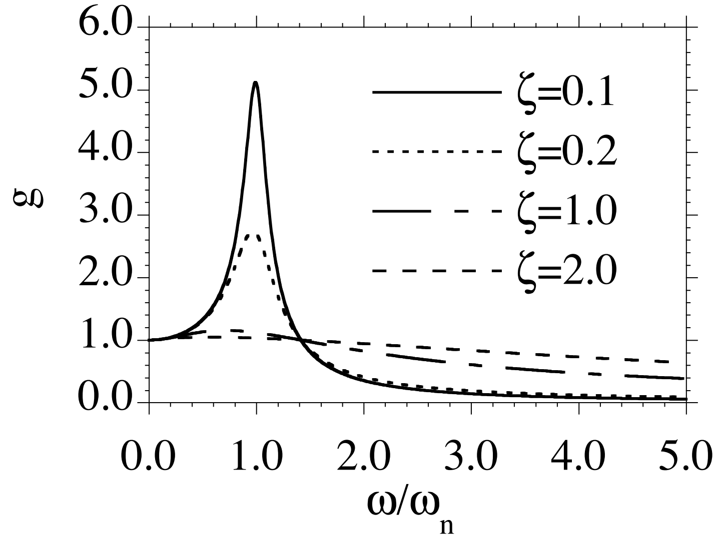

Fig. 17.4

18. Mechanical Power, Work and Energy (Particle)

In this section we consider notions of work and energy in particle mechanics. To this end we first define the <u>mechanical power</u> P

$$P = \mathbf{F} \bullet \mathbf{v} \ , \tag{18.1}$$

of a force $\mathbf{F}$ acting on a particle with absolute velocity $\mathbf{v}$. The mechanical power P is the rate of work done by the force $\mathbf{F}$ <u>on</u> the particle. Letting $U=U_{2/1}$ be the work done on the particle by the force $\mathbf{F}$ during the time interval $t \in [t_1,t_2]$ we may calculate $U_{2/1}$ by the integral

$$U = U_{2/1} = \int_{t_1}^{t_2} P \, dt = \int_{t_1}^{t_2} \mathbf{F} \bullet \mathbf{v} \, dt \ . \tag{18.2}$$

In order to better understand the meaning of mechanical power let us consider the simple case when P is constant so that the work $U_{2/1}$ is given by

$$U = U_{2/1} = P\,(t_2 - t_1) \ . \tag{18.3}$$

It follows form (18.3) that in order to do a given amount of work ($U_{2/1}$ = constant) we need a short time if the mechanical power P is large and a long time if P is small. For this reason it is convenient to measure the strength of motors by the mechanical power that they can supply.

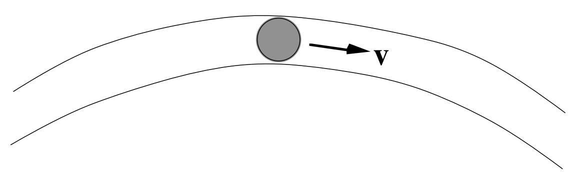

Fig. 18.1

Sometimes it is convenient to think of $\mathbf{F} \bullet \mathbf{v}$ as the projection of the force in the direction of the velocity

$$P = \mathbf{F} \bullet \mathbf{v} = \mathbf{F} \bullet \dot{s} \ \mathbf{e}_t = F_t \, \dot{s} \ . \tag{18.4}$$

For example, when considering a particle moving in a stationary slot (Fig. 18.1) the direction $\mathbf{e}_t$ of the velocity is known to be tangent to the slot path. For this case equation (18.4) means that the component of the force normal to the slot does no work. However, when the slot is not stationary then (18.4) still holds but $\mathbf{e}_t$ is no longer tangent to the path of the slot so the component of the force normal to the slot does work on the particle.

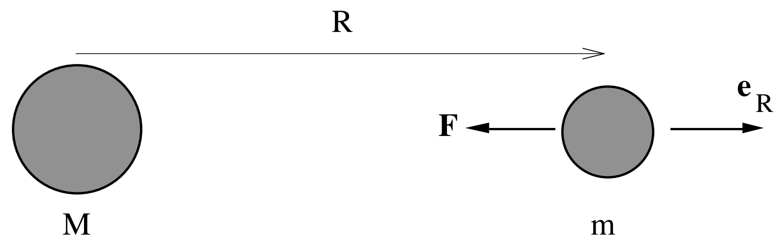

Fig. 18.2

Sometimes it is convenient to think of $\mathbf{F} \bullet \mathbf{v}$ as the projection of the velocity in the direction of the force. For example, when considering the force of gravity acting between two particles of masses M and m (Fig. 18.2) the direction of the force $\mathbf{F}$ is known

$$P = \mathbf{F} \bullet \mathbf{v} = F_R \, (\mathbf{e}_R \bullet \mathbf{v}) \ . \tag{18.4}$$

If the force $\mathbf{F}$ is the total resultant force applied to a particle of mass m then the balance of linear momentum (16.7) may be used to rewrite the mechanical power in the form

$$P \ = \mathbf{F} \bullet \mathbf{v} = m \, \mathbf{a} \bullet \mathbf{v} = \frac{d}{dt}\left(\frac{1}{2} m \, \mathbf{v} \bullet \mathbf{v}\right) = \dot{T}, \tag{18.5}$$

where the <u>kinetic energy</u> T is defined by

$$T = \frac{1}{2} \ m \, \mathbf{v} \bullet \mathbf{v} = \frac{1}{2} \ m \, v^2 \, . \tag{18.6}$$

Now, substitution of (18.5) into (18.2) yields

$$U = U_{2/1} = \int_{t_1}^{t_2} \dot{T} \, dt \ = T_2 - T_1 \ \ , \tag{18.7}$$

which states that the work done by the resultant force applied to a particle is equal to the change in kinetic energy of the particle. In order to emphasize that U in (18.7) is the work done by the <u>resultant</u> force let us consider the example of a motor lifting a mass m from position 1 to position 2 in a gravitational field (Fig. 18.3). Assuming that at the beginning and end of the process the mass is at rest it follows that the kinetic energies T_1 and T_2 vanish so that the work U also vanishes. In order to explain how the work U can vanish when we lift a weight we note that the resultant force $\mathbf{F}$ is composed of the force due to gravity and the force due to the motor. Since the work done by **F** vanishes we may conclude that work done by the motor (which is positive) exactly balances the work done by gravity (which is negative).

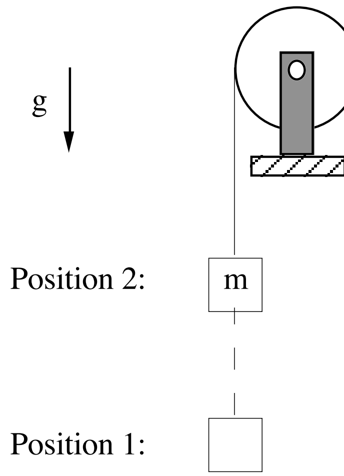

Fig. 18.3

19. Conservative Force Fields

The resultant force $\mathbf{F}$ that acts on a particle may be composed of two types of forces: conservative forces which are denoted by $\mathbf{F}_c$ and nonconservative forces which are denoted by $\mathbf{F}_{nc}$

$$\mathbf{F} = \mathbf{F}_c + \mathbf{F}_{nc} \ . \tag{19.1}$$

It follows that the mechanical power P of the force $\mathbf{F}$ also separates into the mechanical power P_c associated with the conservative forces and the mechanical power P_{nc} associated with the nonconservative forces

$$P = \mathbf{F} \bullet \mathbf{v} = P_c + P_{nc} \ , \ P_c = \mathbf{F}_c \bullet \mathbf{v} \ , \ P_{nc} = \mathbf{F}_{nc} \bullet \mathbf{v} \ . \tag{19.2a,b,c}$$

Conservative forces are of interest because they have special properties that allow the mechanical power P_c to be integrated easily.

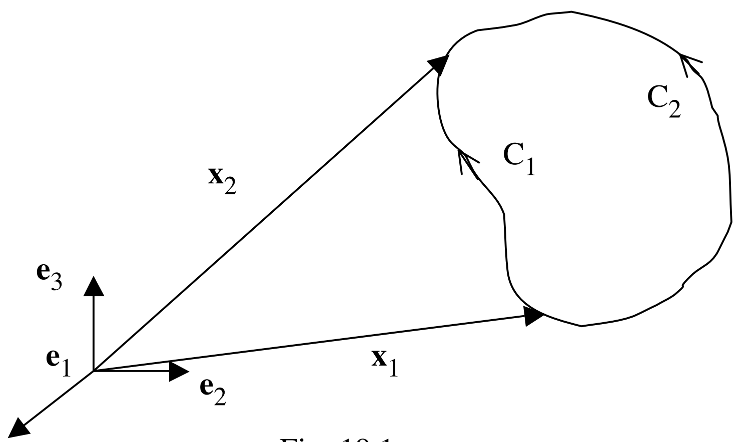

Fig. 19.1

In general a force may be a vector function of position $\mathbf{x}$ and time t. However, if the force is a function of position $\mathbf{x}$ only (independent of time)

$$\mathbf{F}_c = \mathbf{F}_c(\mathbf{x}) \ , \tag{19.3}$$

then the work done by $\mathbf{F}_c$ can be expressed as an integral over position instead of time

$$U_{2/1} = \int_{t_1}^{t_2} \mathbf{F}_c \bullet \mathbf{v} \, dt = \int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{F}_c \bullet d\mathbf{x} \ . \tag{19.4}$$

For a general function $\mathbf{F}_c$ the work done traversing path C_1 from $\mathbf{x}_1$ to $\mathbf{x}_2$ is different from that traversing path C_2 (see Fig. 19.1). However, if the work done by $\mathbf{F}_c$ is path independent

$$\int_{C_1:\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{F}_c \bullet d\mathbf{x} = \int_{C_2:\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{F}_c \bullet d\mathbf{x} \ , \tag{19.5}$$

then $\mathbf{F}_c$ is called a <u>conservative force field</u>. It follows from (19.5) that the work done by a conservative force $\mathbf{F}_c$ depends only on the end points of integration so the integral over an arbitrary closed path vanishes

$$\oint \mathbf{F}_c \bullet d\mathbf{x} = 0 \ . \tag{19.6}$$

It can be shown that $\mathbf{F}_c$ is a conservative force field with (19.6) holding for any closed path if and only if there exists a potential $V(\mathbf{x})$ which is a function of position only such that

$$\mathbf{F}_c = -\nabla V = -\frac{\partial V}{\partial \mathbf{x}} \ , \ F_{ci} = -\frac{\partial V}{\partial x_i} \ , \tag{19.7a,b}$$

where F_{ci} and x_i are the components of $\mathbf{F}_c$ and $\mathbf{x}$, relative to the fixed Cartesian base vectors $\mathbf{e}_i$. Now, substitution of (19.7) into (19.4) yields

$$U_{2/1} = \int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{F}_c \bullet d\mathbf{x} = \int_{\mathbf{x}_1}^{\mathbf{x}_2} [-\frac{\partial V}{\partial \mathbf{x}} \bullet d\mathbf{x}] = \int_{\mathbf{x}_1}^{\mathbf{x}_2} [-dV]$$

$$U_{2/1} = -[\ V(\mathbf{x}_2) - V(\mathbf{x}_1)\ ] = -(V_2 - V_1) \ . \tag{19.8}$$

This means that the work done on the particle by a conservative force equals minus the change in potential energy.

Notice from (19.7a) that since a potential function exists for a conservative force $\mathbf{F}_c$ the conservative force field satisfied the vector equation

$$\nabla \times \mathbf{F}_c = \nabla \times (-\nabla V) = 0 \ . \tag{19.9}$$

It follows that if $\nabla \times \mathbf{F}_c = 0$ and $\mathbf{F}_c$ is independent of time then $\mathbf{F}_c$ is a conservative force field and a potential $V(\mathbf{x})$ exists that is independent of time which is related to $\mathbf{F}_c$ by equation (19.7a). These conditions can be used to check if a given force field is conservative or not. Tables 19.1 and 19.2 summarize useful vector formulas in cylindrical polar coordinates and spherical polar coordinates, respectively.

Notice from (19.2b) and (19.7a) that the mechanical power of a conservative force is equal to minus the rate of change of the potential

$$P_c = \mathbf{F}_c \bullet \mathbf{v} = -\frac{\partial V}{\partial \mathbf{x}} \bullet \dot{\mathbf{x}} = -\dot{V} \ , \tag{19.10}$$

because the potential V is independent of time. In order to understand why a conservative force field must be independent of time we note that if a force field $\mathbf{F}_{nc}(\mathbf{x},t)$ is a function of position and time and yet satisfies the equation

$$\nabla \times \mathbf{F}_{nc}(\mathbf{x},t) = 0 \ , \tag{19.11}$$

then a potential function $V_{nc}(\mathbf{x},t)$ of position and time exits such that

$$\mathbf{F}_{nc}(\mathbf{x},t) = -\nabla\ V_{nc}(\mathbf{x},t) = -\frac{\partial V_{nc}}{\partial \mathbf{x}} \ . \tag{19.12}$$

However, for this case the mechanical power P_{nc} associated with $\mathbf{F}_{nc}$ is not equal to the total derivative of the potential because

$$P_{nc} = \mathbf{F}_{nc} \bullet \mathbf{v} = -\frac{\partial V_{nc}}{\partial \mathbf{x}} \bullet \dot{\mathbf{x}} = -\dot{V}_{nc} + \frac{\partial V_{nc}}{\partial t} \ . \tag{19.13}$$

This means that $\mathbf{F}_{nc}$ is not a conservative force field because the work done traversing a given path depends on the speed at which the path is traversed.

In the following we consider examples of four common conservative force fields and one common nonconservative field.

CONSTANT FORCE FIELD

Let $\mathbf{F}_c = \mathbf{b}$ where $\mathbf{b}$ is a constant vector. Since $\mathbf{F}_c$ is independent of time and independent of space it satisfies equation (19.9) so we know that $\mathbf{F}_c$ is a conservative force field. The functional form for the potential V is obtained by integrating

$$\mathbf{F}_c = -\nabla\ V \ , \ \frac{\partial V}{\partial x_i} = -F_{ci} = -b_i \ , \tag{19.14a,b}$$

to obtain

$$V = V_0 - b_i\, x_i = V_0 - \mathbf{b} \bullet \mathbf{x} \ , \tag{19.15}$$

where V_0 is a constant of integration.

GRAVITATIONAL FORCE BETWEEN TWO MASSES

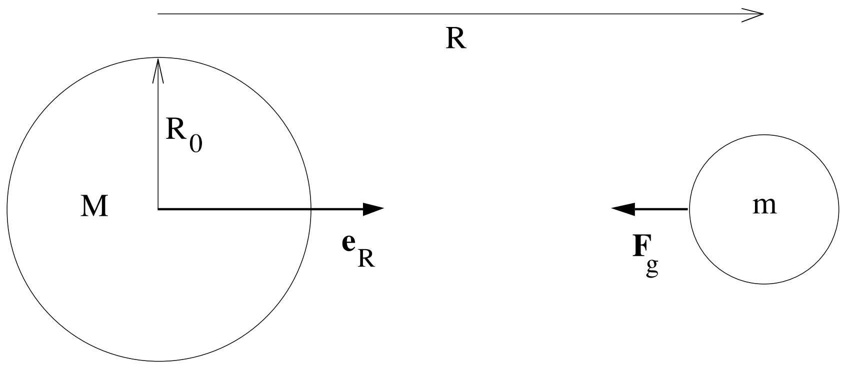

Fig. 19.2

The gravitational force acting between two masses M and m is a central force that acts along the line connecting the center of mass of the two masses. Taking the origin of a spherical coordinate system at the center of mass of the body with mass M, the force $\mathbf{F}_g$ acting on mass m may be represented in the form

$$\mathbf{F}_g = F_{gR}\,\mathbf{e}_R \ , \quad F_{gR} = F_{gR}(R) = -\frac{KMm}{R^2} \ , \qquad (19.16a,b)$$

where K is a universal constant of gravitation. Denoting mg to be the magnitude of the gravitation force at the surface ($R=R_0$) of mass M we have

$$| F_{gR}(R_0) | = \frac{KMm}{R_0^2} = mg \ , \ K = \frac{gR_0^2}{M} \ , \qquad (19.17a,b)$$

so that (19.16b) may be rewritten in the form

$$F_{gR} = -mg\left(\frac{R_0}{R}\right)^2 \ . \qquad (19.18)$$

Now, since $\mathbf{F}_g$ is independent of time and is a central force field it follows from Table 19.2 that equation (19.9) is satisfied so that $\mathbf{F}_g$ is a conservative force field that is related to a potential V_g by the equation

$$\mathbf{F}_g = F_{gR}(R)\,\mathbf{e}_R = -\frac{\partial V_g}{\partial R}\,\mathbf{e}_R \ . \qquad (19.19)$$

Using (19.18) we have

$$V_g = V_\infty - \int_\infty^R F_{gR}(\xi)\, d\xi \;=\; V_\infty + \int_\infty^R \frac{mgR_0^2}{\xi^2}\, d\xi \ , \tag{19.20a}$$

$$V_g = V_\infty - \frac{mgR_0^2}{R} \ , \tag{19.20b}$$

where V_∞ is the value of the gravitational potential V_g at $R=\infty$. It follows that the change in potential energy is given by

$$V_{g2} - V_{g1} = mgR_0^2\left(\frac{1}{R_1} - \frac{1}{R_2}\right) > 0 \ \text{ for } \ R_2 > R_1 \ . \tag{19.21}$$

This means that the gravitational potential increases as the distance between the masses increases.

GRAVITATION CLOSE TO THE EARTH

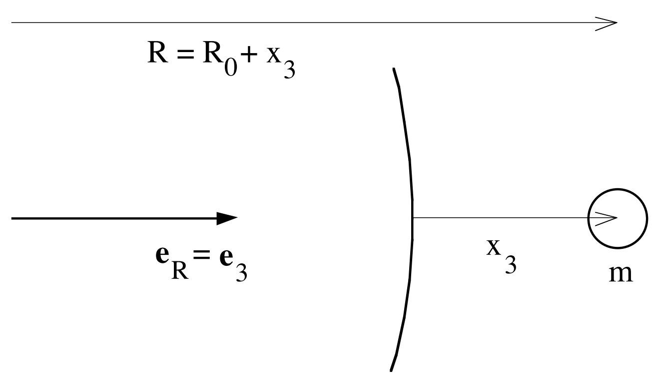

Fig. 19.3

For the gravitational force field close to the surface of the earth the height x_3 above the surface of the earth is quite small relative to the radius R_0 of the earth so that the position R of the mass m may be approximated by

$$R = R_0 + x_3 = R_0\left(1 + \frac{x_3}{R_0}\right) \ , \tag{19.22a}$$

$$\frac{1}{R} \;=\; \frac{1}{R_0\left(1 + \frac{x_3}{R_0}\right)} \;\approx\; \frac{1}{R_0}\left(1 - \frac{x_3}{R_0}\right) \ , \tag{19.22b}$$

so that the gravitational potential (19.20b) may be approximated by

$$V_g = V_\infty - mgR_0\left(1 - \frac{x_3}{R_0}\right) = (V_\infty - mgR_0) + mg\, x_3 \ , \qquad (19.23a)$$

$$V_g = V_0 + mg\, x_3 \ , \ V_0 = V_\infty - mgR_0 \ , \qquad (19.23b,c)$$

where V_0 is the value of the gravitational potential at the surface of the earth. Notice again that the gravitational potential increases as the distance between the mass m and the surface of the earth increases (x_3 increases). At the surface of the earth the force of gravity g per unit mass (often called the acceleration of gravity) is approximately 9.81 m/s^2.

ELASTIC POTENTIAL OF A SPRING

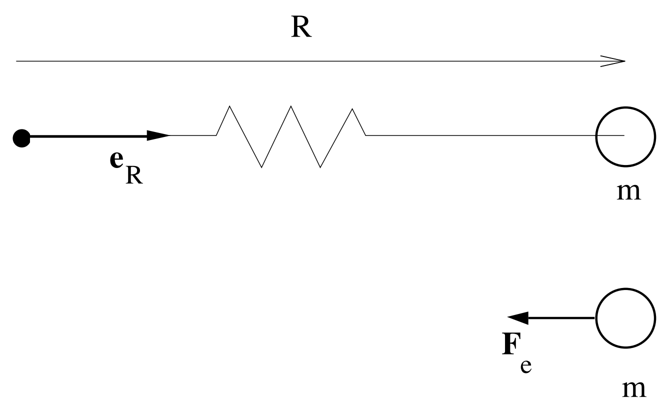

Fig. 19.4

Consider a spring whose free length is L and whose elastic force acting on the mass m is given by

$$\mathbf{F}_e = F_{eR}(R)\, \mathbf{e}_R \ . \qquad (19.24)$$

Since $\mathbf{F}_e$ is independent of time and is a central force field it is a conservative force field with the potential V_e given by

$$V_e = -\int_L^R F_{eR}(\xi)\, d\xi \ , \qquad (19.25)$$

where we have taken $V_e = 0$ when the spring is force free with R=L. For a linear spring we have

$$F_{eR} = -k(R - L) \ , \ V_e = \frac{1}{2} k (R - L)^2 \ . \tag{19.26a,b}$$

NONCONSERVATIVE FORCE FIELD

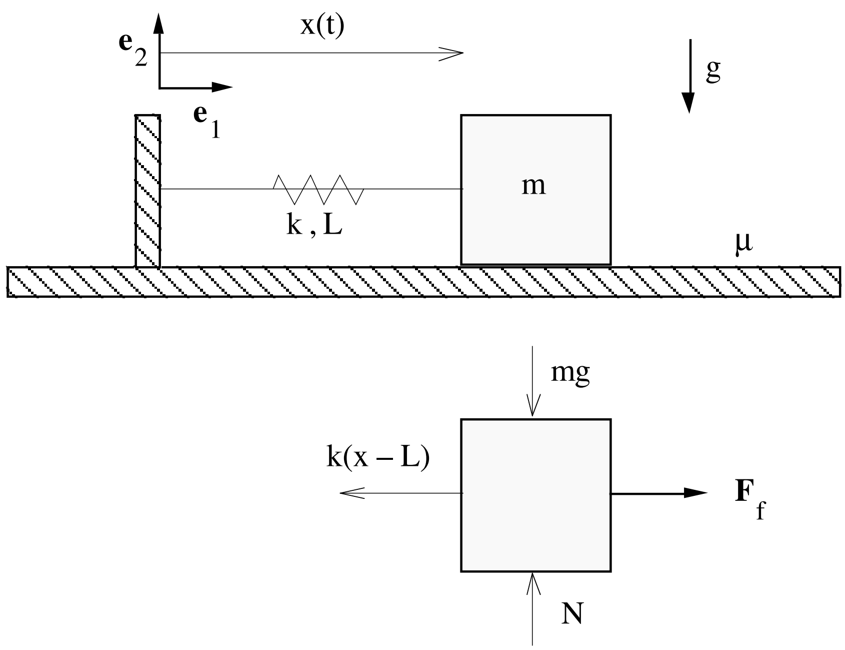

Fig. 19.5

An important example of a nonconservative force field is the force $\mathbf{F}_f$ due to sliding friction. Consider the mass m sliding on a flat plane with friction coefficient μ shown in Fig. 19.4. Since the force of friction always opposes the motion of the mass we may express $\mathbf{F}_f$ in terms of the velocity $\mathbf{v}$ of the mass by the equation

$$\mathbf{F}_f = -\mu N \frac{\mathbf{v}}{|\mathbf{v}|} \quad \text{for} \ \mathbf{v} \neq 0 \ , \tag{19.27}$$

where N is the magnitude of the normal force applied by the plane on the mass. In order to show that the friction force $\mathbf{F}_f$ is nonconservative we calculate the mechanical power P_f due to $\mathbf{F}_f$

$$P_f = \mathbf{F}_f \bullet \mathbf{v} = -\mu N \frac{\mathbf{v} \bullet \mathbf{v}}{|\mathbf{v}|} < 0 \ \text{for} \ \ \mathbf{v} \neq 0 \ . \tag{19.28}$$

Thus the rate of work done by friction on the mass is always negative so energy is always dissipated and the force of friction is nonconservative.

Table 19.1: Cylindrical Polar Coordinates

$$\mathbf{x} = r\,\mathbf{e}_r(\theta) + x_3\,\mathbf{e}_3$$

$$\mathbf{F} = F_r\,\mathbf{e}_r + F_\theta\,\mathbf{e}_\theta + F_3\,\mathbf{e}_3$$

$$\boldsymbol{\nabla} V = \frac{\partial V}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\,\mathbf{e}_\theta + \frac{\partial V}{\partial x_3}\,\mathbf{e}_3$$

$$\boldsymbol{\nabla} \bullet \mathbf{F} = \frac{1}{r}\,\frac{\partial (rF_r)}{\partial r} + \frac{1}{r}\,\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_3}{\partial x_3}$$

$$\boldsymbol{\nabla} \times \mathbf{F} = \left[\frac{1}{r}\frac{\partial F_3}{\partial \theta} - \frac{\partial F_\theta}{\partial x_3}\right]\mathbf{e}_r + \left[\frac{\partial F_r}{\partial x_3} - \frac{\partial F_3}{\partial r}\right]\mathbf{e}_\theta + \left[\frac{1}{r}\frac{\partial (rF_\theta)}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial \theta}\right]\mathbf{e}_3$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial x_3^2}$$

Table 19.2: Spherical Polar Coordinates

$$\mathbf{x} = R\,\mathbf{e}_R(\theta,\phi)$$

$$\mathbf{F} = F_R\,\mathbf{e}_R + F_\theta\,\mathbf{e}_\theta + F_\phi\,\mathbf{e}_\phi$$

$$\boldsymbol{\nabla} V = \frac{\partial V}{\partial R}\,\mathbf{e}_R + \frac{1}{R\cos\phi}\,\frac{\partial V}{\partial \theta}\,\mathbf{e}_\theta + \frac{1}{R}\frac{\partial V}{\partial \phi}\,\mathbf{e}_\phi$$

$$\boldsymbol{\nabla} \bullet \mathbf{F} = \frac{1}{R^2}\,\frac{\partial (R^2 F_R)}{\partial R} + \frac{1}{R\cos\phi}\,\frac{\partial F_\theta}{\partial \theta} + \frac{1}{R\cos\phi}\,\frac{\partial (F_\phi \cos\phi)}{\partial \phi}$$

$$\boldsymbol{\nabla} \times \mathbf{F} = \frac{1}{R\cos\phi}\left[\frac{\partial F_\phi}{\partial \theta} - \frac{\partial (F_\theta \cos\phi)}{\partial \phi}\right]\mathbf{e}_R + \frac{1}{R}\left[\frac{\partial F_R}{\partial \phi} - \frac{\partial (RF_\phi)}{\partial R}\right]\mathbf{e}_\theta$$

$$+ \frac{1}{R\cos\phi}\left[\frac{\partial (RF_\theta \cos\phi)}{\partial R} - \frac{\partial F_R}{\partial \theta}\right]\mathbf{e}_\phi$$

$$\nabla^2 V = \frac{1}{R^2}\,\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\cos^2\phi}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{R^2\cos\phi}\frac{\partial}{\partial \phi}\left(\cos\phi\,\frac{\partial V}{\partial \phi}\right)$$

20. Energy Equation For A Particle

With the help of (18.2) and (18.7) it follows that we can write an energy equation for a particle in the form

$$U_{2/1} = \int_{t_1}^{t_2} \mathbf{F} \bullet \mathbf{v} \, dt = (T_2 - T_1) \ , \tag{20.1}$$

where T_1 and T_2 are values of the kinetic energy T at the times t_1 and t_2, respectively, and $U_{2/1}$ is the work done by <u>all</u> forces acting on the particle. In view of the simple relationship (19.10) between the mechanical power and the time rate of change of the potential associated with a conservative force field it is convenient to separate the total force $\mathbf{F}$ into known conservative force fields $\mathbf{F}_g$ associated with gravity, $\mathbf{F}_e$ associated with elastic springs, and the remainder $\bar{\mathbf{F}}$ such that

$$\mathbf{F} = \mathbf{F}_g + \mathbf{F}_e + \bar{\mathbf{F}} \ , \ \mathbf{F} \bullet \mathbf{v} = -\dot{V}_g - \dot{V}_e + \bar{\mathbf{F}} \bullet \mathbf{v} \ , \tag{20.2a,b}$$

where V_g is the gravitational potential given by (19.20b) or (19.23.b), and V_e is the elastic potential given by (19.26b). Defining the work $\bar{U}_{2/1}$ done by the force $\bar{\mathbf{F}}$

$$\bar{U}_{2/1} = \int_{t_1}^{t_2} \bar{\mathbf{F}} \bullet \mathbf{v} \, dt \ , \tag{20.3}$$

we may rewrite (10.1) in the convenient form

$$\bar{U}_{2/1} = (T_2 - T_1) + (V_{g2} - V_{g1}) + (V_{e2} - V_{e1}) = (E_2 - E_1) \ , \tag{20.4}$$

where the total energy E is equal to the sum of the kinetic energy and the potential energies which are not included in the work done by $\bar{\mathbf{F}}$

$$E = T + V_g + V_e \ . \tag{20.5}$$

Equation (20.5) states that the work done by $\bar{\mathbf{F}}$ <u>on</u> the particle is equal to the change in total energy E. In particular, notice that if all forces acting on the particle are conservative then $\bar{U}_{2/1}$ vanishes and the total energy E remains constant.

When $\bar{U}_{2/1}$ vanishes or the mechanical power $\bar{\mathbf{F}} \bullet \mathbf{v}$ is a simple expression which can be integrated with respect to time, the energy equation (20.4) can be a very convenient equation to use in obtaining the solution of a particular problem. However, we must emphasize that the energy equation (20.4) is merely a scalar integral of the vectorial equations of motion (16.7) and therefore it does not include additional information about the motion of a particle. In this sense the energy equation may be used to replace one of the three scalar equations associated with the balance of linear momentum (16.7) but we must remember that by itself, the energy equation can only provide a single piece of information about the motion of the particle.

21. Linear And Angular Momentum Of A Particle

BALANCE OF LINEAR MOMENTUM

For a single particle it is sometimes convenient to introduce a definition of the linear momentum $\mathbf{G}$ of the particle

$$\mathbf{G} = m\,\mathbf{v} \ , \tag{21.1}$$

and rewrite the balance of linear momentum (16.7) in the form

$$\dot{\mathbf{G}} = \mathbf{F} \ . \tag{21.2}$$

In words equation (21.2) states that the rate of change of linear momentum is equal to the total force acting on the particle. It should be recalled that the velocity $\mathbf{v}$ that appears in the definition (21.1) is the <u>absolute</u> velocity which is measured relative to a <u>fixed</u> point in space.

BALANCE OF ANGULAR MOMENTUM THEOREM

Although the balance of linear momentum (21.2) characterizes the complete motion of a particle it is sometimes desirable to introduce the notion of the angular momentum $\mathbf{H}_o$ of the particle relative to the fixed origin O of a coordinate system by the expression

$$\mathbf{H}_o = \mathbf{x} \times \mathbf{G} \ , \tag{21.3}$$

where $\mathbf{x}(t)$ is the position of the particle relative to the point O. Now differentiating (21.3) with respect to time and using (21.1) and (21.2) we may deduce that

$$\dot{\mathbf{H}}_o = \dot{\mathbf{x}} \times \mathbf{G} + \mathbf{x} \times \dot{\mathbf{G}} = \mathbf{v} \times m\,\mathbf{v} + \mathbf{x} \times \mathbf{F} = \mathbf{x} \times \mathbf{F} \ , \tag{21.4}$$

which means that

$$\dot{\mathbf{H}}_o = \mathbf{M}_o \ , \tag{21.5}$$

where $\mathbf{M}_o$ is the moment of the force $\mathbf{F}$ applied about the fixed point O

$$\mathbf{M}_o = \mathbf{x} \times \mathbf{F} \ . \tag{21.6}$$

In words, the balance of angular momentum (21.5) states that the rate of change of angular momentum about a fixed point is equal to the resultant moment about the same fixed point. It is important to emphasize that for a single particle the balance of angular momentum (21.5) is theorem which has been proved using the balance of linear momentum. In this sense, the balance of angular momentum does not introduce new information about the motion of the particle which is not contained in the balance of linear momentum.

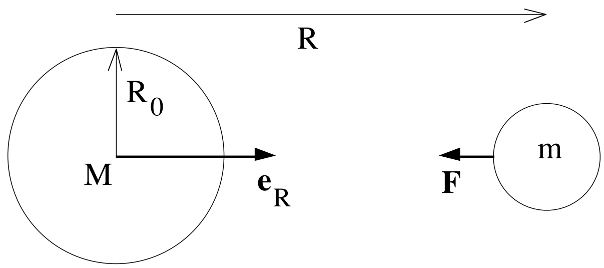

Fig. 21.1

By way of example let us prove that the angular momentum is conserved for a particle that is moving under the action of a central force field only. To this end assume that the earth is fixed in space and that the position of a satellite relative to the center O of the earth is denoted by $\mathbf{x} = R\, \mathbf{e}_R$. Also, let the force of gravity $\mathbf{F}$ applied by the earth on the satellite of mass m be given by $\mathbf{F} = F_R\, \mathbf{e}_R$. It follows that the moment about O of the force $\mathbf{F}$ is given by

$$\mathbf{M}_0 = R\, \mathbf{e}_R \times F_R\, \mathbf{e}_R = 0 \quad . \tag{21.7}$$

Using the balance of angular momentum (21.5) we may deduce that the rate of change of angular momentum vanishes so that $\mathbf{H}_o$ is a constant vector $\mathbf{c}$

$$\mathbf{H}_o = \mathbf{c} \quad . \tag{21.8}$$

However, since

$$\mathbf{H}_o = \mathbf{x} \times m\, \mathbf{v} = \mathbf{c} \quad , \tag{21.9}$$

it follows that $\mathbf{x}$ is always perpendicular to the constant vector $\mathbf{c}$ ($\mathbf{x} \perp \mathbf{c}$) so the particle always moves in the plane which is perpendicular to the constant vector $\mathbf{c}$.

22. Conservation Of Momentum (?)

Sometimes it is easy to observe that one or more components of linear momentum $\mathbf{G}$ or angular momentum $\mathbf{H}_o$ are conserved (remain constant). When this happens part of the equations of motion of a particle integrate simply so it is convenient to use the integrated equations instead of the differential equations. For example, it follows from the balances of linear momentum (21.2) and angular momentum (21.5) that both linear and angular momentum are conserved if the force $\mathbf{F}$ vanishes

$$\mathbf{F} = 0 \;\Rightarrow\; \mathbf{G} = \text{constant} \;, \tag{22.1a}$$

$$\mathbf{F} = 0 \;\Rightarrow\; \mathbf{M}_o = 0 \;\Rightarrow\; \mathbf{H}_o = \text{constant} \;. \tag{22.1b}$$

In view of the simplicity of the equations (22.1a,b) it is reasonable to consider what might happen if only one component of force or moment vanish. In particular, let us consider the case where the component of the force $\mathbf{F}$ in the $\mathbf{b}$ direction vanishes. It follows from (21.2) that

$$\mathbf{b} \bullet \mathbf{F} = 0 \;\Rightarrow\; \dot{\mathbf{G}} \bullet \mathbf{b} = 0 \;. \tag{22.2}$$

The question then arises as to what can be said about the component of linear momentum in the same direction $\mathbf{b}$ that the force component vanishes. To answer this question let us differentiate the component $\mathbf{b} \bullet \mathbf{G}$ to obtain

$$\frac{d(\mathbf{b} \bullet \mathbf{G})}{dt} = \dot{\overline{\mathbf{b} \bullet \mathbf{G}}} = \dot{\mathbf{b}} \bullet \mathbf{G} + \mathbf{b} \bullet \dot{\mathbf{G}} = \dot{\mathbf{b}} \bullet \mathbf{G} \;. \tag{22.3}$$

Equation (22.3) shows that in general the component of linear momentum $\mathbf{G}$ in the direction $\mathbf{b}$ is not constant (conserved) even though the component of force $\mathbf{F}$ in the same direction vanishes. However, if the direction $\mathbf{b}$ is constant so that $\dot{\mathbf{b}}$ vanishes then the component of linear momentum in the direction $\mathbf{b}$ will be conserved. This means that a component of linear or angular momentum is conserved whenever a component of force or moment vanishes in a fixed direction $\mathbf{b}$.

By way of example let us consider the motion of a particle of mass m which moves under the action of a force $\mathbf{F}$ which in cylindrical polar coordinates is expressed in the form

$$\mathbf{F} = F_r \, \mathbf{e}_r \;. \tag{22.4}$$

From the point of view of linear momentum it follows from (22.4) that

$$\mathbf{e}_\theta \bullet \dot{\mathbf{G}} = 0 \;,\; \mathbf{e}_3 \bullet \dot{\mathbf{G}} = 0 \;. \tag{22.5a,b}$$

Since $\mathbf{e}_\theta$ is not a constant vector we do not expect the component $\mathbf{e}_\theta \bullet \mathbf{G}$ to be constant whereas we do expect the component $\mathbf{e}_3 \bullet \mathbf{G}$ to be constant. More specifically, we have

$$\mathbf{x} = r\,\mathbf{e}_r + x_3\,\mathbf{e}_3 \;,\; \mathbf{v} = \dot{r}\;\mathbf{e}_r + r\,\dot{\theta}\,\mathbf{e}_\theta + \dot{x}_3\,\mathbf{e}_3 \;, \tag{22.6a,b}$$

$$\mathbf{G} = m\,(\,\dot{r}\;\mathbf{e}_r + r\,\dot{\theta}\,\mathbf{e}_\theta + \dot{x}_3\,\mathbf{e}_3\,) \;, \tag{22.6c}$$

$$\dot{\mathbf{G}} = m\Big[\,(\ddot{r} - r\,\dot{\theta}^2)\,\mathbf{e}_r + (2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta})\,\mathbf{e}_\theta + \ddot{x}_3\,\mathbf{e}_3\,\Big] \;, \tag{22.6d}$$

Using (22.5) it follows that

$$\mathbf{e}_\theta \bullet \dot{\mathbf{G}} = 0 \;\Rightarrow\; 2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta} = 0 \;\Rightarrow\; \frac{1}{r}\,\frac{d}{dt}(r^2\,\dot{\theta}) = 0 \;, \tag{22.7a}$$

$$\mathbf{e}_3 \bullet \dot{\mathbf{G}} = 0 \;\Rightarrow\; \ddot{x}_3 = 0 \;\Rightarrow\; \frac{d}{dt}(\dot{x}_3) = 0 \;. \tag{22.7b}$$

Consequently,

$$r^2\,\dot{\theta} = \text{constant} \;,\; \dot{x}_3 = \text{constant} \;. \tag{22.8a,b}$$

This means that $\mathbf{e}_\theta \bullet \mathbf{G} = m r\dot{\theta}$ is not necessarily constant whereas $\mathbf{e}_3 \bullet \mathbf{G} = m\dot{x}_3$ is constant. However, the result (22.8a) indicates that something is conserved.

To determine the physical meaning of (22.8a) let us consider the balance of angular momentum. It follows from (22.4) that

$$\mathbf{M}_o = \mathbf{x} \times \mathbf{F} = (r\,\mathbf{e}_r + x_3\,\mathbf{e}_3) \times F_r\,\mathbf{e}_r = x_3\,F_r\,\mathbf{e}_\theta \;. \tag{22.9}$$

Thus there is no moment the $\mathbf{e}_r$ and $\mathbf{e}_3$ directions so we expect the component of angular momentum in the $\mathbf{e}_3$ direction to be preserved. To see this consider

$$\mathbf{H}_o = \mathbf{x} \times m\,\mathbf{v} = (r\,\mathbf{e}_r + x_3\,\mathbf{e}_3) \times m\,(\,\dot{r}\;\mathbf{e}_r + r\,\dot{\theta}\,\mathbf{e}_\theta + \dot{x}_3\,\mathbf{e}_3\,) \;, \tag{22.10a}$$

$$\mathbf{H}_o = m\Big[\,-r\,\dot{\theta}x_3\,\mathbf{e}_r + (\dot{r}\,x_3 - r\,\dot{x}_3)\,\mathbf{e}_\theta + (r^2\,\dot{\theta})\,\mathbf{e}_3\,\Big] \;. \tag{22.10b}$$

Using (22.9) and (22.10b) we have

$$\mathbf{e}_3 \bullet \mathbf{M}_o = 0 \;\Rightarrow\; \mathbf{e}_3 \bullet \mathbf{H}_o = \text{constant} \;\Rightarrow\; r^2\dot{\theta} = \text{constant} \;, \qquad (22.11)$$

which shows that the result (22.8b) indicates that angular momentum about the $\mathbf{e}_3$ axis is conserved.

23. Impulse And Momentum

In view of the simplicity of the balance of linear momentum (21.2) and the balance of angular momentum (21.5) it follows that the changes in momentum from time t_1 to time t_2 can be determined by integrating these balance laws over time. In particular, it is convenient to define $\hat{\mathbf{F}}$ as the impulsive force due to $\mathbf{F}$

$$\hat{\mathbf{F}} = \int_{t_1}^{t_2} \mathbf{F}\, dt \ , \tag{23.1}$$

and use the balance of linear momentum (21.2) to deduce that

$$\hat{\mathbf{F}} = \int_{t_1}^{t_2} \dot{\mathbf{G}}\, dt = \mathbf{G}_2 - \mathbf{G}_1 \ . \tag{23.2}$$

This states that the impulsive force is equal to the change in linear momentum.

Similarly, we can define $\hat{\mathbf{M}}_o$ as the impulsive moment due to $\mathbf{M}_o$

$$\hat{\mathbf{M}}_o = \int_{t_1}^{t_2} \mathbf{M}_o\, dt \ , \tag{23.3}$$

and use the balance of angular momentum (21.5) to deduce that

$$\hat{\mathbf{M}}_o = \int_{t_1}^{t_2} \dot{\mathbf{H}}_o\, dt = \mathbf{H}_{o2} - \mathbf{H}_{o1} \ . \tag{23.4}$$

This states that the impulsive moment about a fixed point is equal to the change in angular momentum about the same fixed point.

By way of example consider the one-dimensional problem of the impact of two masses m_A and m_B. Before impact mass m_A moves with constant velocity v_0 towards mass m_B and mass m_B is stationary. After impact we assume that both masses move together with common velocity v (see Fig. 23.1). Neglecting friction, the only forces acting on the two masses are gravity, and the contact force between themselves and the floor. Let $\mathbf{F}_{A/B}$ be the force applied by mass m_B on mass m_A and let $\mathbf{F}_{B/A}$ be the force applied by mass m_A on mass m_B. Since by Newton's third law these forces must be equal in magnitude and opposite in direction we have

$$\mathbf{F}_{A/B} = -F\,\mathbf{e}_1 \ , \ \mathbf{F}_{B/A} = F\,\mathbf{e}_1 \ , \tag{23.5a,b}$$

where F is the magnitude of the force $\mathbf{F}_{B/A}$. The free body diagrams of each of the masses are shown in Fig. 23.2.

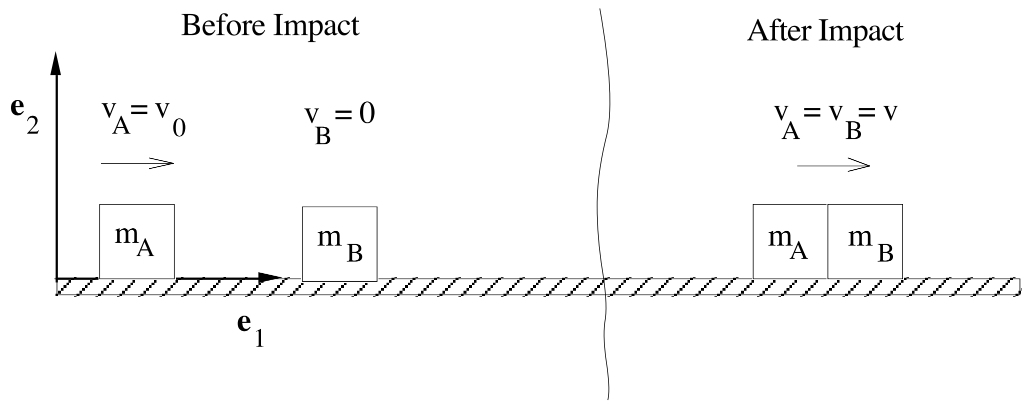

Fig. 23.1

From these free body diagrams it follows that the total force $\mathbf{F}_A$ applied to mass m_A may be written in the form

$$\mathbf{F}_A = -F\,\mathbf{e}_1 + (N_A - m_A g)\,\mathbf{e}_2 \ . \tag{23.6}$$

Similarly, the total force $\mathbf{F}_B$ applied to mass m_B may be written in the form

$$\mathbf{F}_B = F\,\mathbf{e}_1 + (N_B - m_B g)\,\mathbf{e}_2 \ . \tag{23.7}$$

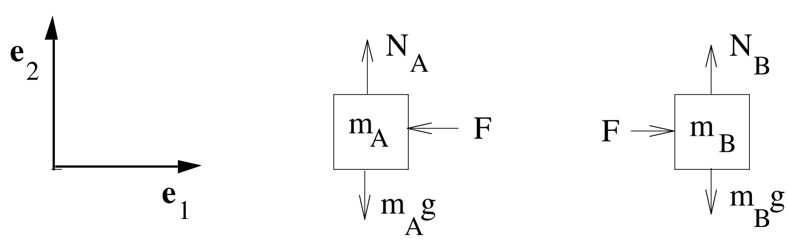

Fig. 23.2

Assuming that the impact occurs over the time interval $[t_1,t_2]$ we may apply equation (23.2) to each of the masses and deduce that

$$\hat{\mathbf{F}}_A = \mathbf{G}_{A2} - \mathbf{G}_{A1} = m_A\,(v - v_0)\,\mathbf{e}_1 \ , \tag{23.8a}$$

$$\hat{\mathbf{F}}_B = \mathbf{G}_{B2} - \mathbf{G}_{B1} = m_B \, v \, \mathbf{e}_1 \ , \tag{23.8b}$$

Next we consider the determination of the impulsive force $\hat{\mathbf{F}}_A$. By definition

$$\hat{\mathbf{F}}_A = \int_{t_1}^{t_2} \left[- F \, \mathbf{e}_1 + (N_A - m_A g) \, \mathbf{e}_2 \right] dt \ . \tag{23.9}$$

During the impact the forces N_A and $m_A g$ remain bounded but the force F may become quite large. As a simple approximation we can assume that the impact occurs over an infinitesimally short time so the impulse due to N_A and $m_A g$ vanish but the impulse due to F does not so that

$$\hat{\mathbf{F}}_A = \underset{t_2 \to t_1}{\text{limit}} \int_{t_1}^{t_2} \left[- F \, \mathbf{e}_1 + (N_A - m_A g) \, \mathbf{e}_2 \right] dt$$

$$= - \underset{t_2 \to t_1}{\text{limit}} \int_{t_1}^{t_2} F \, \mathbf{e}_1 \, dt = - \hat{F} \, \mathbf{e}_1 \ . \tag{23.10}$$

Similarly the impulsive force $\hat{\mathbf{F}}_B$ becomes

$$\hat{\mathbf{F}}_B = \underset{t_2 \to t_1}{\text{limit}} \int_{t_1}^{t_2} F \, \mathbf{e}_1 \, dt = \hat{F} \, \mathbf{e}_1 \ . \tag{23.11}$$

Now, substitution of the results (23.10) and (23.11) into the equations (23.8a,b) we may deduce that

$$\hat{F} = - m_A \, (v - v_0) \ , \ \hat{F} = m_B \, v \ . \tag{23.12a,b}$$

Solving these equations for v and $\hat{F}$ we have

$$v = \left(\frac{m_A}{m_A + m_B} \right) v_0 \ , \ \hat{F} = \left(\frac{m_A \, m_B}{m_A + m_B} \right) v_0 \ . \tag{23.13a,b}$$

Notice that if m_B is small relative to m_A then $v \to v_0$ and $\hat{F} \to 0$, whereas if m_B is much greater than m_A then $v \to 0$ and $\hat{F} \to m_A \, v_0$.

In the above analysis we have considered the dynamics of each mass separately. However, sometimes it is of interest to consider the properties of the system of two masses. To this end, let **G** be the sum of the linear momentum of each of the masses so that before impact we have

$$\mathbf{G} = \mathbf{G}_1 = m_A\, v_0\, \mathbf{e}_1 \ , \tag{23.14}$$

and after impact we have

$$\mathbf{G} = \mathbf{G}_2 = (m_A\, v + m_B\, v)\, \mathbf{e}_1 = (m_A + m_B) \left(\frac{m_A}{m_A + m_B} \right) v_0\, \mathbf{e}_1 = m_A\, v_0\, \mathbf{e}_1 \ . \tag{23.15}$$

This means that the total linear momentum of the system of two masses did not change during the impact. We can also consider what happens to the total kinetic energy T, which is the sum of the kinetic energies of the two masses. Before impact we have

$$T = T_1 = \frac{1}{2}\ m_A\, v_0^2 \ , \tag{23.16}$$

whereas after impact we have

$$T = T_2 = \frac{1}{2}\ m_A\, v^2 + \frac{1}{2}\ m_B\, v^2 = \left(\frac{m_A}{m_A + m_B} \right) \frac{1}{2}\, m_A\, v_0^2 \ , \tag{23.17}$$

so that kinetic energy is lost during the impact

$$T_2 - T_1 = -\left(\frac{m_B}{m_A + m_B} \right) \frac{1}{2}\, m_A\, v_0^2 < 0 \ . \tag{23.18}$$

In a more complete analysis it could be shown that kinetic energy is converted into heat during the dissipative process which causes the two masses to stick together after impact.

24. Kinetics Of Systems Of Particles

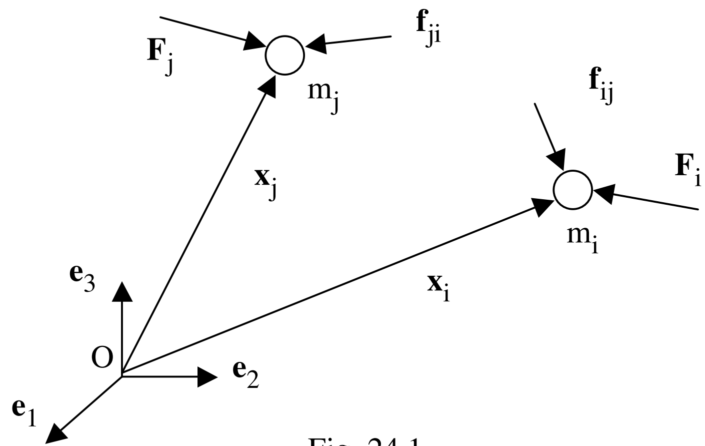

Fig. 24.1

Consider a set of N particles and let a typical particle, called the i'th particle have mass m_i, and position $\mathbf{x}_i$, relative to a fixed origin. Furthermore, let $\mathbf{F}_i$ be the external force (external to the system) applied to the i'th particle and let $\mathbf{f}_{ij}$ be the internal force applied by the j'th particle on the i'th particle. Then the balance of linear momentum for the i'th particle may be expressed in the form

$$\dot{\mathbf{G}}_i = \frac{d}{dt}(m_i\, \mathbf{v}_i) = \mathbf{F}_i + \sum_{j=1}^{N} \mathbf{f}_{ij} \quad i=1,2,...,N \text{ (no sum on i)} \ . \tag{24.1}$$

In the above it is assumed that $\mathbf{f}_{ii}$ vanishes (i.e. the i'th particle does not apply an internal force on itself). Notice that the N vector equations of motion (24.1) are sufficient to determine the motion of all the N particles in the system. Sometimes it is convenient to determine what can be said about the system as a whole. To this end, let us define $\mathbf{F}$ to be the total external force applied to the system, $\mathbf{f}$ to be the total internal force, and $\mathbf{G}$ to be the total linear momentum of the system

$$\mathbf{G} = \sum_{i=1}^{N} \mathbf{G}_i \ , \ \mathbf{F} = \sum_{i=1}^{N} \mathbf{F}_i \ , \ \mathbf{f} = \sum_{i=1}^{N}\sum_{j=1}^{N} \mathbf{f}_{ij} \ . \tag{24.2a,b,c}$$

It follows by summing the equations (24.1) that we may write

$$\dot{\mathbf{G}} = \mathbf{F} + \mathbf{f} \ , \tag{24.3}$$

which is a single vector equation characterizing the balance of linear momentum of the system of particles.

A vector equation representing the balance of angular momentum of the system of particles can be obtained by defining the total moment $\mathbf{M}_o$ of external forces about the fixed point O, the total moment $\mathbf{m}_o$ of internal forces about O, and the total angular momentum $\mathbf{H}_o$ about O

$$\mathbf{H}_o = \sum_{i=1}^{N} \mathbf{x}_i \times \mathbf{G}_i \ , \tag{24.4a}$$

$$\mathbf{M}_o = \sum_{i=1}^{N} \mathbf{x}_i \times \mathbf{F}_i \ , \ \mathbf{m}_o = \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{x}_i \times \mathbf{f}_{ij} \ , \tag{24.4b,c}$$

Now, taking the cross product of each of equations (24.1) with the position vector $\mathbf{x}_i$ and summing the resulting equations we may deduce that

$$\dot{\mathbf{H}}_o = \mathbf{M}_o + \mathbf{m}_o \ . \tag{24.5}$$

Before continuing let us reconsider the quantities $\mathbf{f}$ and $\mathbf{m}_o$. In particular by changing the indices for the summations we have

$$\mathbf{f} = \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{f}_{ij} = \sum_{j=1}^{N} \sum_{i=1}^{N} \mathbf{f}_{ji} \ , \tag{24.6a}$$

$$\mathbf{m}_o = \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{x}_i \times \mathbf{f}_{ij} = \sum_{j=1}^{N} \sum_{i=1}^{N} \mathbf{x}_j \times \mathbf{f}_{ji} \ . \tag{24.6b}$$

Now, interchanging the order of the summations we may deduce that

$$\mathbf{f} = \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{f}_{ij} = \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{f}_{ji} \ , \tag{24.7a}$$

$$\mathbf{m}_o = \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{x}_i \times \mathbf{f}_{ij} = \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{x}_j \times \mathbf{f}_{ji} \ . \tag{24.7b}$$

By expressing $\mathbf{f}$ and $\mathbf{m}_o$ as the averages of the two representations in (24.7a,b) we may write

$$\mathbf{f} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (\mathbf{f}_{ij} + \mathbf{f}_{ji}) \quad , \tag{24.8a}$$

$$\mathbf{m}_o = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (\mathbf{x}_i \times \mathbf{f}_{ij} + \mathbf{x}_j \times \mathbf{f}_{ji}) \ . \tag{24.8b}$$

Now, in view of Newton's third law, the force $\mathbf{f}_{ij}$ applied by the mass m_j on the mass m_i is equal in magnitude and opposite in direction to the force $\mathbf{f}_{ji}$ applied by the mass m_i on the mass m_j so that

$$\mathbf{f}_{ij} = - \mathbf{f}_{ji} \ . \tag{24.9}$$

Substituting (24.9) into (24.8) we have

$$\mathbf{f} = 0 \ , \ \mathbf{m}_o = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (\mathbf{x}_i - \mathbf{x}_j) \times \mathbf{f}_{ij} \ . \tag{24.10a,b}$$

Using the result (24.10a) together with equation (24.3) we obtain the equation

$$\dot{\mathbf{G}} = \mathbf{F} \quad , \tag{24.11}$$

which states that the resultant external force applied to the system of particles is equal to the rate of change of linear momentum of the system. Since equation (24.11) is identical to the balance of linear momentum (21.2) of a single particle it is sometimes convenient to think of the system of particles as a single particle with total mass m located at the center of mass $\bar{\mathbf{x}}$, such that

$$m = \sum_{i=1}^{N} m_i \ , \ m \, \bar{\mathbf{x}} = \sum_{i=1}^{N} m_i \, \mathbf{x}_i \ . \tag{24.12a,b}$$

Notice that the location of the center of mass of the system is defined as the mass-weighted average of the position of the particles. Furthermore, the linear momentum $\mathbf{G}$ and rate of change of linear momentum $\dot{\mathbf{G}}$ may be expressed in terms of the velocity $\bar{\mathbf{v}} = \dot{\bar{\mathbf{x}}}$ and acceleration $\bar{\mathbf{a}} = \dot{\bar{\mathbf{v}}}$ of the center of mass of the system

$$\mathbf{G} = \sum_{i=1}^{N} m_i \, \mathbf{v}_i = m \, \bar{\mathbf{v}} \quad , \ \dot{\mathbf{G}} = \sum_{i=1}^{N} m_i \, \mathbf{a}_i = m \, \bar{\mathbf{a}} \ . \tag{24.13a,b}$$

Thus, the balance of linear momentum may be written in the form

$$\dot{\mathbf{G}} = \frac{d}{dt}(m\,\bar{\mathbf{v}}) = m\,\bar{\mathbf{a}} = \mathbf{F} \quad . \tag{24.14}$$

Equation (24.14) is sometimes called the Principle of Motion of the Center of Mass. In words it states that relative to an inertial frame of reference the mass center of a system of particles moves as though the system were a single particle of mass equal to the total mass of the system moving under the action of the total external force applied to the system. It is important to note that (24.14) is only a single vector equation so it can only provide limited information about the motion of the system of N particles.

Notice that even with the help of Newton's third law (24.9) the internal moment $\mathbf{m}_o$ does not necessarily vanish (24.10b). Nevertheless, it is of interest to consider three cases when the internal moment $\mathbf{m}_o$ vanishes:

Case I: A simple system of particles with the internal forces $\mathbf{f}_{ij}$ which are central forces that act along the line of centers of the masses so that

$$\mathbf{f} = f(x_{ij})\,(\mathbf{x}_i - \mathbf{x}_j) \quad , \quad x_{ij} = (\mathbf{x}_i - \mathbf{x}_j) \bullet (\mathbf{x}_i - \mathbf{x}_j) \ , \quad \text{(no sum on i,j)} \tag{24.15a,b}$$

where x_{ij} is the square of the distance between the masses m_i and m_j and the function $f(x_{ij})$ characterizes the force acting between the masses m_i and m_j.

Case II: Forces perpendicular to the line of centers of the masses are allowed but the masses are assumed to be in contact so that for some values of i and j we have

$$\mathbf{x}_i = \mathbf{x}_j \quad . \tag{24.16}$$

Case III: The trivial case when there are no internal forces

$$\mathbf{f}_{ij} = 0 \quad . \tag{24.17}$$

For each of these cases the internal moment $\mathbf{m}_o$ vanishes and the balance of angular momentum of the system of particles (24.5) reduces to

$$\dot{\mathbf{H}}_o = \mathbf{M}_o \quad , \tag{24.18}$$

which is identical to the balance of angular momentum (21.5) of a single particle. However, in contrast with the angular momentum equation for a single particle which was derived as a theorem, the balance of angular momentum (24.18) contains independent information about the system of particles. For example it gives information about the rotation of the system of particles about its center of mass.

25. Alternative Formulation of the Balance Laws

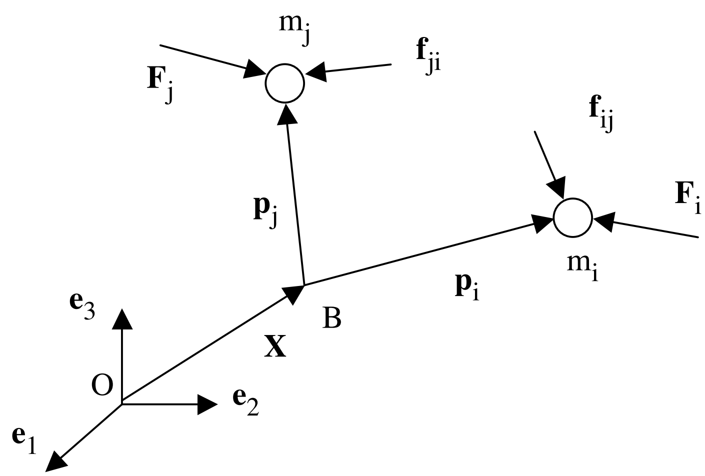

Fig. 25.1

It is often convenient to formulate the balances of linear and angular momentum relative to an arbitrary moving point B. To this end, let $\mathbf{X}$ denote the location of point B relative to a fixed origin O and let $\mathbf{p}_i$ be the location of mass m_i relative to B. It follows that the position $\mathbf{r}_i$, velocity $\mathbf{v}_i$ and acceleration $\mathbf{a}_i$ of the mass m_i are given by

$$\mathbf{x}_i = \mathbf{X} + \mathbf{p}_i \ , \ \mathbf{v}_i = \dot{\mathbf{X}} + \dot{\mathbf{p}}_i \ , \ \mathbf{a}_i = \ddot{\mathbf{X}} + \ddot{\mathbf{p}}_i \ . \tag{25.1a,b,c}$$

The objective here is to use the balance laws (24.11) and (24.18) to obtain alternative forms expressed in terms of quantities referred to the moving point B.

Recalling the definition (24.2a) of the linear momentum of the system of particles we may write

$$\mathbf{G} = \sum_{i=1}^{N} m_i \, \mathbf{v}_i = \sum_{i=1}^{N} m_i \, (\dot{\mathbf{X}} + \dot{\mathbf{p}}_i) = m \, (\dot{\mathbf{X}} + \dot{\bar{\mathbf{p}}}) \ , \tag{25.2}$$

where m is the total mass of the system and $\bar{\mathbf{p}}$ denotes the location of the center of mass of the system relative to the point B

$$m \, \bar{\mathbf{p}} = \sum_{i=1}^{N} m_i \, \mathbf{p}_i \ . \tag{25.3}$$

Thus, the absolute position of the center of mass $\bar{\mathbf{x}}$ and velocity of the center of mass $\bar{\mathbf{v}}$ become

$$\bar{\mathbf{x}} = \mathbf{X} + \bar{\mathbf{p}} \ , \ \bar{\mathbf{v}} = \dot{\mathbf{X}} + \dot{\bar{\mathbf{p}}} \ . \tag{25.4a,b}$$

It follows that the balance of linear momentum may be written in the alternative form

$$\mathrm{m}\,(\ddot{\mathbf{X}} + \ddot{\bar{\mathbf{p}}}) = \mathbf{F} \ . \tag{25.5}$$

It is important to notice that the acceleration which appears in (25.5) is the <u>absolute</u> acceleration and not the acceleration $\ddot{\bar{\mathbf{p}}}$ of the center of mass of the system relative to the moving point B. In this sense, the form of the balance of linear momentum is changed when it is referred to a moving point instead of a fixed point. Also notice that when the point B moves with constant velocity the balance of linear momentum reduces to

$$\mathrm{m}\,\ddot{\bar{\mathbf{p}}} = \mathbf{F} \ \text{ for } \ \dot{\mathbf{X}} = \text{constant} \ . \tag{25.6}$$

This means that the balance of linear momentum remains invariant (unchanged in form) to a superposed constant velocity $\dot{\mathbf{X}}$.

To develop the alternative form of the balance of angular momentum we substitute (25.1) into the definition (24.2a) and obtain the expressions

$$\mathbf{H}_o = \sum_{i=1}^{N} (\mathbf{X} + \mathbf{p}_i) \times \mathrm{m_i}\, \mathbf{v}_i = \mathbf{X} \times \sum_{i=1}^{N} \mathrm{m_i}\, \mathbf{v}_i + \sum_{i=1}^{N} \mathbf{p}_i \times \mathrm{m_i}\, \mathbf{v}_i \ ,$$

$$\mathbf{H}_o = \mathbf{X} \times \mathrm{m}\, \bar{\mathbf{v}} + \sum_{i=1}^{N} \mathbf{p}_i \times \mathrm{m_i}\, (\dot{\mathbf{X}} + \dot{\mathbf{p}}_i) \ ,$$

$$\mathbf{H}_o = \mathbf{X} \times \mathrm{m}\, \bar{\mathbf{v}} + \left(\sum_{i=1}^{N} \mathrm{m_i}\, \mathbf{p}_i \right) \times \dot{\mathbf{X}} + \sum_{i=1}^{N} \mathbf{p}_i \times \mathrm{m_i}\, \dot{\mathbf{p}}_i \ ,$$

$$\mathbf{H}_o = \mathbf{X} \times \mathrm{m}\, \bar{\mathbf{v}} + \mathrm{m}\, \bar{\mathbf{p}} \times \dot{\mathbf{X}} + \mathbf{H}_B \ , \tag{25.7}$$

where the relative angular momentum $\mathbf{H}_B$ about the point B is defined by replacing the absolute position $\mathbf{r}_i$ and velocity $\mathbf{v}_i$ in (24.4a) by the relative position $\mathbf{p}_i$ and relative velocity $\dot{\mathbf{p}}_i$ so that

$$\mathbf{H}_B = \sum_{i=1}^{N} \mathbf{p}_i \times m_i \dot{\mathbf{p}}_i \,. \tag{25.8}$$

Furthermore, the derivative of $\mathbf{H}_o$ may be expressed in the form

$$\dot{\mathbf{H}}_o = \dot{\mathbf{X}} \times m\,\bar{\mathbf{v}} + \mathbf{X} \times m\,\bar{\mathbf{a}} + m\,\dot{\bar{\mathbf{p}}} \times \dot{\mathbf{X}} + m\,\bar{\mathbf{p}} \times \ddot{\mathbf{X}} + \dot{\mathbf{H}}_B \;,$$

$$\dot{\mathbf{H}}_o = \mathbf{X} \times m\,\bar{\mathbf{a}} + m\,\bar{\mathbf{p}} \times \ddot{\mathbf{X}} + \dot{\mathbf{H}}_B + \dot{\mathbf{X}} \times m\,(\bar{\mathbf{v}} - \dot{\bar{\mathbf{p}}}) \;,$$

$$\dot{\mathbf{H}}_o = \mathbf{X} \times \mathbf{F} + m\,\bar{\mathbf{p}} \times \ddot{\mathbf{X}} + \dot{\mathbf{H}}_B + \dot{\mathbf{X}} \times m\,\dot{\mathbf{X}} \,,$$

$$\dot{\mathbf{H}}_o = \mathbf{X} \times \mathbf{F} + m\,\bar{\mathbf{p}} \times \ddot{\mathbf{X}} + \dot{\mathbf{H}}_B \;, \tag{25.9}$$

where use has been made of (24.14) and (25.4b). Now, substituting (25.1) into (24.4b) the expression for the moment $\mathbf{M}_o$ about the origin becomes

$$\mathbf{M}_o = \sum_{i=1}^{N} (\mathbf{X} + \mathbf{p}_i) \times \mathbf{F}_i = \mathbf{X} \times \left(\sum_{i=1}^{N} \mathbf{F}_i \right) + \sum_{i=1}^{N} \mathbf{p}_i \times \mathbf{F}_i \;,$$

$$\mathbf{M}_o = \mathbf{X} \times \mathbf{F} + \mathbf{M}_B \;, \tag{25.10}$$

where the moment $\mathbf{M}_B$ of the external forces about the point B is defined by

$$\mathbf{M}_B = \sum_{i=1}^{N} \mathbf{p}_i \times \mathbf{F}_i \;. \tag{25.11}$$

In words, equation (25.10) states that the moment about the origin is equal to the sum of the moment of the resultant force $\mathbf{F}$ as if it were applied at the point B, and the moment $\mathbf{M}_B$ of the external forces about the point B. Finally, substitution of (25.9) and (25.11) into (24.18) yields the balance of angular momentum in the alternative form

$$\dot{\mathbf{H}}_B + m\,\bar{\mathbf{p}} \times \ddot{\mathbf{X}} = \mathbf{M}_B \;. \tag{25.12}$$

Notice that for a general moving point B the balance of angular momentum changes form. However, for the special case when $\bar{\mathbf{p}} \times \ddot{\mathbf{X}}$ vanishes, the balance of angular momentum remains invariant with

$$\dot{\mathbf{H}}_B = \mathbf{M}_B \quad \text{for} \;\; \bar{\mathbf{p}} \times \ddot{\mathbf{X}} = 0 \,. \tag{25.13}$$

This happens for the following three cases.

Case I: The point B moves with constant velocity

$$\dot{\mathbf{X}} = \text{constant} \ , \tag{25.14}$$

so the balance of angular momentum remains invariant to a superposed constant velocity.

Case II: The point B is the center of mass of the system

$$\bar{\mathbf{p}} = 0 \ , \tag{25.15}$$

and the balance of angular momentum may be written in the form

$$\dot{\bar{\mathbf{H}}} = \bar{\mathbf{M}} \ , \tag{25.16}$$

where for convenience we have denoted the value of $\mathbf{H}_B$ by $\bar{\mathbf{H}}$ and the value of $\mathbf{M}_B$ by $\bar{\mathbf{M}}$, so that $\bar{\mathbf{H}}$ is the relative angular momentum about the center of mass and $\bar{\mathbf{M}}$ is the moment due to external forces about the center of mass. It is important to emphasize that equation (25.16) holds even when the center of mass accelerates.

Case III: The point B accelerates towards or away from the center of mass

$$\ddot{\mathbf{X}} \parallel \bar{\mathbf{p}} \ . \tag{25.17}$$

Before closing this section it is convenient to use the result (25.7) and consider the special case when B is taken to be the center of mass with

$$\mathbf{X} = \bar{\mathbf{x}} \ , \quad \bar{\mathbf{p}} = 0 \ , \quad \mathbf{H}_B = \bar{\mathbf{H}} \ , \tag{25.18a,b,c}$$

to derive the result that

$$\mathbf{H}_o = (\bar{\mathbf{x}} \times m\, \bar{\mathbf{v}}) + \bar{\mathbf{H}} \ .$$

This means that the angular momentum about the origin O is equal to the sum of the angular momentum of the center of mass $(\bar{\mathbf{x}} \times m\, \bar{\mathbf{v}})$ about O and the angular momentum $\bar{\mathbf{H}}$ of the system about the center of mass.

26. Impulse And Momentum (System Of Particles)

In view of the simplicity of the balance of linear momentum (24.11) and the two forms of the balance of angular momentum (24.18) and (25.16) it follows that changes in momentum from time t_1 to time t_2 can be determined by integrating these balance laws. In particular we can define the impulsive force $\hat{\mathbf{F}}$ and impulsive moments $\hat{\mathbf{M}}_o$ and $\hat{\bar{\mathbf{M}}}$ such that

$$\hat{\mathbf{F}} = \int_{t_1}^{t_2} \mathbf{F}\, dt \quad , \tag{26.1a}$$

$$\hat{\mathbf{M}}_o = \int_{t_1}^{t_2} \mathbf{M}_o\, dt \quad , \tag{26.1b}$$

$$\hat{\bar{\mathbf{M}}} = \int_{t_1}^{t_2} \bar{\mathbf{M}}\, dt \quad . \tag{26.1c}$$

so that integration of these balance laws yields the equations

$$\hat{\mathbf{F}} = \int_{t_1}^{t_2} \dot{\mathbf{G}}\, dt = \mathbf{G}_2 - \mathbf{G}_1 \quad , \tag{26.2a}$$

$$\hat{\mathbf{M}}_o = \int_{t_1}^{t_2} \dot{\mathbf{H}}_o\, dt = \mathbf{H}_{o2} - \mathbf{H}_{o1} \quad , \tag{26.2b}$$

$$\hat{\bar{\mathbf{M}}} = \int_{t_1}^{t_2} \dot{\bar{\mathbf{H}}}\, dt = \bar{\mathbf{H}}_2 - \bar{\mathbf{H}}_1 \quad . \tag{26.2c}$$

27. Mechanical Power And Kinetic Energy (System Of Particles)

In order to discuss the energy equation for a system of particles we first define the mechanical power P_i and the kinetic energy T_i associated with mass m_i by the formulas

$$P_i = \left(\mathbf{F}_i + \sum_{j=1}^{N} \mathbf{f}_{ij} \right) \bullet \mathbf{v}_i \;\; , \;\; T_i = \frac{1}{2} \;\; m_i \, \mathbf{v}_i \bullet \mathbf{v}_i \;\; . \quad \text{(no sum on i)} \qquad (27.1a,b)$$

Recalling the balance of linear momentum (24.1) for each mass it follows that the mechanical power is equal to the rate of change of kinetic energy of the mass m_i so that

$$P_i = \dot{T}_i \;\; , \qquad (27.2)$$

It now is convenient to define the total mechanical power P and the total kinetic energy T of the system of particles by

$$P = \sum_{i=1}^{N} P_i \;\; , \quad T = \sum_{i=1}^{N} T_i \;\; , \qquad (27.3a,b)$$

so that

$$P = \dot{T} \quad . \qquad (27.4)$$

Notice that this is the same result as that obtained for a single particle (18.5). It follows that when some of the external forces applied to the system of particles are conservative forces then we can write an energy equation for the system of particles. In particular if we separate the external effects of gravity and springs we can write an energy equation of the type (20.4) for the system of particles

$$\bar{U}_{2/1} = (T_2 - T_1) + (V_{g2} - V_{g1}) + (V_{e2} - V_{e1}) \;\; . \qquad (27.5)$$

By representing the motion of each mass relative to the center of mass $\bar{\mathbf{x}}$ of the system of particles

$$\mathbf{x}_i = \bar{\mathbf{x}} + \mathbf{p}_i \;\; , \;\; \mathbf{v}_i = \bar{\mathbf{v}} + \dot{\mathbf{p}}_i \;\; , \qquad (27.6a,b)$$

we may rewrite the kinetic energy in the form

$$T = \sum_{i=1}^{N} \frac{1}{2} \;\; m_i \, (\mathbf{v}_i \bullet \mathbf{v}_i) = \sum_{i=1}^{N} \frac{1}{2} \;\; m_i \, (\bar{\mathbf{v}} + \dot{\mathbf{p}}_i) \bullet (\bar{\mathbf{v}} + \dot{\mathbf{p}}_i) \;\; ,$$

$$T = \sum_{i=1}^{N} \frac{1}{2} \; m_i \left[(\bar{\mathbf{v}} \bullet \bar{\mathbf{v}}) + 2\,(\bar{\mathbf{v}} \bullet \dot{\mathbf{p}}_i) + (\dot{\mathbf{p}}_i \bullet \dot{\mathbf{p}}_i) \right] ,$$

$$T = \frac{1}{2} \left(\sum_{i=1}^{N} m_i \right) (\bar{\mathbf{v}} \bullet \bar{\mathbf{v}}) \; + \; \bar{\mathbf{v}} \bullet \left(\sum_{i=1}^{N} m_i \, \dot{\mathbf{p}}_i \right) + \sum_{i=1}^{N} \frac{1}{2} \; m_i \, (\dot{\mathbf{p}}_i \bullet \dot{\mathbf{p}}_i) \; . \tag{27.7}$$

But since the motion has been referred to the center of mass we have

$$\sum_{i=1}^{N} m_i = m \;\; , \;\; \sum_{i=1}^{N} m_i \, \mathbf{p}_i = m \, \bar{\mathbf{p}} = 0 \;\; , \;\; \sum_{i=1}^{N} m_i \, \dot{\mathbf{p}}_i = 0 \;\; , \tag{27.8a,b,c}$$

so that (27.7) reduces to

$$T = \frac{1}{2} \;\; m \, (\bar{\mathbf{v}} \bullet \bar{\mathbf{v}}) + \sum_{i=1}^{N} \frac{1}{2} \;\; m_i \, (\dot{\mathbf{p}}_i \bullet \dot{\mathbf{p}}_i) \;\; . \tag{27.8}$$

In words, equation (27.8) states that the kinetic energy of the system of particles is equal to the sum of the kinetic energy of center of mass of the system and the kinetic energy of the motion of the masses relative to the center of mass. This split of kinetic energy is sometimes used in atomic theories which relate the second term in (27.8) with the temperature of the system.

Next it is desirable to reconsider the expression for total mechanical power P and in particular consider the role of the internal forces $\mathbf{f}_{ij}$. To this end, we use Newton's third law (24.9) and follow the arguments of section 24 to obtain

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{f}_{ij} \bullet \mathbf{v}_i = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{f}_{ij} \bullet (\mathbf{v}_i - \mathbf{v}_j) \;\; , \tag{27.9}$$

so that the total mechanical power P becomes

$$P = \sum_{i=1}^{N} \mathbf{F}_i \bullet \mathbf{v}_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{f}_{ij} \bullet (\mathbf{v}_i - \mathbf{v}_j) \;\; . \tag{27.10}$$

Notice that in general the internal forces do work and thus contribute to the total mechanical power. For the special case of central forces $\mathbf{f}_{ij}$ given by (24.15) the expression (27.10) further reduces to

$$P = \sum_{i=1}^{N} \mathbf{F}_i \bullet \mathbf{v}_i + \frac{1}{4} \sum_{i=1}^{N} \sum_{j=1}^{N} f(x_{ij})\, \dot{x}_{ij} \;, \tag{27.11}$$

where the time derivative $\dot{x}_{ij}$ of the relative distance squared between the masses m_i and m_j is given by

$$\dot{x}_{ij} = 2\,(\mathbf{x}_i - \mathbf{x}_j) \bullet (\mathbf{v}_i - \mathbf{v}_j) \;\; . \tag{27.12}$$

Since the function $f(x_{ij})$ depends only on the relative distance between the two masses m_i and m_j a potential function $\psi(x_{ij})$ exists such that

$$\frac{d\psi(x_{ij})}{dx_{ij}} = -\frac{1}{4}\; f(x_{ij}) \;\; . \tag{27.13}$$

Thus, (27.11) can be rewritten in the simpler form

$$P = \sum_{i=1}^{N} \mathbf{F}_i \bullet \mathbf{v}_i - \sum_{i=1}^{N} \sum_{j=1}^{N} \dot{\psi}(x_{ij}) \;\; . \tag{27.14}$$

These results show that the internal forces do work and thus contribute to the change in kinetic energy of the system of particles.

For the special case of a system of particles which are rigidly connected with no external moments (i.e. a special rigid body) the relative distance between any two masses remains constant so that

$$\dot{x}_{ij} = 0 \;\; , \tag{27.15}$$

and the internal forces do no work. Consequently, only the external forces contribute to the mechanical power so that

$$P = \sum_{i=1}^{N} \mathbf{F}_i \bullet \mathbf{v}_i \;\; \text{for a rigid body} \;\; . \tag{27.16}$$

Finally, we emphasize that the velocity which appears in the expression (27.16) is the velocity of the particular point of application of the force $\mathbf{F}_i$.

28. Impact of two particles

The objective of this section is to develop a simple empirical model for analyzing the impact of two particles. To this end, consider the case of two particles of masses m_A and m_B which at time t_1 just before impact are moving with absolute velocities $\mathbf{v}_{A1}$ and $\mathbf{v}_{B1}$, respectively. Also, the particles are assumed to collide at a single point at which the unit outward normal to particle A is denoted by **n** (see Fig. 28.1). The impact process occurs during the time period $[t_1,t_2]$ and at time t_2 just after impact the particles have absolute velocities $\mathbf{v}_{A2}$ and $\mathbf{v}_{B2}$, respectively.

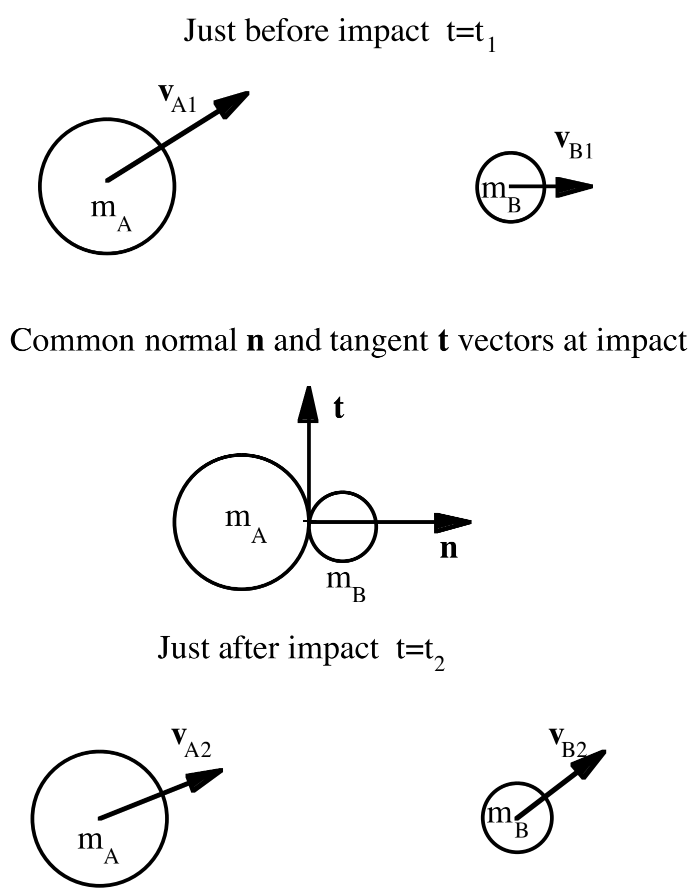

Fig. 28.1 Two particles just before ($t=t_1$) and just after ($t=t_2$) impact. During the impact process the velocities of the particles change abruptly.

Given the state of the two masses just before impact

$$\{ m_A , \mathbf{v}_{A1} , m_B , \mathbf{v}_{B1} \} , \tag{28.1}$$

the objective is to determine the velocities

$$\{ \mathbf{v}_{A2} , \mathbf{v}_{B2} \} \ , \tag{28.2}$$

just after impact.

During the impact process a number of very complicated interactions occur between the two particles which can cause energy to be dissipated. In particular, the particles can deform elastically, or plastically and friction can act, with or without slip. Analyzing the details of these interactions is beyond the scope of particle dynamics. Consequently, for simplicity, it is assumed that the impact process occurs over a very short time so that the impact process can be modeled as an abrupt change in momentum of each particle. This means that it is only necessary to consider the impulse $\hat{\mathbf{F}}$ of the force $\mathbf{F}$(t) applied by particle B on particle A during the impact process

$$\hat{\mathbf{F}} = \int_{t_1}^{t_2} \mathbf{F}(t) \, dt \ . \tag{28.3}$$

Now, by integrating the balance of linear momentum associated with each of the particles it can be shown that

$$m_A \, (\mathbf{v}_{A2} - \mathbf{v}_{A1}) = \hat{\mathbf{F}} \ , \ m_B \, (\mathbf{v}_{B2} - \mathbf{v}_{B1}) = - \hat{\mathbf{F}} \ , \tag{28.4a,b}$$

where use has been made of Newton's third law which states that $(-\mathbf{F})$ is the force applied by particle A on particle B (see Fig. 28.2). Thus, the velocities $\mathbf{v}_{A2}$ and $\mathbf{v}_{B2}$ are determined by the equations

$$\mathbf{v}_{A2} = \mathbf{v}_{A1} + \frac{1}{m_A} \hat{\mathbf{F}} \ , \ \ \mathbf{v}_{B2} = \mathbf{v}_{B1} - \frac{1}{m_B} \hat{\mathbf{F}} \ . \tag{28.5a,b}$$

This means that the complicated phenomena occurring during impact can be modeled by proposing an equation for the impulse $\hat{\mathbf{F}}$.

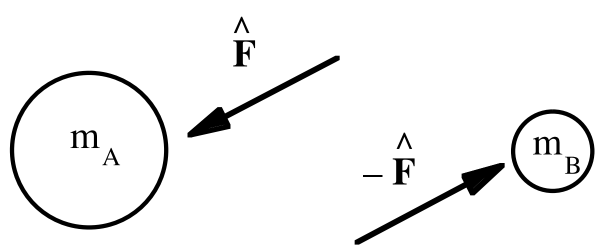

Fig. 28.2 Impulse acting during the impact process.

Simple analysis (Coefficient of restitution)

In the simple analysis, the two particles are treated as a system of two particles which is not influenced by external forces during the impact period. This means that the linear momentum of the system remains constant

$$\mathbf{G}_1 = m_A\, \mathbf{v}_{A1} + m_B\, \mathbf{v}_{B1} = m_A\, \mathbf{v}_{A2} + m_B\, \mathbf{v}_{B2} = \mathbf{G}_2 \ . \tag{28.6}$$

Moreover, in the simple analysis it is common to introduce an empirical constant e called the coefficient of restitution which specifies the ratio of the separation velocities to the approach velocities of the masses

$$e = \frac{\mathbf{n} \bullet (\mathbf{v}_{B2} - \mathbf{v}_{A2})}{\mathbf{n} \bullet (\mathbf{v}_{A1} - \mathbf{v}_{B1})} = \frac{\text{separation velocity}}{\text{approach velocity}} \ . \tag{28.7}$$

This empirical constant attempts to model the net effect of the complicated interactions during impact.

Next, taking the normal component of the linear momentum equation (28.6) it follows that

$$m_A\, (\mathbf{v}_{A2} \bullet \mathbf{n}) + m_B\, (\mathbf{v}_{B2} \bullet \mathbf{n}) = m_A\, (\mathbf{v}_{A1} \bullet \mathbf{n}) + m_B\, (\mathbf{v}_{B1} \bullet \mathbf{n}) \ . \tag{28.8}$$

Also, rewriting the (28.7) in the form

$$-(\mathbf{v}_{A2} \bullet \mathbf{n}) + (\mathbf{v}_{B2} \bullet \mathbf{n}) = e\, (\mathbf{v}_{A1} \bullet \mathbf{n}) - e\, (\mathbf{v}_{B1} \bullet \mathbf{n}) \ , \tag{28.9}$$

these equations can be solved to obtain

$$(\mathbf{v}_{A2} \bullet \mathbf{n}) = \frac{m_A - e\, m_B}{m_A + m_B} (\mathbf{v}_{A1} \bullet \mathbf{n}) + \frac{m_B(1+e)}{m_A + m_B} (\mathbf{v}_{B1} \bullet \mathbf{n}) \ ,$$

$$(\mathbf{v}_{B2} \bullet \mathbf{n}) = \frac{m_A(1+e)}{m_A + m_B} (\mathbf{v}_{A1} \bullet \mathbf{n}) + \frac{m_B - e\, m_A}{m_A + m_B} (\mathbf{v}_{B1} \bullet \mathbf{n}) \ . \tag{28.10}$$

For smooth particles, the impulsive force $\hat{\mathbf{F}}$ has no component in the tangential direction $\mathbf{t}$

$$\hat{\mathbf{F}} \bullet \mathbf{t} = 0 \ , \tag{28.11}$$

so that from (28.4) it follows that the component of linear momentum in the tangential direction is preserved for each of the particles

$$(\mathbf{v}_{A2} \bullet \mathbf{t}) = (\mathbf{v}_{A1} \bullet \mathbf{t}) \ , \quad (\mathbf{v}_{B2} \bullet \mathbf{t}) = (\mathbf{v}_{B1} \bullet \mathbf{t}) \ . \tag{28.12}$$

Thus, with the help of (28.5a) it can be shown that for this case, the impulse $\hat{\mathbf{F}}$ is given by

$$\hat{\mathbf{F}} = -\,m^* \,(1+e)\,\left[(\mathbf{v}_{A1} - \mathbf{v}_{B1}) \bullet \mathbf{n}\right] \mathbf{n} \quad . \tag{28.13}$$

where m^* is the effective mass

$$m^* = \frac{m_A m_B}{m_A + m_B} \quad . \tag{28.14}$$

Now, the total kinetic energy T_1 before impact and T_2 after impact are given by

$$T_1 = \frac{1}{2}\, m_A \left[(\mathbf{v}_{A1} \bullet \mathbf{n})^2 + (\mathbf{v}_{A1} \bullet \mathbf{t})^2\right] + \frac{1}{2}\, m_B \left[(\mathbf{v}_{B1} \bullet \mathbf{n})^2 + (\mathbf{v}_{B1} \bullet \mathbf{t})^2\right] \ ,$$

$$T_2 = \frac{1}{2}\, m_A \left[(\mathbf{v}_{A2} \bullet \mathbf{n})^2 + (\mathbf{v}_{A2} \bullet \mathbf{t})^2\right] + \frac{1}{2}\, m_B \left[(\mathbf{v}_{B2} \bullet \mathbf{n})^2 + (\mathbf{v}_{B2} \bullet \mathbf{t})^2\right] \ , \tag{28.15}$$

so with the help of thee results for smooth particles it can be shown that the that the loss of kinetic energy during impact becomes

$$T_1 - T_2 = (1 - e^2)\,\frac{1}{2}\, m^* \left[(\mathbf{v}_{A1} - \mathbf{v}_{B1}) \bullet \mathbf{n}\right]^2 \quad . \tag{28.16}$$

Note, that energy is preserved if e=1

$$T_1 - T_2 = 0 \quad \text{for} \;\; e = 1 \ , \tag{28.17}$$

and that the maximum energy is lost if e=0

$$T_1 - T_2 = \frac{1}{2}\, m^* \left[(\mathbf{v}_{A1} - \mathbf{v}_{B1}) \bullet \mathbf{n}\right]^2 > 0 \quad \text{for} \;\; e = 0 \ . \tag{28.18}$$

Moreover, since energy cannot be created during the impact, the value of e is taken in the range

$$0 \le e \le 1 \ . \tag{28.19}$$

More general analysis

In a more general analysis it is of interest to analyze physical restrictions on an equation for $\hat{\mathbf{F}}$. To this end, it is convenient to define the relative velocities of the particles just before and just after the impact event by the formulas

$$\Delta\mathbf{v}_1 = \mathbf{v}_{A1} - \mathbf{v}_{B1} \ , \ \ \Delta\mathbf{v}_2 = \mathbf{v}_{A2} - \mathbf{v}_{B2} \ . \tag{28.20a,b}$$

Thus, using (28.5) it can be shown that

$$\Delta\mathbf{v}_2 = \Delta\mathbf{v}_1 + \frac{1}{m^*}\,\hat{\mathbf{F}} \ , \tag{28.21a,b}$$

where m^* is the effective mass (28.14).

Next, it is assumed that the impulse $\hat{\mathbf{F}}$ must satisfy the following four physical restrictions:

(P1) The impulse $\hat{\mathbf{F}}$ must have a component that resists the relative velocity $\Delta\mathbf{v}_1$

$$\hat{\mathbf{F}} \bullet (-\Delta\mathbf{v}_1) > 0 \ , \tag{28.22a}$$

(P2) The impulse $\hat{\mathbf{F}}$ must have a component that resists penetration and aids separation of the two particles

$$\hat{\mathbf{F}} \bullet (-\,\mathbf{n}) > 0 \ , \tag{28.22b}$$

(P3) The two particles have a tendency to separate after the impact event

$$\Delta\mathbf{v}_2 \bullet (-\,\mathbf{n}) \geq 0 \ , \tag{28.22c}$$

(P4) The dissipation of kinetic energy of the two particle system during the impact event is nonnegative

$$T_1 - T_2 \geq 0 \ . \tag{28.22d}$$

Also, it is noted that impact will not occur unless the component of the approach velocity $\Delta\mathbf{v}_1$ in the normal direction $\mathbf{n}$ is positive

$$\Delta\mathbf{v}_1 \bullet \mathbf{n} = (\mathbf{v}_{A1} - \mathbf{v}_{B1}) \bullet \mathbf{n} > 0 \ . \tag{28.23}$$

Now, in order to analyze implications of these physical restrictions it is convenient to write $\hat{\mathbf{F}}$ in terms of its magnitude f and its direction $\mathbf{f}$ such that

$$\hat{\mathbf{F}} = f\,\mathbf{f} \ , \ f \geq 0 \ , \ \mathbf{f} \bullet \mathbf{f} = 1 \ . \tag{28.24a,b,c}$$

Then, the restrictions (28.22a,b) limit the direction of $\mathbf{f}$ by the expressions

$$\mathbf{f} \bullet (-\Delta\mathbf{v}_1) > 0 \ , \ \mathbf{f} \bullet (-\,\mathbf{n}) \geq 0 \ . \tag{28.25a,b}$$

It will be shown presently that it is convenient to express the magnitude f of the impulse in terms of another parameter η through the formula

$$f = m^* (1+\eta)\,(-\,\mathbf{f} \bullet \Delta\mathbf{v}_1) \ . \tag{28.26}$$

Thus, with the help of (28.24) and (28.26) the equation (28.21a) becomes

$$\Delta\mathbf{v}_2 = \Delta\mathbf{v}_1 + (1+\eta)\,(-\,\mathbf{f} \bullet \Delta\mathbf{v}_1)\,\mathbf{f} \ . \tag{28.27}$$

Moreover, by taking the dot product of (28.27) with **f** it can be shown that

$$\eta = \frac{(\mathbf{f} \bullet \Delta\mathbf{v}_2)}{(-\,\mathbf{f} \bullet \Delta\mathbf{v}_1)} = \frac{\mathbf{f} \bullet (\mathbf{v}_{B2} - \mathbf{v}_{A2})}{\mathbf{f} \bullet (\mathbf{v}_{A1} - \mathbf{v}_{B1})} \ . \tag{28.28}$$

Physically, this means that η is the ratio of the component of the separation velocity ($-\Delta\mathbf{v}_2 = \mathbf{v}_{B2} - \mathbf{v}_{A2}$) in the direction of impulse **f** to the component of the approach velocity ($\Delta\mathbf{v}_1 = \mathbf{v}_{A1} - \mathbf{v}_{B1}$) in the direction of **f**.

Next, consider the expressions for energy T_1 before impact and T_2 after impact

$$T_1 = \frac{1}{2}\, m_A\, \mathbf{v}_{A1} \bullet \mathbf{v}_{A1} + \frac{1}{2}\, m_B\, \mathbf{v}_{B1} \bullet \mathbf{v}_{B1} \ , \tag{28.29a}$$

$$T_2 = \frac{1}{2}\, m_A\, \mathbf{v}_{A2} \bullet \mathbf{v}_{A2} + \frac{1}{2}\, m_B\, \mathbf{v}_{B2} \bullet \mathbf{v}_{B2} \ . \tag{28.29b}$$

However, with the help of (28.5) and (28.26) these equations (28.5) can be rewritten in the forms

$$\mathbf{v}_{A2} = \mathbf{v}_{A1} + \left[\frac{m^*(1+\eta)(-\mathbf{f} \bullet \Delta\mathbf{v}_1)}{m_A}\right] \mathbf{f} \ , \tag{28.30a}$$

$$\mathbf{v}_{B2} = \mathbf{v}_{B1} - \left[\frac{m^*(1+\eta)(-\mathbf{f} \bullet \Delta\mathbf{v}_1)}{m_B}\right] \mathbf{f} \ . \tag{28.30b}$$

Thus, the kinetic energy after impact can be expressed in the form

$$\begin{aligned} T_2 = T_1 &+ m^*(1+\eta)(-\mathbf{f} \bullet \Delta\mathbf{v}_1)(\mathbf{f} \bullet \mathbf{v}_{A1}) - m^*(1+\eta)(-\mathbf{f} \bullet \Delta\mathbf{v}_1)(\mathbf{f} \bullet \mathbf{v}_{B1}) \\ &+ \frac{1}{2}\, m_A\left[\frac{m^*(1+\eta)(-\mathbf{f} \bullet \Delta\mathbf{v}_1)}{m_A}\right]^2 + \frac{1}{2}\, m_B\left[\frac{m^*(1+\eta)(-\mathbf{f} \bullet \Delta\mathbf{v}_1)}{m_B}\right]^2 \ , \end{aligned} \tag{28.31a}$$

$$T_2 = T_1 - m^*(1+\eta)(-\mathbf{f} \bullet \Delta\mathbf{v}_1)^2 + \frac{1}{2}\left[\frac{1}{m_A} + \frac{1}{m_B}\right]\left[m^*(1+\eta)(-\mathbf{f} \bullet \Delta\mathbf{v}_1)\right]^2 \ , \tag{28.31b}$$

$$T_2 = T_1 - m^*(1+\eta)(-\mathbf{f} \bullet \Delta\mathbf{v}_1)^2 + \frac{1}{2}\, m^*(1+\eta)^2(-\mathbf{f} \bullet \Delta\mathbf{v}_1)^2 \ , \tag{28.31c}$$

$$T_2 = T_1 - \frac{1}{2}\, m^*\,(1-\eta^2)\,(-\mathbf{f} \bullet \Delta\mathbf{v}_1)^2 \ , \tag{28.31d}$$

so the restriction (28.22d) on the energy dissipation reduces to

$$T_1 - T_2 = \frac{1}{2}\, m^*\,(1-\eta^2)\,(-\mathbf{f} \bullet \Delta\mathbf{v}_1)^2 \geq 0 \ . \tag{28.32}$$

This restriction can easily be satisfied by limiting the value of η by

$$\eta^2 \leq 1 \ . \tag{28.33}$$

However, in view of the physical interpretation (28.28) of η it does not seem reasonable to allow η to be negative so η is restricted to the range

$$0 \leq \eta \leq 1 \ . \tag{28.34}$$

In particular, note that the maximum energy dissipation (for fixed **f**) occurs when η=0 and the energy is conserved if η=1. For this reason the collision is called elastic if η=1 and kinetic energy is preserved. Also, note that if **f** is taken in the special direction which is parallel to $\Delta\mathbf{v}_1$ then it can be shown that

$$\Delta\mathbf{v}_2 = -\,\eta\,\Delta\mathbf{v}_1 \quad \text{for} \quad \mathbf{f} = -\frac{\Delta\mathbf{v}_1}{|\Delta\mathbf{v}_1|} \ . \tag{28.35a,b}$$

Thus, for this specification of the direction of the impulse the relative velocity $\Delta\mathbf{v}_2$ remains parallel to $\Delta\mathbf{v}_1$, its direction is changed and its magnitude is reduced by the value of η.

In summary, the physical restrictions (28.22) can be rewritten in the forms

$$\mathbf{f} \bullet (-\Delta\mathbf{v}_1) > 0 \ , \ \mathbf{f} \bullet (-\,\mathbf{n}) \geq 0 \ , \tag{28.36a,b}$$

$$\Delta\mathbf{v}_2 \bullet (-\,\mathbf{n}) = -\,(\Delta\mathbf{v}_1 \bullet \mathbf{n}) + (1+\eta)\,(-\,\mathbf{f} \bullet \Delta\mathbf{v}_1)\,\mathbf{f} \bullet (-\,\mathbf{n}) \geq 0 \ , \tag{28.36c}$$

$$0 \leq \eta \leq 1 \ . \tag{28.36d}$$

where use has been made of the expression (28.27). Fig. 28.3 shows that the restrictions (28.36a,b) require the impulse direction **f** to lie in a conical region. However, it will be seen below that the condition (28.36c) is more difficult to satisfy in general.

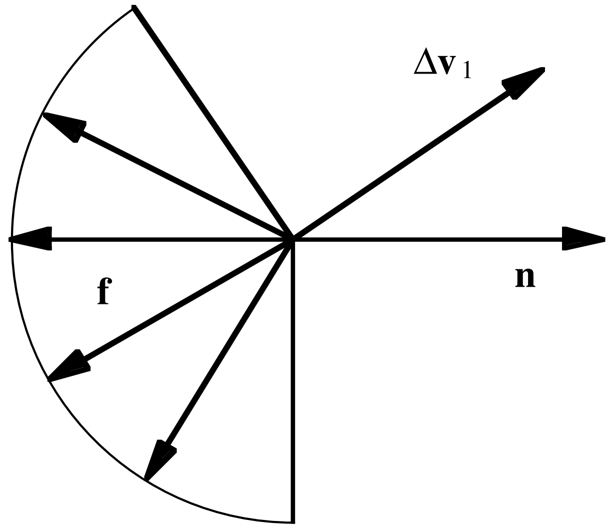

Fig. 28.3 Conical region satisfying the restrictions (28.36) on the direction **f** of the impulse $\hat{\mathbf{F}}$ acting on particle A during the impact process.

SMOOTH PARTICLES

For the special case of smooth particles with no friction acting during the impact process the direction of impulse must be normal to the surface of impact so that

$$\mathbf{f} = -\mathbf{n} \ . \tag{28.37}$$

It then follows that the restrictions (28.36a,b,c) are automatically satisfied whenever η satisfies the restriction (28.36d). Also, it follows that (28.27) and (28.28) reduce to

$$\Delta \mathbf{v}_2 = \Delta \mathbf{v}_1 - (1+\eta)\,(\mathbf{n} \bullet \Delta \mathbf{v}_1)\,\mathbf{n} \ , \tag{28.38}$$

and the equation

$$\eta = \frac{(\mathbf{n} \bullet \Delta \mathbf{v}_2)}{(-\,\mathbf{n} \bullet \Delta \mathbf{v}_1)} = \frac{\mathbf{n} \bullet (\mathbf{v}_{B2} - \mathbf{v}_{A2})}{\mathbf{n} \bullet (\mathbf{v}_{A1} - \mathbf{v}_{B1})} \ . \tag{28.39}$$

Thus, η is seen to be a generalized coefficient of restitution e defined in (28.7).

NONSMOOTH PARTICLES

For the more general case of nonsmooth particles with friction acting it is reasonable to specify the direction of impulse **f** to lie in the plane of the relative velocity $\Delta \mathbf{v}_1$ and the normal **n**. To this end, let **t** be a unit vector in the contact plane ($\mathbf{t} \bullet \mathbf{t}$=1, $\mathbf{t} \bullet \mathbf{n}$=0) with a nonnegative component in the $\Delta \mathbf{v}_1$ direction such that

$$\Delta \mathbf{v}_1 = \Delta v_1 \left[\cos\theta \, \mathbf{n} + \sin\theta \, \mathbf{t}\right] \ , \ \Delta v_1 > 0 \ , \ 0 \le \theta \le \frac{\pi}{2} \ . \qquad (28.40a,b,c)$$

Also, define the direction **f** by the friction angle ϕ such that

$$\mathbf{f} = -\left[\cos\phi \, \mathbf{n} + \sin\phi \, \mathbf{t}\right] \ . \qquad (28.41)$$

Now, substitution of the expressions (28.40a) and (28.41) into the restrictions (28.36a,b,c) yields the expressions

$$\mathbf{f} \bullet (-\Delta \mathbf{v}_1) = \Delta v_1 \cos(\phi-\theta) > 0 \ \Rightarrow \ -\frac{\pi}{2} + \theta < \phi < \frac{\pi}{2} + \theta \ , \qquad (28.42a,b)$$

$$\mathbf{f} \bullet (-\mathbf{n}) = \cos\phi \ge 0 \ \Rightarrow \ -\frac{\pi}{2} \le \phi \le \frac{\pi}{2} \ , \qquad (28.42c,d)$$

$$\Delta \mathbf{v}_2 \bullet (-\mathbf{n}) = \Delta v_1 \, F(\theta,\phi) \ge 0 \ , \qquad (28.42e)$$

$$F(\theta,\phi) = (1+\eta) \cos(\phi-\theta) \cos\phi - \cos\theta \ . \qquad (28.42f)$$

The restrictions (28.42a,c) are satisfied whenever ϕ lies in the range

$$-\frac{\pi}{2} + \theta < \phi < \frac{\pi}{2} \ . \qquad (28.43)$$

However, it can easily be shown that at the boundaries of the this range the restriction (28.42e) is violated

$$\Delta \mathbf{v}_2 \bullet (-\mathbf{n}) = \Delta v_1 \, F(\theta, -\frac{\pi}{2} + \theta)) = -\Delta v_1 \cos\theta < 0 \ , \qquad (28.44a)$$

$$\Delta \mathbf{v}_2 \bullet (-\mathbf{n}) = \Delta v_1 \, F(\theta, \frac{\pi}{2})) = -\Delta v_1 \cos\theta < 0 \ . \qquad (28.44b)$$

This means that the acceptable range of ϕ is completely determined by the restriction (28.42e).

To determine the bounds on ϕ consider the value ϕ^* of ϕ which the equality holds in (28.42e)

$$\Delta \mathbf{v}_2 \bullet (-\mathbf{n}) = \Delta v_1 \, F(\theta,\phi^*) = \Delta v_1 \left[(1+\eta) \cos(\phi^*-\theta) \cos\phi^* - \cos\theta\right] = 0 \ . \qquad (28.45)$$

Thus, ϕ^* is determined by the equation

$$(1+\eta) \cos(\phi^*-\theta) \cos\phi^* - \cos\theta = 0 \ , \qquad (28.46a)$$

$$(1+\eta) \left[\cos\phi^* \cos\theta + \sin\phi^* \sin\theta \right] \cos\phi^* - \cos\theta = 0 \ , \qquad (28.46b)$$

$$\left[(1+\eta)\cos^2\phi^* - 1\right]\cos\theta + (1+\eta)\left[\sin\phi^*\cos\phi^*\right]\sin\theta = 0 \;, \tag{28.46c}$$

$$\left[(1+\eta)\{1+\cos(2\phi^*)\} - 2\right]\cos\theta + (1+\eta)\sin(2\phi^*)\sin\theta = 0 \;, \tag{28.46d}$$

$$(1+\eta)\cos(2\phi^*) - (1-\eta) = -(1+\eta)\sin(2\phi^*)\tan\theta \;. \tag{28.46e}$$

Squaring both sides of (28.46e) yields a quadratic equation for $\cos(2\phi^*)$ of the form

$$(1+\eta)^2\cos^2(2\phi^*) - 2(1-\eta^2)\cos(2\phi^*) + (1-\eta)^2 = (1+\eta)^2\tan^2\theta\left[1-\cos^2(2\phi^*)\right] , \tag{28.47a}$$

$$(1+\eta)^2(1+\tan^2\theta)\cos^2(2\phi^*) - 2(1-\eta^2)\cos(2\phi^*) + \left[(1-\eta)^2 - (1+\eta)^2\tan^2\theta\right] = 0 \tag{28.47b}$$

$$\left[\frac{1+\eta}{\cos\theta}\right]^2\cos^2(2\phi^*) - 2(1-\eta^2)\cos(2\phi^*) + \left[(1-\eta)^2 - (1+\eta)^2\tan^2\theta\right] = 0 \;. \tag{28.47c}$$

Thus, the solutions of this quadratic equation become

$$\cos(2\phi^*) = \frac{(1-\eta^2) \pm \sqrt{(1-\eta^2)^2 - \left\{\frac{1+\eta}{\cos\theta}\right\}^2\left\{(1-\eta)^2 - (1+\eta)^2\tan^2\theta\right\}}}{\left\{\frac{1+\eta}{\cos\theta}\right\}^2} \;. \tag{28.48}$$

However, the term under the radical can be simplified to obtain

$$(1-\eta^2)^2 - \left\{\frac{1+\eta}{\cos\theta}\right\}^2\left\{(1-\eta)^2 - (1+\eta)^2\tan^2\theta\right\}$$

$$= (1-\eta^2)^2\left[1 - \frac{1}{\cos^2\theta}\right] + \left[\frac{(1+\eta)^4}{\cos^2\theta}\right]\tan^2\theta$$

$$= -(1-\eta^2)^2\tan^2\theta + (1+\eta)^4\left[\frac{\tan^2\theta}{\cos^2\theta}\right]$$

$$= \left[\frac{(1+\eta)^4\tan^2\theta}{\cos^2\theta}\right]\left[1 - \left\{\frac{1-\eta}{1+\eta}\right\}^2\cos^2\theta\right] \geq 0 \;. \tag{28.49}$$

Since this term is nonnegative, the solutions of (28.48) are real and the relevant solutions ϕ_1 and ϕ_2 of (28.45) can be written in the simplified forms

$$\phi_1 = -\frac{1}{2}\cos^{-1}\left[\left\{\frac{1-\eta}{1+\eta}\right\}\cos^2\theta + \left\{1 - \left\{\frac{1-\eta}{1+\eta}\right\}^2\cos^2\theta\right\}^{1/2}\sin\theta\right] \;, \tag{28.50a}$$

$$\phi_2 = \frac{1}{2}\cos^{-1}\left[\left\{\frac{1-\eta}{1+\eta}\right\}\cos^2\theta - \left\{1 - \left\{\frac{1-\eta}{1+\eta}\right\}^2\cos^2\theta\right\}^{1/2}\sin\theta\right] \;. \tag{28.50b}$$

Moreover, it follows from (28.25) that if the impulse is parallel to the approach velocity (ϕ=θ) then the restriction (28.42e) is satisfied since

$$\Delta \mathbf{v}_2 \bullet (-\mathbf{n}) = \Delta v_1 \, F(\theta,\theta) = \eta \, \Delta v_1 \cos\theta \geq 0 \quad \text{for} \quad \phi = \theta \; . \tag{28.51}$$

This means that θ is within the bounds on the friction angle ϕ

$$\phi_1 \leq \theta \leq \phi_2 \; . \tag{28.52}$$

Finally, it then can be shown that all of the physical restrictions (28.22) are satisfied provided that

$$\phi_1 \leq \phi \leq \phi_2 \; , \; 0 \leq \eta \leq 1 \; . \tag{28.53}$$

Fig. 28.4 plots the acceptable the bounds on ϕ for three different values of η.

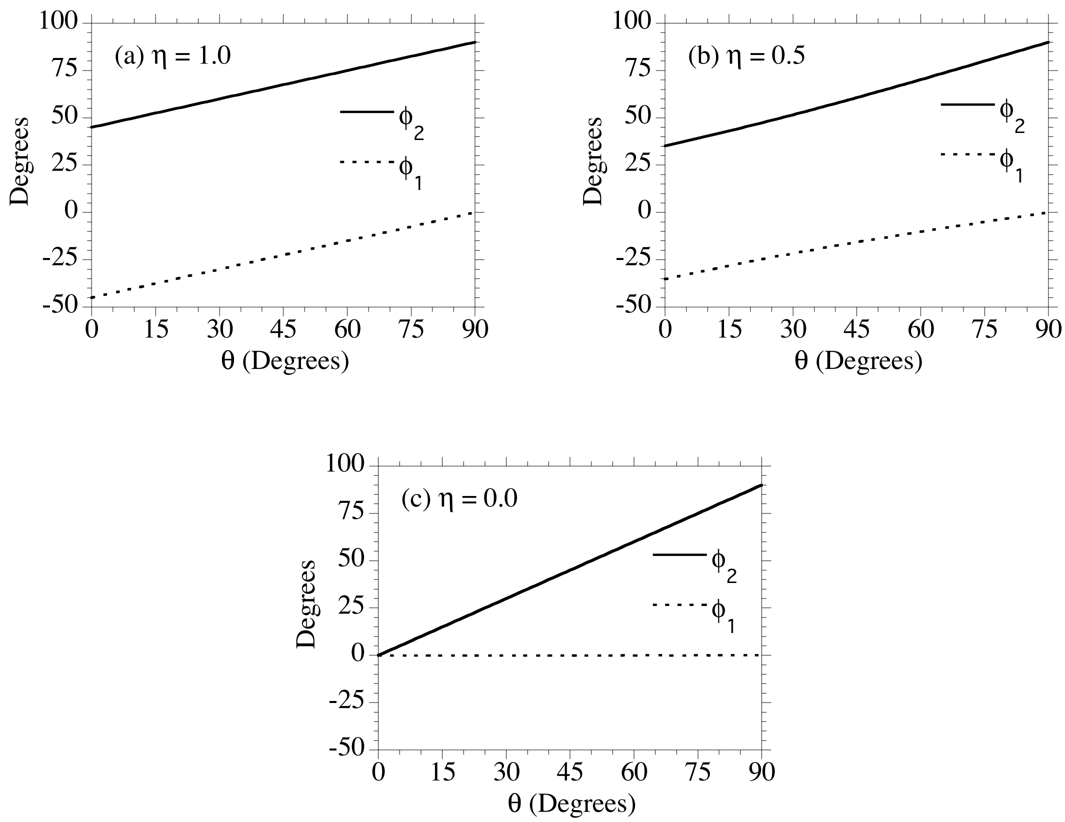

Fig. 28.4 Plots of the minimum (ϕ_1) and maximum (ϕ_2) values of the friction angle ϕ as function of the angle θ of the approach velocity for three different values of the coefficient of restitution η

29. Equations Of Motion Of A Rigid Body

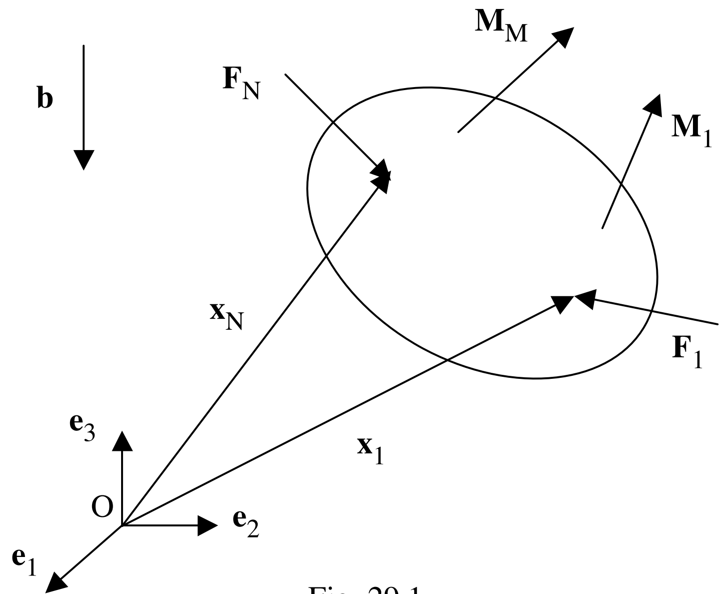

Fig. 29.1

To describe the dynamics of rigid bodies we define the mass m, linear momentum $\mathbf{G}$, angular momentum $\mathbf{H}_o$ (about the origin). Motivated by the definitions of m, $\mathbf{G}$, $\mathbf{H}_o$ for a system of particles we replace the summation process by integration over the body of an elemental mass dm and write

$$m = \int dm \ , \tag{29.1a}$$

$$\mathbf{G} = \int \mathbf{v} \, dm \ , \tag{29.1b}$$

$$\mathbf{H}_o = \int \mathbf{x} \times \mathbf{v} \, dm \ , \tag{29.1c}$$

where $\mathbf{v}$ is the linear momentum per unit mass, and $\mathbf{x} \times \mathbf{v}$ is the angular momentum (about the origin) per unit mass. In rigid body dynamics we define two types of external forces that act on the body: (1) <u>body forces</u> per unit mass denoted by the vector $\mathbf{b}$ (like the force of gravity), and (2) concentrated forces $\mathbf{F}_i$ acting at the points $\mathbf{x}_i$ (see Fig. 29.1). Then the resultant external force $\mathbf{F}$ applied to the rigid body is given by

$$\mathbf{F} = \int \mathbf{b} \, dm + \sum_{i=1}^{N} \mathbf{F}_i \ . \tag{29.2}$$

Similarly, the resultant moment $\mathbf{M}_o$ (about the origin) is given by

$$\mathbf{M}_o = \int \mathbf{x} \times \mathbf{b}\, dm + \sum_{i=1}^{N} \mathbf{x}_i \times \mathbf{F}_i + \sum_{i=1}^{M} \mathbf{M}_i \ , \tag{29.3}$$

where $\mathbf{M}_i$ (i=1,2,...,M) are M moments applied to the rigid body at various points.

We now assume that the equations of motion of a rigid body have the same form as those for a single particle and a simple system of particles so we postulate the conservation of mass

$$\dot{m} = 0 \ , \tag{29.4}$$

the balance of linear momentum

$$\dot{\mathbf{G}} = \mathbf{F} \ , \tag{29.5}$$

and the balance of angular momentum

$$\dot{\mathbf{H}}_o = \mathbf{M}_o \ . \tag{29.6}$$

Notice that these balance laws are postulated instead of being derived directly from the dynamics of a system of particles, and they are consistent with the dynamics of a system of particles if the internal forces are such that they apply no net moment $\mathbf{m}_o$ to the body.

Ultimately the validity of these assumptions can only be verified by comparing theoretical predictions with results of experiments. In this regard, we note that numerous experiments have proven these equations of motion of a rigid body to be quite accurate for nonrelativistic velocities. In fact, the generalizations (29.4)-(29.6) hold even for continuous deformable media like fluids and elastic solids.

It is important to emphasize that since a rigid body has six degrees of freedom (three translational characterized by the position vector of a point on the body, and three rotational characterized by the angular velocity vector) both the balances of linear momentum and angular momentum must be used to determine the position of a point on the body and its angular orientation in space. This should be contrasted with the fact that the balance of angular momentum of a single particle was derived as a theorem using the balance of linear momentum, so the balance of angular momentum contains no information that is not already contained in the balance of linear momentum.

In our study of dynamics of rigid bodies we will confine attention to values of $\mathbf{b}$ which are constant (like the case of a constant gravitational field, e.g. $\mathbf{b} = -\, g\, \mathbf{e}_3$). For this case the body force $\mathbf{F}_b$ and the moment $\mathbf{M}_{bo}$ of the body force may be expressed in the simple forms

$$\mathbf{F}_b = \int \mathbf{b}\, dm = m\, \mathbf{b} \;\;,\;\; \mathbf{M}_{bo} = \int \mathbf{x} \times \mathbf{b}\, dm = \bar{\mathbf{x}} \times m\, \mathbf{b} \;\;, \tag{29.7}$$

where $\bar{\mathbf{x}}$ denotes the location of the center of mass of the body

$$m\, \bar{\mathbf{x}} = \int \mathbf{x}\, dm \;\;. \tag{29.8}$$

The results (29.7) indicates that when $\mathbf{b}$ is constant the body force $\mathbf{F}_b$ is merely the total mass of the body m times $\mathbf{b}$ and the body force $\mathbf{F}_b$ acts at the center of mass of the body. This proves that the gravitational force acts through the center of mass of the body.

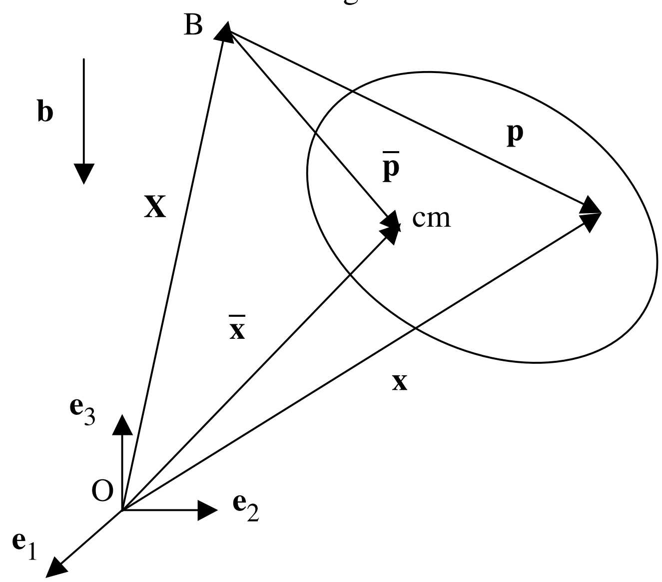

Fig. 29.2

Sometimes it is convenient to refer the equations of motion to a moving point. To this end we let $\mathbf{X}$ be the position vector from the fixed origin to an arbitrary moving point B and let $\mathbf{p}$ be the vector from B to any material point in the body (see Fig. 29.2) so that

$$\mathbf{x} = \mathbf{X} + \mathbf{p} \;\;,\;\; \bar{\mathbf{x}} = \mathbf{X} + \bar{\mathbf{p}} \;\;,\;\; \mathbf{x}_i = \mathbf{X} + \mathbf{p}_i \;\;, \tag{29.9a,b,c}$$

where $\bar{\mathbf{p}}$ is the location of the center of mass of the body relative to B. Using (29.1b) and differentiating (29.8) we may deduce that the linear momentum of a rigid body is the mass of the body times the absolute velocity of its center of mass

$$\mathbf{G} = m\,\bar{\mathbf{v}} \ . \tag{29.10}$$

Now, with the help of (29.9) and (29.10) the balance of linear momentum (29.5) may be written in the alternative form

$$m\,(\ddot{\mathbf{X}} + \ddot{\bar{\mathbf{p}}}) = \mathbf{F} \ . \tag{29.11}$$

This is identical to equation (25.5) for a system of particles so it follows that the balance of linear momentum of a rigid body remains invariant (unchanged in form) to a superposed constant velocity $\dot{\mathbf{X}}$ [see equation (25.6)].

To develop the alternative form of the balance of angular momentum we substitute (29.9a) into the definition (29.1c) and obtain the expressions

$$\mathbf{H}_o = \int (\mathbf{X} + \mathbf{p}) \times \mathbf{v}\ dm = \mathbf{X} \times \int \mathbf{v}\ dm + \int \mathbf{p} \times \mathbf{v}\ dm \ ,$$

$$\mathbf{H}_o = \mathbf{X} \times m\,\bar{\mathbf{v}} + \int \mathbf{p} \times (\dot{\mathbf{X}} + \dot{\mathbf{p}})\ dm \ ,$$

$$\mathbf{H}_o = \mathbf{X} \times m\,\bar{\mathbf{v}} + \int \mathbf{p}\ dm \times \dot{\mathbf{X}} + \int \mathbf{p} \times \dot{\mathbf{p}}\ dm \ ,$$

$$\mathbf{H}_o = \mathbf{X} \times m\,\bar{\mathbf{v}} + m\,\bar{\mathbf{p}} \times \dot{\mathbf{X}} + \mathbf{H}_B \ , \tag{29.12}$$

where $\mathbf{H}_B$ is the relative angular momentum about the point B

$$\mathbf{H}_B = \int \mathbf{p} \times \dot{\mathbf{p}}\ dm \ . \tag{29.13}$$

Furthermore, following the development of (25.9) we may differentiate (29.12) to obtain

$$\dot{\mathbf{H}}_o = \mathbf{X} \times \mathbf{F} + m\,\bar{\mathbf{p}} \times \ddot{\mathbf{X}} + \dot{\mathbf{H}}_B \ . \tag{29.14}$$

Now, substituting (29.9a,c) into (29.3), the expression for the moment $\mathbf{M}_o$ about the origin becomes

$$\mathbf{M}_o = \int (\mathbf{X} + \mathbf{p}) \times \mathbf{b}\ dm + \sum_{i=1}^{N} (\mathbf{X} + \mathbf{p}_i) \times \mathbf{F}_i + \sum_{i=1}^{M} \mathbf{M}_i \ ,$$

$$\mathbf{M}_o = \mathbf{X} \times [\int \mathbf{b}\ dm + \sum_{i=1}^{N} \mathbf{F}_i] + \int \mathbf{p} \times \mathbf{b}\ dm + \sum_{i=1}^{N} \mathbf{p}_i \times \mathbf{F}_i + \sum_{i=1}^{M} \mathbf{M}_i \ ,$$

$$\mathbf{M}_o = \mathbf{X} \times \mathbf{F} + \mathbf{M}_B \ , \tag{29.15}$$

where the moment $\mathbf{M}_B$ of the external forces about the point B is defined by

$$\mathbf{M}_B = \int \mathbf{p} \times \mathbf{b} \, dm + \sum_{i=1}^{N} \mathbf{p}_i \times \mathbf{F}_i + \sum_{i=1}^{M} \mathbf{M}_i \ . \tag{29.16}$$

Furthermore, if $\mathbf{b}$ is constant then (29.16) reduces to

$$\mathbf{M}_B = \int \mathbf{p} \, dm \times \mathbf{b} + \sum_{i=1}^{N} \mathbf{p}_i \times \mathbf{F}_i + \sum_{i=1}^{M} \mathbf{M}_i \ ,$$

$$\mathbf{M}_B = \bar{\mathbf{p}} \times m \, \mathbf{b} + \sum_{i=1}^{N} \mathbf{p}_i \times \mathbf{F}_i + \sum_{i=1}^{M} \mathbf{M}_i \ , \tag{29.17}$$

which shows that $\mathbf{b}$ acts through the center of mass. Finally, substitution of (29.14) and (29.15) into (29.6) yields the balance of angular momentum in the alternative form

$$\dot{\mathbf{H}}_B + m \, \bar{\mathbf{p}} \times \ddot{\mathbf{X}} = \mathbf{M}_B \ . \tag{29.18}$$

Notice that for a general moving point B the balance of angular momentum changes form. However, for the special case when $\bar{\mathbf{p}} \times \ddot{\mathbf{X}}$ vanishes, the balance of angular momentum remains invariant with

$$\dot{\mathbf{H}}_B = \mathbf{M}_B \quad \text{for} \quad \bar{\mathbf{p}} \times \ddot{\mathbf{X}} = 0 \ . \tag{29.19}$$

This happens for the following three cases.

Case I: The point B moves with constant velocity

$$\dot{\mathbf{X}} = \text{constant} \ , \tag{29.20}$$

so the balance of angular momentum remains invariant to a superposed constant velocity.

Case II: The point B is the center of mass of the system

$$\bar{\mathbf{p}} = 0 \ , \tag{29.21}$$

and the balance of angular momentum may be written in the form

$$\dot{\bar{\mathbf{H}}} = \bar{\mathbf{M}} \ , \tag{29.22}$$

where for convenience we have denoted the value of $\mathbf{H}_B$ by $\bar{\mathbf{H}}$ and the value of $\mathbf{M}_B$ by $\bar{\mathbf{M}}$, so that $\bar{\mathbf{H}}$ is the relative angular momentum about the center of mass and $\bar{\mathbf{M}}$ is the moment due to external forces about the center of mass. It is important to emphasize that equation (29.22) holds even when the center of mass accelerates.

<u>Case III</u>: The point B accelerates towards or away from the center of mass

$$\ddot{\mathbf{X}} \parallel \bar{\mathbf{p}} \quad . \tag{29.23}$$

Before closing this section it is convenient to use the result (29.12) and consider the special case when B is taken to be the center of mass with

$$\mathbf{X} = \bar{\mathbf{x}} \ , \ \bar{\mathbf{p}} = 0 \ , \tag{29.24a,b}$$

$$\mathbf{H}_B = \bar{\mathbf{H}} = \int (\mathbf{x} - \bar{\mathbf{x}}) \times (\mathbf{v} - \bar{\mathbf{v}}) \, dm \ , \tag{29.24c}$$

to derive the result that

$$\mathbf{H}_o = (\bar{\mathbf{x}} \times m\,\bar{\mathbf{v}}) + \bar{\mathbf{H}} \ . \tag{29.25}$$

This means that the angular momentum about the origin O is equal to the sum of the angular momentum of the center of mass $(\bar{\mathbf{x}} \times m\,\bar{\mathbf{v}})$ about O and the angular momentum $\bar{\mathbf{H}}$ of the rigid body about its center of mass. Also, we may solve (29.12) for $\mathbf{H}_B$ and substitute (29.25) into the resulting expression to obtain

$$\mathbf{H}_B = \mathbf{H}_o - \mathbf{X} \times m\,\bar{\mathbf{v}} - m\,\bar{\mathbf{p}} \times \dot{\mathbf{X}} = (\bar{\mathbf{x}} \times m\,\bar{\mathbf{v}}) + \bar{\mathbf{H}} - \mathbf{X} \times m\,\bar{\mathbf{v}} - m\,\bar{\mathbf{p}} \times \dot{\mathbf{X}} \ ,$$

$$\mathbf{H}_B = (\mathbf{X} + \bar{\mathbf{p}}) \times m\,\bar{\mathbf{v}} + \bar{\mathbf{H}} - \mathbf{X} \times m\,\bar{\mathbf{v}} - m\,\bar{\mathbf{p}} \times \dot{\mathbf{X}} \ ,$$

$$\mathbf{H}_B = \bar{\mathbf{p}} \times m\,\bar{\mathbf{v}} + \bar{\mathbf{H}} - m\,\bar{\mathbf{p}} \times \dot{\mathbf{X}} = \bar{\mathbf{p}} \times m\,(\dot{\mathbf{X}} + \dot{\bar{\mathbf{p}}}) + \bar{\mathbf{H}} - m\,\bar{\mathbf{p}} \times \dot{\mathbf{X}} \ ,$$

$$\mathbf{H}_B = \bar{\mathbf{p}} \times m\,\dot{\bar{\mathbf{p}}} + \bar{\mathbf{H}} \ . \tag{29.26}$$

which is similar to (29.25).

30. Inertia Tensor

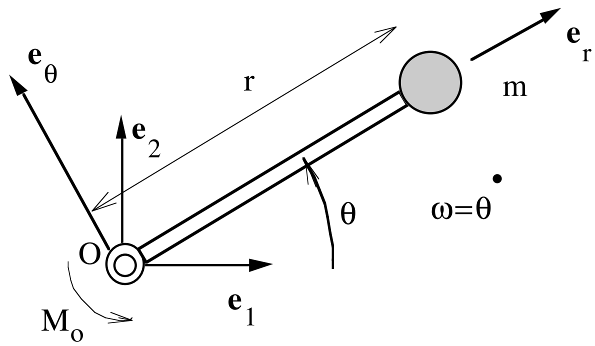

Fig. 30.1

By way of introduction to the concept of the inertia tensor of a rigid body consider the case of a rigid body which is connected by a frictionless joint at the fixed point O and rotates with angular velocity $\boldsymbol{\omega} = \omega\, \mathbf{e}_3$ due to the moment $\mathbf{M}_o = M_o\, \mathbf{e}_3$ (see Fig. 30.1). For simplicity, let the mass m of the rigid body be concentrated at a distance r from the point O. Letting $\mathbf{x}$ be the vector from O to the concentrated mass we may use polar coordinates to write

$$\mathbf{x} = r\, \mathbf{e}_r \ . \tag{30.1}$$

Since the mass is concentrated at a point, the integral in (29.1c) for the angular momentum $\mathbf{H}_o$ about O may be simply evaluated to obtain

$$\mathbf{H}_o = \mathbf{x} \times m\, \mathbf{v} = r\, \mathbf{e}_r \times m\, (r\omega\, \mathbf{e}_\theta) = (m\, r^2)\, \omega\, \mathbf{e}_3 = H_o\, \mathbf{e}_3 \ , \tag{30.2}$$

where $\mathbf{v} = \dot{\mathbf{x}}$ is the velocity of the mass and H_o is the component of $\mathbf{H}_o$ in the $\mathbf{e}_3$ direction. Since the mass m and the distance r are constant we have

$$\dot{\mathbf{H}}_o = \dot{H}_o\, \mathbf{e}_3 = (m\, r^2)\, \dot{\omega}\, \mathbf{e}_3 \ , \tag{30.3}$$

so that the balance of angular momentum (29.6) yields the equations

$$\dot{\mathbf{H}}_o = \dot{H}_o\, \mathbf{e}_3 = (m\, r^2)\, \dot{\omega}\, \mathbf{e}_3 = M_o\, \mathbf{e}_3 = \mathbf{M}_o \ . \tag{30.4}$$

For this simple case we can define I_o to be the moment of inertia of the rigid body about the $\mathbf{e}_3$ axis

$$I_o = m\, r^2 \ , \tag{30.5}$$

and write the angular momentum H_o in the form

$$H_o = I_o \, \omega \ , \tag{30.6}$$

and the balance of angular momentum in the scalar form

$$\dot{H}_o = M_o \ . \tag{30.7}$$

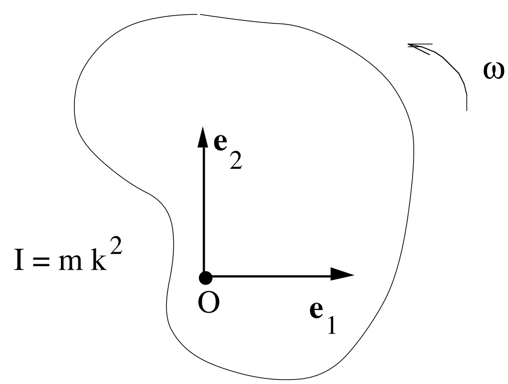

Fig. 30.2

Formulas of the type (30.6) and (30.7) hold for special cases of a rigid body rotating about the $\mathbf{e}_3$ axis even when the mass is not concentrated (see Fig. 30.2). For this case we sometimes use the analogy of a concentrated mass (30.5) and define the <u>radius of gyration</u> k by the formulas

$$I_o = m\,k^2 \ , \quad k = \left[\frac{I_o}{m}\right]^{1/2} \ . \tag{30.8}$$

Thus the radius of gyration k is the radius at which the mass of the body would have to be concentrated in order to obtain the correct value of the moment of inertia I_o for rotation about the origin.

To describe three-dimensional motion of a rigid body we must generalize the scalar equation (30.6) to a vector equation. It will be shown presently that the appropriate vector equation can be written in the form

$$\mathbf{H}_o = \mathbf{I}_o \, \boldsymbol{\omega} \ , \tag{30.9}$$

where $\boldsymbol{\omega}$ is the angular velocity vector of the rigid body and $\mathbf{I}_o$ is the inertia tensor of the rigid body about the point O. The inertia tensor $\mathbf{I}_o$ is a second order tensor that operates on the vector $\boldsymbol{\omega}$ to give the vector $\mathbf{H}_o$. Writing (30.9) in the component form relative to the basis $\mathbf{e}_i$

$$H_{oi}\,\mathbf{e}_i = I_{oij}\,\omega_j\,\mathbf{e}_i\ ,\quad H_{oi} = I_{oij}\,\omega_j\ , \tag{30.10a,b}$$

we may observe that the (i,j) components I_{oij} of $\mathbf{I}_o$ may be identified as a square matrix so that (30.10) merely represents a matrix I_{oij} multiplying a vector ω_j to obtain another vector H_{oi}. More generally we can write the angular momentum $\bar{\mathbf{H}}$ about the center of mass and the angular momentum $\mathbf{H}_B$ about a moving point B in terms of the inertia $\bar{\mathbf{I}}$ about the center of mass and the inertia $\mathbf{I}_B$ about B in the forms

$$\bar{\mathbf{H}} = \bar{\mathbf{I}}\ \boldsymbol{\omega}\ ,\quad \mathbf{H}_B = \mathbf{I}_B\,\boldsymbol{\omega}\ ,\ H_{Bi} = I_{Bij}\,\omega_j\ . \tag{30.12a,b,c}$$

To derive explicit expressions for the components of the inertia tensor let B be an arbitrary <u>material point</u> attached to the rigid body and recall the definition (29.13) for $\mathbf{H}_B$

$$\mathbf{H}_B = \int \mathbf{p}\times\dot{\mathbf{p}}\ dm\ , \tag{30.13}$$

where $\mathbf{p}$ is the vector from the point B to any point in the body. Since B is a material point and the vector $\mathbf{p}$ connects two material points on the rigid body it follows that

$$\dot{\mathbf{p}} = \boldsymbol{\omega}\times\mathbf{p}\ , \tag{30.14}$$

so that (30.13) may be written in the form

$$\mathbf{H}_B = \int \mathbf{p}\times(\boldsymbol{\omega}\times\mathbf{p})\ dm = \int \left[(\mathbf{p}\bullet\mathbf{p})\ \boldsymbol{\omega} - (\mathbf{p}\bullet\boldsymbol{\omega})\ \mathbf{p}\right] dm\ . \tag{30.15}$$

It is important to note that the angular velocity $\boldsymbol{\omega}(t)$ is a function of time only so it is not affected by the integration over the mass of the body and therefore $\boldsymbol{\omega}$ can be factored out of the integral. This can be done by defining the tensor product $\otimes$ such that for any vectors $\mathbf{a},\mathbf{b},\mathbf{c}$, the quantity $(\mathbf{a}\otimes\mathbf{b})$ is a second order tensor and $\mathbf{d} = (\mathbf{a}\otimes\mathbf{b})\ \mathbf{c}$ is a vector having the properties

$$\mathbf{d} = (\mathbf{a}\otimes\mathbf{b})\ \mathbf{c} = \mathbf{a}\ (\mathbf{b}\bullet\mathbf{c}) = (\mathbf{b}\bullet\mathbf{c})\ \mathbf{a}\ , \tag{30.16a}$$

$$d_i = (a_i\,b_j)\,c_j = a_i\,(b_j\,c_j) = (b_j\,c_j)\,a_i\ . \tag{30.16b}$$

Thus, we may write

$$(\mathbf{p} \bullet \boldsymbol{\omega})\, \mathbf{p} = (\mathbf{p} \otimes \mathbf{p})\, \boldsymbol{\omega} \quad , \quad \boldsymbol{\omega} = \mathbf{I}\, \boldsymbol{\omega} \quad , \tag{30.17a,b}$$

where $\mathbf{I}$ denotes the unit tensor and should not be confused with inertia tensors which are denoted by the same symbol with either an over bar $\overline{\mathbf{I}}$ or a subscript $\mathbf{I}_B$. Now substituting (30.17) into (30.15) we may deduce that

$$\mathbf{H}_B = \mathbf{I}_B\, \boldsymbol{\omega} \quad , \quad \mathbf{I}_B = \int \left[(\mathbf{p} \bullet \mathbf{p})\, \mathbf{I} - (\mathbf{p} \otimes \mathbf{p}) \right] dm \, . \tag{30.18a,b}$$

Notice that the inertia tensor $\mathbf{I}_B$ is a property of the body and is independent of the motion of the body. It follows that it is most convenient to calculate the components of $\mathbf{I}_B$ relative to a body coordinate system $\mathbf{e}_i'$ which rotates with the body

$$\dot{\mathbf{e}}_i' = \boldsymbol{\omega} \times \mathbf{e}_i' \ . \tag{30.19}$$

Letting $p_i' = \mathbf{e}_i' \bullet \mathbf{p}$ be the coordinates of the body coordinate system, P' be the region of space occupied by the body, $dV = (dp_1'\ dp_2'\ dp_3')$ be the element of volume, and $\rho(p_1')$ be the positive mass density (mass per unit volume), we may write

$$\mathbf{I}_B = \int_{P'} \rho \left[(\mathbf{p} \bullet \mathbf{p})\, \mathbf{I} - (\mathbf{p} \otimes \mathbf{p}) \right] dV \quad , \tag{30.20a}$$

$$I_{Bij}' = \int_{P'} \rho \left[(p_m'\, p_m')\, \delta_{ij} - (p_i'\, p_j') \right] dV \quad , \tag{30.20b}$$

where the Kronecker delta δ_{ij} are the components of the unit tensor $\mathbf{I}$ and I_{Bij}' are the components of $\mathbf{I}_B$ relative to the basis $\mathbf{e}_i'$. Notice that since we have chosen a body coordinate system the limits of integration in (30.20b) are independent of time (independent of the motion of the body).

In view of either of the representations (30.20a,b) it can be seen that the inertia tensor $\mathbf{I}_B$ is a symmetric tensor (i.e. its transpose $\mathbf{I}_B^T$ is equal to itself)

$$\mathbf{I}_B^T = \mathbf{I}_B \quad , \quad I_{Bji} = I_{Bij} \quad , \tag{30.21a,b}$$

and it is a positive definite tensor. By positive definite we mean that for an arbitrary nonzero constant vector $\mathbf{a}$ the scalar product $\mathbf{a} \bullet \mathbf{I}_B\, \mathbf{a}$ is positive

$$\mathbf{a} \bullet \mathbf{I}_B\, \mathbf{a} = a_i\, I_{Bij}\, a_j > 0 \quad \text{for} \quad \mathbf{a} \neq 0 \ . \tag{30.22}$$

This can be proved by substituting (30.20a) into (30.22) to obtain

$$\mathbf{a} \bullet \mathbf{I}_B\, \mathbf{a} = \int_{P'} \rho \left[(\mathbf{p} \bullet \mathbf{p})\, (\mathbf{a} \bullet \mathbf{a}) - (\mathbf{p} \bullet \mathbf{a})(\mathbf{p} \bullet \mathbf{a}) \right] dV > 0 \ , \tag{30.23}$$

which is positive since the constant vector **a** can only be equal to **p** at at most one point in the body.

In order to discuss the physical meaning of the inertia tensor $\mathbf{I}_B$ it is convenient to consider the expanded forms of the components (30.20b). In particular the diagonal components may be written in the forms

$$I'_{B11} = \int_{P'} \rho \left[(p'_2)^2 + (p'_3)^2 \right] dV \ , \tag{30.24a}$$

$$I'_{B22} = \int_{P'} \rho \left[(p'_1)^2 + (p'_3)^2 \right] dV \ , \tag{30.24b}$$

$$I'_{B33} = \int_{P'} \rho \left[(p'_1)^2 + (p'_2)^2 \right] dV \ , \tag{30.24c}$$

and the off-diagonal components may be written as

$$I'_{B12} = -\int_{P'} \rho \left[p'_1\, p'_2 \right] dV \ , \tag{30.25a}$$

$$I'_{B13} = -\int_{P'} \rho \left[p'_1\, p'_3 \right] dV \ , \tag{30.25a}$$

$$I'_{B23} = -\int_{P'} \rho \left[p'_2\, p'_3 \right] dV \ , \tag{30.25a}$$

Physically the integrands of the diagonal components of inertia can be interpreted as the square of the distance of the material point from the coordinate axes $\mathbf{e}'_i$.

It is also convenient to expand equation (30.12c) for the angular momentum to obtain

$$H'_{Bi} = I'_{Bi1}\, \omega'_1 + I'_{Bi2}\, \omega'_2 + I'_{Bi3}\, \omega'_3 \ , \tag{30.26a}$$

$$\begin{pmatrix} H'_{B1} \\ H'_{B2} \\ H'_{B3} \end{pmatrix} = \begin{pmatrix} I'_{B11} & I'_{B12} & I'_{B13} \\ I'_{B12} & I'_{B22} & I'_{B23} \\ I'_{B13} & I'_{B23} & I'_{B33} \end{pmatrix} \begin{pmatrix} \omega'_1 \\ \omega'_2 \\ \omega'_3 \end{pmatrix} \ . \tag{30.26b}$$

Notice from (30.26b) that in general the inertia tensor has six independent components and the angular momentum vector has a different direction from the angular velocity vector. This means that if you rotate the body about a given direction then in general you will get angular momentum components in the direction of rotation and in the plane normal to the rotation. From a physical point of view it makes sense to ask the question whether there are special directions in which you can rotate the body and only get angular momentum in the direction of rotation. For this case we would have

$$\mathbf{H}_B = \mathbf{I}_B\, \boldsymbol{\omega} = \mathrm{I}\, \boldsymbol{\omega} \ , \quad H'_{Bi} = I'_{Bij}\, \omega'_j = \mathrm{I}\, \omega'_i \ , \tag{30.27a,b}$$

where I is a scalar to be determined. Rewriting (30.27) we have

$$(\mathbf{I}_B - \mathrm{I}\, \mathbf{I})\, \boldsymbol{\omega} = 0 \ , \quad (I'_{Bij} - \mathrm{I}\, \delta_{ij})\, \omega'_j = 0 \ , \tag{30.28a,b}$$

or in expanded form

$$\begin{pmatrix} I'_{B11}-\mathrm{I} & I'_{B12} & I'_{B13} \\ I'_{B12} & I'_{B22}-\mathrm{I} & I'_{B23} \\ I'_{B13} & I'_{B23} & I'_{B33}-\mathrm{I} \end{pmatrix} \begin{pmatrix} \omega'_1 \\ \omega'_2 \\ \omega'_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \ . \tag{30.29}$$

Thus, we have reduced the problem to a standard eigenvalue problem. Since the tensor I'_{Bij} is real and symmetric there exists three <u>eigenvalues</u> I which are determined by the characteristic equation

$$\det\, (I'_{Bij} - \mathrm{I}\, \delta_{ij}) = 0 \ , \tag{30.30}$$

and three associated <u>eigenvectors</u> which are determined by solving (30.29) for the associated directions of ω'_i. These eigenvectors can be ordered to form a right-handed orthonormal set of vectors which we denote by $\mathbf{e}''_i$. Then letting I''_{Bij} be the components of $\mathbf{I}_B$ relative to $\mathbf{e}''_i$ we have

$$I''_{Bij} = \begin{pmatrix} I''_{B11} & 0 & 0 \\ 0 & I''_{B22} & 0 \\ 0 & 0 & I''_{B33} \end{pmatrix} \ , \tag{30.31}$$

with I''_{B11}, I''_{B22}, I''_{B33} being positive constants, since for example

$$I''_{B11} = \int_{P'} \rho\, \left[\, (p''_2)^2 + (p''_3)^2 \,\right] dV \ . \tag{30.32}$$

Sometimes the eigenvalues are called <u>principal values</u> and the eigenvectors are called <u>principal directions</u> of the tensor $\mathbf{I}_B$.

In general the components I'_{Bij} relative to $\mathbf{e}'_i$, and I''_{Bij} relative to $\mathbf{e}''_i$ of the tensor $\mathbf{I}_B$ must satisfy tensor transformation rules in order for $\mathbf{I}_B$ to be independent of our choice of the coordinate system. In particular, let A_{ij} be the transformation tensor (direction cosines) characterizing the relationship between the orientations of $\mathbf{e}_i'$ and $\mathbf{e}_i''$ such that

$$A_{ij} = \mathbf{e}_i'' \bullet \mathbf{e}_j' \quad , \quad \mathbf{e}_i'' = A_{ij}\, \mathbf{e}_j' \quad , \quad \mathbf{e}_i' = A_{ji}\, \mathbf{e}_j'' \quad . \qquad (30.33a,b,c)$$

Notice that by definition the first index of A_{ij} always refers to the double primed system and the second index always refers to the single primed system. It follows that A_{ij} is an orthogonal tensor because

$$\delta_{ij} = \mathbf{e}_i'' \bullet \mathbf{e}_j'' = A_{im}\, \mathbf{e}_m' \bullet A_{jn}\, \mathbf{e}_n' = A_{im}\, A_{jn}\, \delta_{mn} = A_{im}\, A_{jm} = A_{im}\, A_{mj}^T \quad , \qquad (30.34a)$$

$$\delta_{ij} = \mathbf{e}_i' \bullet \mathbf{e}_j' = A_{mi}\, \mathbf{e}_m'' \bullet A_{nj}\, \mathbf{e}_n'' = A_{mi}\, A_{nj}\, \delta_{mn} = A_{mi}\, A_{mj} = A_{im}^T\, A_{mj} \, . \qquad (30.34b)$$

Recalling that the components of $\mathbf{I}_B$ satisfy the equations

$$I_{Bij}' = \mathbf{e}_i' \bullet \mathbf{I}_B\, \mathbf{e}_j' \quad , \quad I_{Bij}'' = \mathbf{e}_i'' \bullet \mathbf{I}_B\, \mathbf{e}_j'' \quad , \qquad (30.35a,b)$$

we may use (30.33) to derive the transformation relations

$$I_{Bij}'' = A_{im}\, \mathbf{e}_m' \bullet \mathbf{I}_B\, A_{jn}\, \mathbf{e}_n' = A_{im}\, (\mathbf{e}_m' \bullet \mathbf{I}_B\, \mathbf{e}_n')\, A_{jn} \, , \qquad (30.36a)$$

$$I_{Bij}'' = A_{im}\, I_{Bmn}'\, A_{jn} = A_{im}\, I_{Bmn}'\, A_{nj}^T \quad , \qquad (30.36b)$$

$$\begin{pmatrix} I_{B11}'' & I_{B12}'' & I_{B13}'' \\ I_{B12}'' & I_{B22}'' & I_{B23}'' \\ I_{B13}'' & I_{B23}'' & I_{B33}'' \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} I_{B11}' & I_{B12}' & I_{B13}' \\ I_{B12}' & I_{B22}' & I_{B23}' \\ I_{B13}' & I_{B23}' & I_{B33}' \end{pmatrix} \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

(30.36c)

and

$$I_{Bij}' = A_{mi}\, \mathbf{e}_m'' \bullet \mathbf{I}_B\, A_{nj}\, \mathbf{e}_m'' = A_{im}\, (\mathbf{e}_m'' \bullet \mathbf{I}_B\, \mathbf{e}_m'')\, A_{nj} \, , \qquad (30.37a)$$

$$I_{Bij}' = A_{mi}\, I_{Bmn}''\, A_{nj} = A_{im}^T\, I_{Bmn}''\, A_{nj} \quad , \qquad (30.37b)$$

$$\begin{pmatrix} I_{B11}' & I_{B12}' & I_{B13}' \\ I_{B12}' & I_{B22}' & I_{B23}' \\ I_{B13}' & I_{B23}' & I_{B33}' \end{pmatrix} = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} \begin{pmatrix} I_{B11}'' & I_{B12}'' & I_{B13}'' \\ I_{B12}'' & I_{B22}'' & I_{B23}'' \\ I_{B13}'' & I_{B23}'' & I_{B31}'' \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

(30.37c)

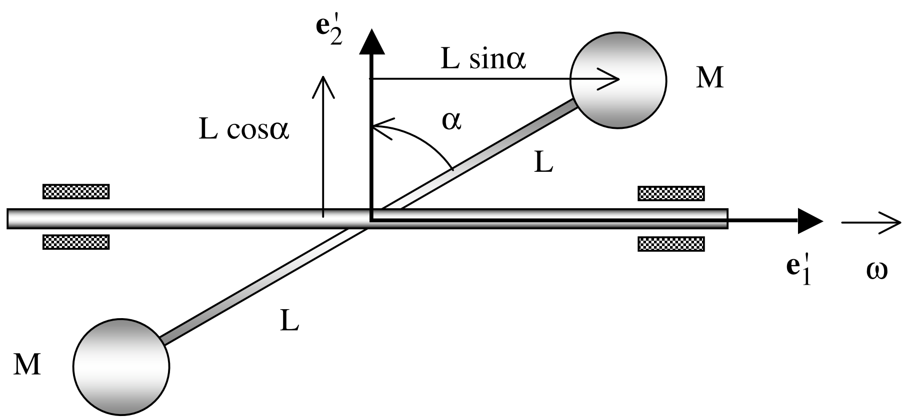

Fig. 30.3

In order to understand the physical meaning of the off-diagonal terms of the inertia tensor consider the case of a body with two concentrated masses M, which rotates with constant angular velocity ω about the $\mathbf{e}_1'$ axes (see Fig. 30.3). Letting $\mathbf{e}_i'$ be the base vectors of a body coordinate system and letting $\boldsymbol{\omega}$ be the angular velocity of the body we have

$$\dot{\mathbf{e}}_i' = \boldsymbol{\omega} \times \mathbf{e}_i' \ , \ \boldsymbol{\omega} = \omega \, \mathbf{e}_1' \ . \tag{30.38a,b}$$

Then referred to the basis $\mathbf{e}_i'$, the components I_{oij}' of the inertia tensor about the fixed origin become

$$I_{o11}' = \int_{P'} \left[(p_2')^2 + (p_3')^2 \right] dm \ = 2 \, M \, L^2 \cos^2 \alpha \ , \tag{30.39a}$$

$$I_{o22}' = \int_{P'} \left[(p_1')^2 + (p_3')^2 \right] dm \ = 2 \, M \, L^2 \sin^2 \alpha \ , \tag{30.39b}$$

$$I_{o33}' = \int_{P'} \left[(p_1')^2 + (p_2')^2 \right] dm \ = 2 \, M \, L^2, \tag{30.39c}$$

$$I_{o12}' = -\int_{P'} \left[p_1' \, p_2' \right] dm \ = - M \, (L \sin \alpha)(L \cos \alpha)$$

$$- M \, (- L \sin \alpha)(- L \cos \alpha) = - M \, L^2 \sin 2\alpha \ , \tag{30.39d}$$

$$I'_{o13} = -\int_{P'} [\ p'_1\, p'_3\ ]\, dm\ = 0\ , \tag{30.39e}$$

$$I'_{o23} = -\int_{P'} [\ p'_2\, p'_3\ ]\, dm\ = 0\ , \tag{30.39f}$$

since the masses are located in the $p'_3 = 0$ plane. Substituting (30.38b) and (30.39) into (30.10) the angular momentum $\mathbf{H}_o$ may be written in the form

$$\mathbf{H}_o = I'_{oij}\, \omega'_j\, \mathbf{e}'_i = I'_{oi1}\, \omega\, \mathbf{e}'_i = (I'_{o11}\, \omega)\, \mathbf{e}'_1 + (I'_{o21}\, \omega)\, \mathbf{e}'_2 + (I'_{o31}\, \omega)\, \mathbf{e}'_3\ ,$$

$$\mathbf{H}_o = (2\, M\, L^2 \cos^2 \alpha)\, \omega\, \mathbf{e}'_1 - (M\, L^2 \sin 2\alpha)\, \omega\, \mathbf{e}'_2\ . \tag{30.40}$$

Notice first that the off-diagonal term I'_{o12} causes the angular momentum $\mathbf{H}_o$ to have a term in the $\mathbf{e}'_2$ direction even though the body is only rotating about the $\mathbf{e}'_1$ direction. Recalling that $\mathbf{H}_o$ is expressed in terms of the rotating basis $\mathbf{e}'_i$, the rate of change of angular momentum becomes

$$\dot{\mathbf{H}}_o = \boldsymbol{\omega} \times \mathbf{H}_o = -\, (M\, L^2 \sin 2\alpha)\, \omega^2\, \mathbf{e}'_3\ , \tag{30.41}$$

since ω is constant. Thus the balance of angular momentum (29.6) yields an expression for the moment $\mathbf{M}_o$ applied to the body of the form

$$\mathbf{M}_o = I'_{o12}\, \omega^2\, \mathbf{e}'_3 = -\, (M\, L^2 \sin 2\alpha)\, \omega^2\, \mathbf{e}'_3\ . \tag{30.42}$$

Notice from (30.42) that if the angle α vanishes then $\mathbf{e}'_i$ are principal axes of inertia so the off-diagonal terms of the inertia tensor vanish and there is no moment required to rotate the body with constant angular velocity about the principal $\mathbf{e}'_1$ axes. This means that in general there is no moment required to rotate a body with constant angular velocity about any of its principal directions. Notice also, that if $0 < \alpha < \pi/2$ then the moment $\mathbf{M}_o$ is directed along the negative $\mathbf{e}'_3$ axes which is consistent with the physical notion that if masses were allowed to rotate freely in the $\mathbf{e}'_1$–$\mathbf{e}'_2$ plane then the angle α would tend to zero.

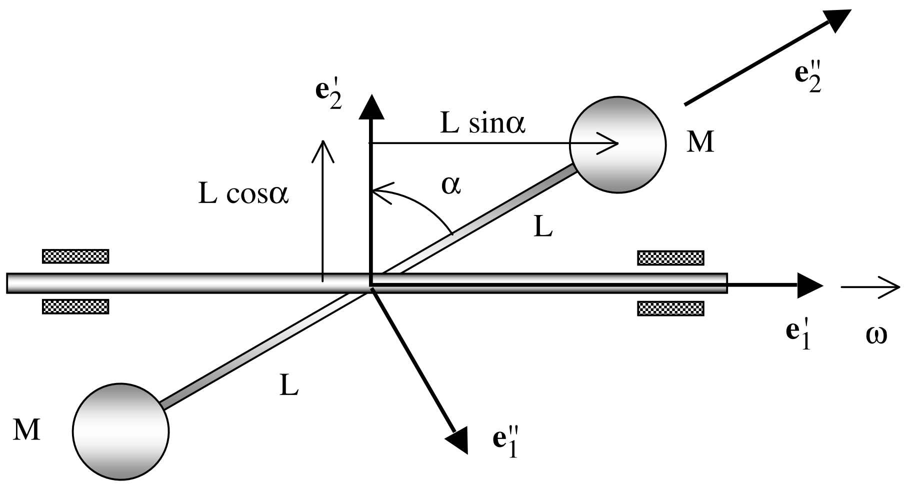

Fig. 30.4

Since we can always choose a body coordinate system in which the inertia tensor is diagonalized it is of interest to reconsider the previous example and examine what simplification and complications occur when we refer all tensors to the vectors $\mathbf{e}_i''$ which are oriented along the principal directions of the body (see Fig. 30.4). For this case the base vectors $\mathbf{e}_i''$ also rotate with angular velocity $\boldsymbol{\omega}$ such that

$$\dot{\mathbf{e}}_i'' = \boldsymbol{\omega} \times \mathbf{e}_i'' \ , \ \ \boldsymbol{\omega} = \omega\,(\cos\alpha\,\mathbf{e}_1'' + \sin\alpha\,\mathbf{e}_2'') \ . \tag{30.43a,b}$$

Since $\mathbf{e}_i''$ are principal axes of inertia the components I_{oij}'' of the inertia tensor referred to $\mathbf{e}_i''$ become

$$I_{o11}'' = I_{o33}'' = 2\,M\,L^2 \ , \ \ I_{o22}'' = I_{o12}'' = I_{o13}'' = I_{o23}'' = 0 \ . \tag{30.44a,b}$$

Recalling that any vector or tensor can be referred to any coordinate system, the angular momentum vector $\mathbf{H}_o$ may be written in the form

$$\mathbf{H}_o = I_{oij}''\,\omega_i''\,\mathbf{e}_i'' = (I_{o11}''\,\omega_1'')\,\mathbf{e}_1'' + (I_{o22}''\,\omega_2'')\,\mathbf{e}_2'' + (I_{o33}''\,\omega_3'')\,\mathbf{e}_3'' \ ,$$

$$\mathbf{H}_o = (I_{o11}''\,\omega_1'')\,\mathbf{e}_1'' = (2\,M\,L^2)\,(\omega\cos\alpha)\,\mathbf{e}_1'' \ . \tag{30.45}$$

Since ω is constant we have

$$\dot{\mathbf{H}}_o = \boldsymbol{\omega} \times \mathbf{H}_o = -\,(M\,L^2 \sin 2\alpha)\,\omega^2\,\mathbf{e}_3'' \ , \tag{30.46}$$

which gives the same moment $\mathbf{M}_o$ as in (30.42). Notice that here the components I''_{oij} of the inertia tensor are simple but the components of the angular velocity $\boldsymbol{\omega}$ referred to $\mathbf{e}''_i$ are more complicated. Since both choices of the coordinate system yield rather simple analyses the particular choice becomes merely a matter of preference.

31. Transfer Theorem For The Inertia Tensor

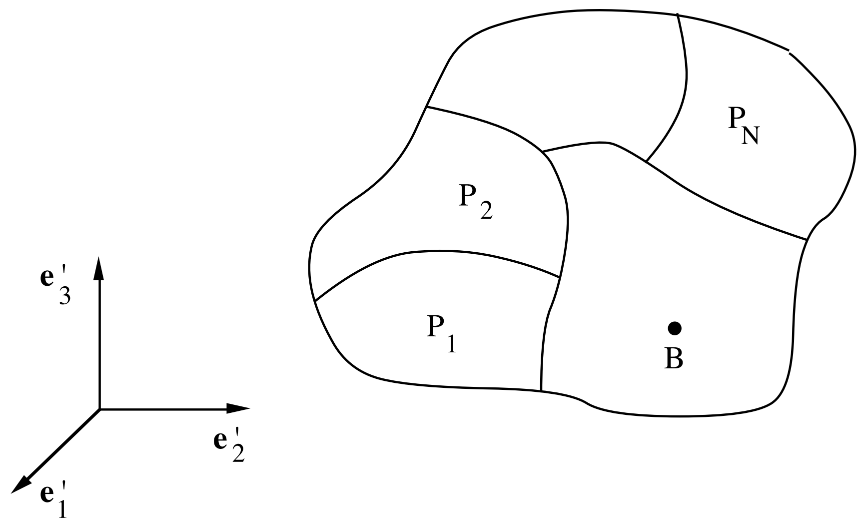

Fig. 31.1

If a body is composed of N parts P_I' (I=1,2,...,N) (see Fig. 31.1), then by the additive property of integration we obtain the additive property of the inertia tensor referred to an arbitrary point B in the body

$$P' = P'_1 \cup P'_2 \cup ... \cup P'_N \quad , \tag{31.1a}$$

$$\mathbf{I}_B = {}_1\mathbf{I}_B + {}_2\mathbf{I}_B + ... + {}_N\mathbf{I}_B \quad , \tag{31.1b}$$

where $\mathbf{I}_B$ is the inertia tensor of the whole body relative to B and ${}_I\mathbf{I}_B$ is the inertia tensor of the part P'_I of body relative to B

$$\mathbf{I}_B = \int_P \rho \left[(\mathbf{p} \bullet \mathbf{p})\, \mathbf{I} - \mathbf{p} \otimes \mathbf{p} \right] dV \quad , \tag{31.2a}$$

$${}_I\mathbf{I}_B = \int_{P_I} \rho \left[(\mathbf{p} \bullet \mathbf{p})\, \mathbf{I} - \mathbf{p} \otimes \mathbf{p} \right] dV \quad . \tag{31.2b}$$

Furthermore letting I'_{Bij} be the components of $\mathbf{I}_B$ and ${}_I I'_{Bij}$ be the components of ${}_I\mathbf{I}_B$ relative to the body base vectors $\mathbf{e}'_i$ we have

$$I'_{Bij} = {}_1 I'_{Bij} + {}_2 I'_{Bij} + ... + {}_N I'_{Bij} \quad . \tag{31.3}$$

It is important to emphasize that the components of tensors can only be added if they are referred to the <u>same</u> base vectors.

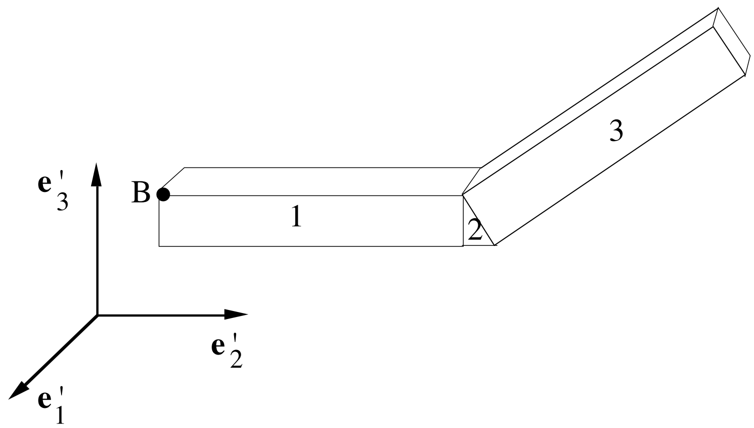

Fig. 31.2

Sometimes a body is composed of simple parts such as the body shown in Fig. 31.2. Since the each of the parts is a simple geometric shape it is usually possible to find the principal values of inertia about the part's center of mass in tables found in dynamics books. However since the centers of mass and principal directions of the parts do not necessarily coincide the components found in the tables cannot be added directly. It is necessary to calculate the inertia tensor of each part relative to the common point B and also transform the components to a common coordinate system before we can use the formula (31.3). In the previous section we discussed the transformation of components of a tensor from the primed system $\mathbf{e}'_i$ to the double primed system $\mathbf{e}''_i$ [see equations (30.37) and (30.37)] so here we can focus on the transfer theorem which allows us to calculate the inertia tensor relative to an arbitrary point B attached to the body, given the inertia tensor $\overline{\mathbf{I}}$ of the body relative to its center of mass.

To this end let: B be an arbitrary point attached rigidly to the body; $\overline{\mathbf{p}}$ be the position vector of the center of mass cm relative to B; $\mathbf{p}$ be the position vector of an arbitrary point in the body relative to B; and $\boldsymbol{\xi}$ be the position vector of an arbitrary point in the body relative to the center of mass so that (see Fig. 31.3)

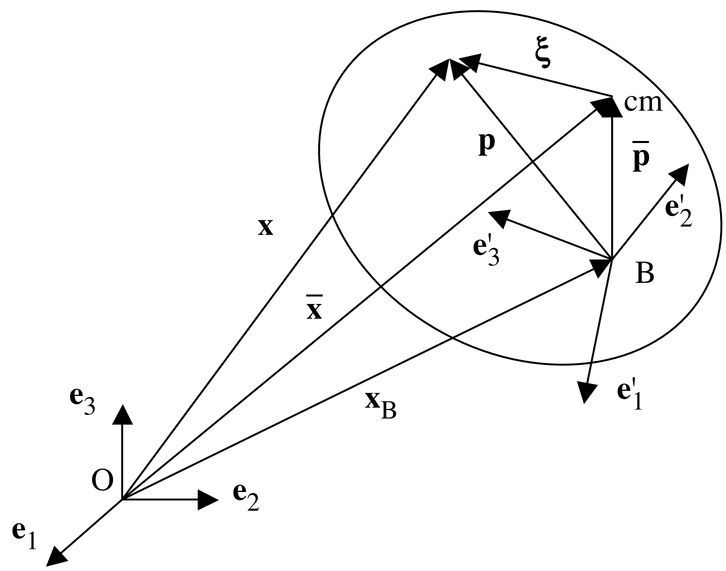

Fig. 31.3

$$\mathbf{p} = \bar{\mathbf{p}} + \boldsymbol{\xi} \quad . \tag{31.4}$$

Using the formula (31.2a) for $\mathbf{I}_B$, and the representation (31.4) we may write

$$\mathbf{I}_B = \int_P \rho \left[\{ (\bar{\mathbf{p}} + \boldsymbol{\xi}) \bullet (\bar{\mathbf{p}} + \boldsymbol{\xi}) \} \mathbf{I} - (\bar{\mathbf{p}} + \boldsymbol{\xi}) \otimes (\bar{\mathbf{p}} + \boldsymbol{\xi}) \right] dV \ ,$$

$$\mathbf{I}_B = \int_P \rho \left[(\bar{\mathbf{p}} \bullet \bar{\mathbf{p}} + 2\,\bar{\mathbf{p}} \bullet \boldsymbol{\xi} + \boldsymbol{\xi} \bullet \boldsymbol{\xi}) \mathbf{I} - (\bar{\mathbf{p}} \otimes \bar{\mathbf{p}} + \bar{\mathbf{p}} \otimes \boldsymbol{\xi} + \boldsymbol{\xi} \otimes \bar{\mathbf{p}} + \boldsymbol{\xi} \otimes \boldsymbol{\xi}) \right] dV \ ,$$

$$\mathbf{I}_B = \int_P \rho \left[(\boldsymbol{\xi} \bullet \boldsymbol{\xi}) \mathbf{I} - \boldsymbol{\xi} \otimes \boldsymbol{\xi} \right] dV + \int_P \rho \left[(\bar{\mathbf{p}} \bullet \bar{\mathbf{p}}) \mathbf{I} - \bar{\mathbf{p}} \otimes \bar{\mathbf{p}} \right] dV$$

$$+ \int_P \rho \left[(2\bar{\mathbf{p}} \bullet \boldsymbol{\xi}) \mathbf{I} - (\bar{\mathbf{p}} \otimes \boldsymbol{\xi} + \boldsymbol{\xi} \otimes \bar{\mathbf{p}}) \right] dV \ , \tag{31.5}$$

However, since $\boldsymbol{\xi}$ is measured from the center of mass of the body

$$\int_P \rho\, \boldsymbol{\xi} \; dV = 0 \ , \tag{31.6}$$

and $\bar{\mathbf{p}}$ is independent of the integration equation (31.5) reduces to

$$\mathbf{I}_B = m \left[(\bar{\mathbf{p}} \bullet \bar{\mathbf{p}}) \mathbf{I} - \bar{\mathbf{p}} \otimes \bar{\mathbf{p}} \right] + \bar{\mathbf{I}} \quad , \tag{31.7}$$

where $\bar{\mathbf{I}}$ is the inertia tensor relative to the center of mass. This is the transfer theorem for the inertia tensor. Now, if we refer all tensors to the base vectors $\mathbf{e}'_i$ we may write the component form of (31.7) as

$$I'_{Bij} = \bar{I}'_{ij} + m\,[(\bar{p}'_m\,\bar{p}'_m)\,\delta_{ij} - \bar{p}'_i\,\bar{p}'_j]\ , \tag{31.8a}$$

and the expanded form as

$$I'_{B11} = \bar{I}'_{11} + m\,[(\bar{p}'_2)^2 + (\bar{p}'_3)^2]\ , \tag{31.9a}$$

$$I'_{B22} = \bar{I}'_{22} + m\,[(\bar{p}'_1)^2 + (\bar{p}'_3)^2]\ , \tag{31.9b}$$

$$I'_{B33} = \bar{I}'_{33} + m\,[(\bar{p}'_1)^2 + (\bar{p}'_2)^2]\ , \tag{31.9c}$$

$$I'_{B12} = \bar{I}'_{12} - m\,[\bar{p}'_1\,\bar{p}'_2]\ , \tag{31.9d}$$

$$I'_{B13} = \bar{I}'_{13} - m\,[\bar{p}'_1\,\bar{p}'_3]\ , \tag{31.9e}$$

$$I'_{B23} = \bar{I}'_{23} - m\,[\bar{p}'_2\,\bar{p}'_3]\ . \tag{31.9f}$$

Notice that even if $\mathbf{e}'_i$ are parallel to the principal directions of inertia about the center of mass (i.e. $\bar{I}'_{ij}$ is a diagonal tensor) they may no longer be parallel to the principal directions of inertia relative to the point B (i.e. I'_{Bij} may not be a diagonal tensor). Finally, we emphasize that although equation (31.7) holds even if the point B is not rigidly attached to the body the resulting tensor $\mathbf{I}_B$ cannot be used in the expression $\mathbf{H}_B = \mathbf{I}_B\,\boldsymbol{\omega}$ for the angular momentum of the body since then $\bar{\mathbf{p}}$ will not be a vector of constant length rotating with angular velocity $\boldsymbol{\omega}$ (i.e. $\dot{\bar{\mathbf{p}}} \neq \boldsymbol{\omega} \times \bar{\mathbf{p}}$).

32. Planar Motion

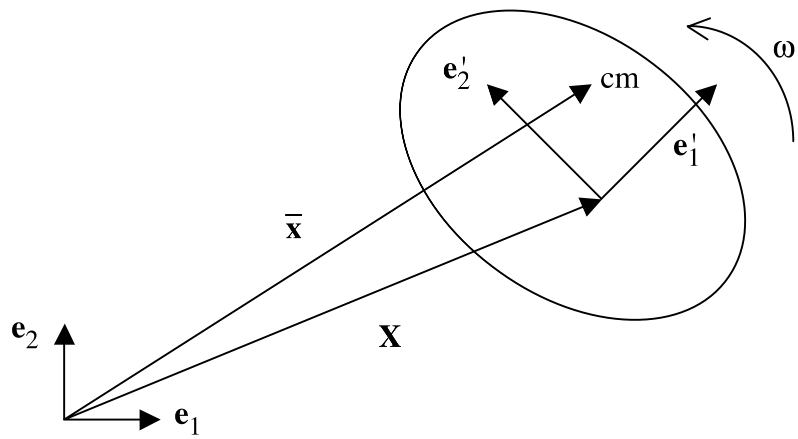

Fig. 32.1

GENERAL PLANAR MOTION

For general planar motion of a rigid body relative to the plane $\mathbf{e}_1$–$\mathbf{e}_2$ we require all points in the rigid body to move in planes parallel to the plane $\mathbf{e}_1$–$\mathbf{e}_2$. Mathematically this means the velocity of every point in the $\mathbf{e}_3$ direction vanishes

$$\mathbf{e}_3 \bullet \mathbf{v} = 0 \ . \tag{32.1}$$

Letting: $\mathbf{p}$ be the position of an arbitrary point in the body relative to the center of mass; $\bar{\mathbf{v}}$ be the velocity of the center of mass; and $\boldsymbol{\omega}$ be the angular velocity of the rigid body, the condition (32.1) requires

$$\mathbf{e}_3 \bullet (\bar{\mathbf{v}} + \boldsymbol{\omega} \times \mathbf{p}) = \mathbf{e}_3 \bullet \bar{\mathbf{v}} + (\mathbf{e}_3 \times \boldsymbol{\omega}) \bullet \mathbf{p} = 0 \ , \tag{32.2}$$

for every point $\mathbf{p}$. It follows that for planar motion we require

$$\mathbf{e}_3 \bullet \bar{\mathbf{v}} = 0 \ , \ \mathbf{e}_3 \times \boldsymbol{\omega} = 0 \ , \tag{32.3a,b}$$

which means that the center of mass moves in the $\mathbf{e}_1$–$\mathbf{e}_2$ plane and the body only rotates about the $\mathbf{e}_3$ direction so the angular velocity becomes

$$\boldsymbol{\omega} = \omega \, \mathbf{e}_3 \ . \tag{32.4}$$

Since the center of mass only moves in the plane it follows that it has no acceleration in the $\mathbf{e}_3$ direction so from the balance of linear momentum we may deduce that the force $\mathbf{F}$ is also planar with

$$\mathbf{F} = \mathrm{F}_1 \, \mathbf{e}_1 + \mathrm{F}_2 \, \mathbf{e}_2 \ , \ \bar{\mathbf{a}} = \bar{\mathrm{a}}_1 \, \mathbf{e}_1 + \bar{\mathrm{a}}_2 \, \mathbf{e}_2 \ . \tag{32.5a,b}$$

Thus, for planar motion the balance of linear momentum reduces to the two scalar equations

$$m\ \bar{a}_1 = F_1 \ ,\ m\ \bar{a}_2 = F_2 \ . \tag{32.6a,b}$$

We can also refer the vectors to the rotating body coordinate system $\mathbf{e}_i'$ defined by

$$\dot{\mathbf{e}}_i' = \boldsymbol{\omega} \times \mathbf{e}_i' \ ,\ \boldsymbol{\omega} = \omega\, \mathbf{e}_3 = \omega\, \mathbf{e}_3' \ , \tag{32.7a,b}$$

and write the balance of linear momentum in the form

$$m\ \bar{a}_1' = F_1' \ ,\ m\, \bar{a}_2' = F_2' \ . \tag{32.8a,b}$$

Letting $\bar{I}_{ij}'$ be the components of the inertia tensor $\bar{\mathbf{I}}$ relative to the center of mass we may express the angular momentum $\bar{\mathbf{H}}$ relative to the center of mass in the form

$$\bar{\mathbf{H}} = \bar{\mathbf{I}}\ \boldsymbol{\omega} = \bar{I}_{ij}'\ \omega_j'\, \mathbf{e}_i' = \bar{I}_{i1}'\ \omega_1'\ \mathbf{e}_i' + \bar{I}_{i2}'\ \omega_2'\ \mathbf{e}_i' + \bar{I}_{i3}'\ \omega_3'\ \mathbf{e}_i' \ ,$$

$$\bar{\mathbf{H}} = \bar{\mathbf{I}}\ \boldsymbol{\omega} = \bar{I}_{i3}'\ \omega\, \mathbf{e}_i' \ = \bar{I}_{13}'\ \omega\, \mathbf{e}_1' + \bar{I}_{23}'\ \omega\, \mathbf{e}_2' + \bar{I}_{33}'\ \omega\, \mathbf{e}_3' \ . \tag{32.9}$$

Thus, the rate of change of angular momentum becomes

$$\dot{\bar{\mathbf{H}}} = \frac{\delta \bar{\mathbf{H}}}{\delta t} + \boldsymbol{\omega} \times \bar{\mathbf{H}} = \frac{\delta \bar{\mathbf{H}}}{\delta t} + \omega \mathbf{e}_3' \times \bar{\mathbf{H}} \ . \tag{32.10}$$

However since $\mathbf{e}_i'$ is a body coordinate system the components $\bar{I}_{ij}'$ are constants so that

$$\frac{\delta \bar{\mathbf{H}}}{\delta t} = \bar{I}_{i3}'\ \dot{\omega}\, \mathbf{e}_i' \ . \tag{32.11}$$

Expanding the cross product in (32.10) we deduce that

$$\dot{\bar{\mathbf{H}}} = (\bar{I}_{13}'\dot{\omega} - \bar{I}_{23}'\ \omega^2)\ \mathbf{e}_1' + (\bar{I}_{23}'\dot{\omega} + \bar{I}_{13}'\ \omega^2)\ \mathbf{e}_2' + (\bar{I}_{33}'\ \dot{\omega})\ \mathbf{e}_3' \ . \tag{32.12}$$

Thus, the component form of the balance of angular momentum (29.22) may be written as

$$\bar{I}_{13}'\dot{\omega} - \bar{I}_{23}'\ \omega^2 = \bar{M}_1' \ , \tag{32.13a}$$

$$\bar{I}_{23}'\dot{\omega} + \bar{I}_{13}'\ \omega^2 = \bar{M}_2' \ , \tag{32.13b}$$

$$\bar{I}_{33}'\ \dot{\omega} = \bar{M}_3' \ . \tag{32.13c}$$

It is important to emphasize that since the body is rotating about an axis which is not parallel to a principal axis of inertia the off diagonal components of the inertia tensor do not vanish so we must apply moments $\bar{M}_1'$ and $\bar{M}_2'$ in order to maintain planar motion relative to the $\mathbf{e}_1'$–$\mathbf{e}_2'$ plane.

PURELY PLANAR MOTION

If the plane of motion $\mathbf{e}_1'$–$\mathbf{e}_2'$ is perpendicular to a principal direction of the inertia tensor then two of the off-diagonal components of inertia vanish (note that $\bar{I}_{12}'$ need not vanish and will depend on our choice of $\mathbf{e}_i'$)

$$\bar{I}_{13}' = 0 \ , \ \bar{I}_{23}' = 0 \ , \qquad (32.14a,b)$$

so the balance of angular momentum reduces to a single scalar equation (32.13c) which can be rewritten in the simpler form ($\bar{I} = \bar{I}_{33}'$)

$$\dot{\bar{H}} = \bar{I}\ \dot{\omega} = \bar{M} \ , \ \bar{H} = \bar{I}\ \omega \ . \qquad (32.15a,b)$$

For this case the motion is called purely planar motion since the only moment that need be supplied is a moment in the direction perpendicular to the plane of motion.

33. Impulse On A Rigid Body

In view of the simplicity of the balance of linear momentum (29.5) and the two forms of the balance of angular momentum (29.6) and (29.2) it follows that changes in momentum from time t_1 to time t_2 can be determined by integrating these balance laws. In particular, we can define the impulsive force $\hat{\mathbf{F}}$ and impulsive moments $\hat{\mathbf{M}}_o$, $\hat{\bar{\mathbf{M}}}$ such that

$$\hat{\mathbf{F}} = \int_{t_1}^{t_2} \mathbf{F} \, dt \quad , \tag{33.1a}$$

$$\hat{\mathbf{M}}_o = \int_{t_1}^{t_2} \mathbf{M}_o \, dt \quad , \tag{33.1b}$$

$$\hat{\bar{\mathbf{M}}} = \int_{t_1}^{t_2} \bar{\mathbf{M}} \, dt \quad . \tag{33.1c}$$

so that integration of these balance laws yields the equations

$$\hat{\mathbf{F}} = \int_{t_1}^{t_2} \dot{\mathbf{G}} \, dt = \mathbf{G}_2 - \mathbf{G}_1 \quad , \tag{33.2a}$$

$$\hat{\mathbf{M}}_o = \int_{t_1}^{t_2} \dot{\mathbf{H}}_o \, dt = \mathbf{H}_{o2} - \mathbf{H}_{o1} \quad , \tag{33.2b}$$

$$\hat{\bar{\mathbf{M}}} = \int_{t_1}^{t_2} \dot{\bar{\mathbf{H}}} \, dt = \bar{\mathbf{H}}_2 - \bar{\mathbf{H}}_1 \quad . \tag{33.2c}$$

As an example let us consider the purely planar motion of a rigid body that is acted upon by an impulsive force $\hat{\mathbf{R}}$ (see Fig. 33.1). The body has mass m, moment of inertia $\bar{I} = m \, \bar{k}^2$ about its center of mass and is initially at rest. The impulsive force $\hat{\mathbf{R}}$ is applied in the $\mathbf{e}_1'$ direction

$$\hat{\mathbf{R}} = \hat{R}\,\mathbf{e}'_1 \ , \tag{33.3}$$

at the position $\mathbf{x}_1$ relative to the center of mass

$$\mathbf{x}_1 = -\,s\,\mathbf{e}'_1 - h\,\mathbf{e}'_2 \quad . \tag{33.4}$$

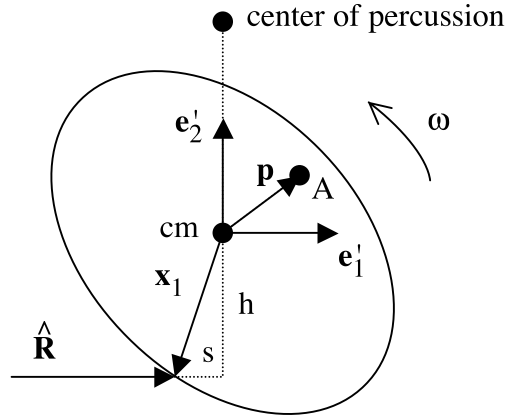

Fig. 33.1

Letting $\bar{\mathbf{v}}_2$ be the velocity of the center of mass after the application of the impulsive force, equation (33.2a) can be used to deduce that

$$\hat{\mathbf{F}} = \hat{\mathbf{R}} = \hat{R}\,\mathbf{e}'_1 = m\,(\bar{\mathbf{v}}_2 - \bar{\mathbf{v}}_1) = m\,\bar{\mathbf{v}}_2 \ , \ \bar{\mathbf{v}}_2 = \frac{\hat{R}}{m}\,\mathbf{e}'_1 \ , \tag{33.5}$$

which means that the center of mass moves in the direction of the impulsive force. Next, we consider the consequence of the balance of angular momentum about the center of mass and let $\boldsymbol{\omega}_2 = \omega_2\,\mathbf{e}'_3$ be the angular velocity of the body after the application of the impulsive force, so that (33.2c) yields

$$\hat{\bar{\mathbf{M}}} = \bar{\mathbf{H}}_2 - \bar{\mathbf{H}}_1 = \bar{I}\,\omega_2\,\mathbf{e}'_3 \quad . \tag{33.6}$$

However, if we assume that the impulsive force is applied to the body over an infinitesimally small time interval then the vector $\mathbf{x}_1$ remains nearly constant during the application of the impulsive force so that $\hat{\bar{\mathbf{M}}}$ may be approximated by

$$\hat{\bar{\mathbf{M}}} = \int_{t_1}^{t_2} \mathbf{x}_1 \times \mathbf{R}(t)\,dt = \mathbf{x}_1 \times \hat{\mathbf{R}} = h\,\hat{R}\,\mathbf{e}'_3 \ , \tag{33.7}$$

where $\mathbf{R}$(t) is the force associated with the impulsive force $\hat{\mathbf{R}}$. Substituting (33.7) into (33.6) we may calculate the value of the angular velocity $\boldsymbol{\omega}_2$

$$\boldsymbol{\omega}_2 = \omega_2\, \mathbf{e}_3' \ , \ \omega_2 = \frac{h\,\hat{R}}{\bar{I}} \ . \tag{33.8a,b}$$

Next, we calculate the velocity of an arbitrary point A which is attached to the body and which is located by the position vector $\mathbf{p}$ relative to the center of mass. Thus, after application of the impulsive force we have

$$\mathbf{v}_A = \bar{\mathbf{v}} + \dot{\mathbf{p}} = \bar{\mathbf{v}} + \boldsymbol{\omega} \times \mathbf{p} \ ,$$

$$\mathbf{v}_A = \frac{\hat{R}}{m}\, \mathbf{e}_1' + \frac{h\,\hat{R}}{\bar{I}}\, \mathbf{e}_3' \times (p_1'\, \mathbf{e}_1' + p_2'\, \mathbf{e}_2') \ ,$$

$$\mathbf{v}_A = [\frac{\hat{R}}{m} - \frac{h\,\hat{R}}{\bar{I}}\, p_2']\, \mathbf{e}_1' + [\frac{h\,\hat{R}}{\bar{I}}\, p_1']\, \mathbf{e}_2' \ . \tag{33.9}$$

Recall that the body instantaneously rotates about its instantaneous center of zero velocity which is located by the values of p_1' and p_2' associated with the point A whose velocity vanishes

$$p_1' = 0 \ , \ p_2' = \frac{\bar{I}}{m\,h} = \frac{\bar{k}^2}{h} \ . \tag{33.10a,b}$$

The result (33.10a) states that the instantaneous center of zero velocity lies along the line perpendicular to the velocity of the center of mass of the body as it should. Also, note that for impulsive problems this point is called the <u>center of percussion</u> because if the body were hinged there the impulsive reaction at the hinge would vanish.

34. Energy Equation For A Rigid Body

Recall from (18.5) that we proved that for a single particle the rate of work of all external forces is equal to the rate of change of kinetic energy

$$P = \mathbf{F} \bullet \mathbf{v} = \dot{T} = \frac{d}{dt}\left[\frac{1}{2} m\, \mathbf{v} \bullet \mathbf{v}\right] . \tag{34.1}$$

Similarly we proved for a system of particles rigidly connected by central forces that the internal forces do no work so the rate of work done by external forces again equals the rate of change of kinetic energy [see (27.4) and (27.16)]

$$P = \sum_{i=1}^{N} \mathbf{F}_i \bullet \mathbf{v}_i = \dot{T} = \sum_{i=1}^{N} T_i \ , \ T_i = \frac{1}{2} \ m_i\, \mathbf{v}_i \bullet \mathbf{v}_i \ . \tag{34.2a,b}$$

Before developing the energy equation for a rigid body it is desirable to first develop an expression for the kinetic energy of the rigid body. It follows from the definition (27.3b) for the kinetic energy of a system of particles that the kinetic energy of a rigid body should be defined by

$$T = \int \frac{1}{2} \mathbf{v} \bullet \mathbf{v} \, dm \ . \tag{34.3}$$

Next, referring the motion to the center of mass we may write

$$\mathbf{x} = \bar{\mathbf{x}} + \mathbf{p} \ , \ \mathbf{v} = \bar{\mathbf{v}} + \dot{\mathbf{p}} = \bar{\mathbf{v}} + \boldsymbol{\omega} \times \mathbf{p} \ , \tag{34.4a,b}$$

where $\mathbf{p}$ is the vector from the center of mass $\bar{\mathbf{x}}$ to an arbitrary point in the body and $\boldsymbol{\omega}$ is the angular velocity of the body. It follows from (34.4b) that

$$\mathbf{v} \bullet \mathbf{v} = (\bar{\mathbf{v}} + \boldsymbol{\omega} \times \mathbf{p}) \bullet (\bar{\mathbf{v}} + \boldsymbol{\omega} \times \mathbf{p}) \ ,$$

$$\mathbf{v} \bullet \mathbf{v} = \bar{\mathbf{v}} \bullet \bar{\mathbf{v}} + 2\, \bar{\mathbf{v}} \bullet (\boldsymbol{\omega} \times \mathbf{p}) + (\boldsymbol{\omega} \times \mathbf{p}) \bullet (\boldsymbol{\omega} \times \mathbf{p}) \ ,$$

$$\mathbf{v} \bullet \mathbf{v} = \bar{\mathbf{v}} \bullet \bar{\mathbf{v}} + 2\, \bar{\mathbf{v}} \bullet (\boldsymbol{\omega} \times \mathbf{p}) + \boldsymbol{\omega} \bullet [\, \mathbf{p} \times (\boldsymbol{\omega} \times \mathbf{p})\,] \ ,$$

$$\mathbf{v} \bullet \mathbf{v} = \bar{\mathbf{v}} \bullet \bar{\mathbf{v}} + 2\, \bar{\mathbf{v}} \bullet (\boldsymbol{\omega} \times \mathbf{p}) + \boldsymbol{\omega} \bullet [\, (\mathbf{p} \bullet \mathbf{p})\, \boldsymbol{\omega} - (\mathbf{p} \bullet \boldsymbol{\omega})\, \mathbf{p}\,] \ . \tag{34.5}$$

Thus, the kinetic energy becomes

$$T = \int \frac{1}{2} \ \bar{\mathbf{v}} \bullet \bar{\mathbf{v}} \, dm + 2\, \bar{\mathbf{v}} \bullet \left[\boldsymbol{\omega} \times \int \frac{1}{2} \ \mathbf{p} \, dm\right]$$

$$+ \frac{1}{2} \ \boldsymbol{\omega} \bullet \int [\, (\mathbf{p} \bullet \mathbf{p})\, \boldsymbol{\omega} - (\mathbf{p} \bullet \boldsymbol{\omega})\, \mathbf{p}\,] \, dm \ ,$$

$$T = \frac{1}{2}\ m\,\bar{\mathbf{v}} \bullet \bar{\mathbf{v}} + \bar{T}\ , \tag{34.6}$$

where the kinetic energy of motion relative to the center of mass $\bar{T}$ is given by

$$\bar{T} = \frac{1}{2}\,\boldsymbol{\omega} \bullet \bar{\mathbf{H}} = \frac{1}{2}\,\boldsymbol{\omega} \bullet \bar{\mathbf{I}}\,\boldsymbol{\omega} = \frac{1}{2}\,\omega_i'\,\bar{I}_{ij}'\,\omega_j'\ . \tag{34.7}$$

Note that in words, equation (34.6) states that the kinetic energy of the rigid body is equal to the kinetic energy of the center of mass plus the kinetic energy of motion relative to the center of mass.

If the body rotates about a fixed point O (see Fig. 34.1) then the kinetic energy T may be expressed in the alternative form

$$T = \int \frac{1}{2}\ \mathbf{v} \bullet \mathbf{v}\,dm = \frac{1}{2}\int (\boldsymbol{\omega} \times \mathbf{x} \bullet \mathbf{v})\,dm = \frac{1}{2}\int (\boldsymbol{\omega} \bullet \mathbf{x} \times \mathbf{v})\,dm\ ,$$

$$T = \frac{1}{2}\ \boldsymbol{\omega} \bullet \int (\mathbf{x} \times \mathbf{v})\,dm = \frac{1}{2}\ \boldsymbol{\omega} \bullet \mathbf{H}_o = \frac{1}{2}\ \boldsymbol{\omega} \bullet \mathbf{I}_o\,\boldsymbol{\omega} = \frac{1}{2}\,\omega_i'\,\bar{I}_{oij}'\,\omega_j'\ . \tag{34.8}$$

One of the differences between a rigid body and a system of particles is that it is possible to apply external moments at points on a rigid body which can do work on the body. Therefore, in order to develop the general form of the energy equation for a rigid body we generalize the procedure used to obtain (34.2) which took the scalar product of the equations of linear momentum of each particle with its associated velocity and summed the result. Thus, for a rigid body we multiply the linear momentum equation (29.5) by the velocity of the center of mass $\bar{\mathbf{v}}$ and multiply the balance of angular momentum (29.22) by the angular velocity $\boldsymbol{\omega}$ and sum the results to obtain

$$[\int \mathbf{b}\,dm + \sum_{i=1}^{N} \mathbf{F}_i\,] \bullet \bar{\mathbf{v}} + [\int \mathbf{p} \times \mathbf{b}\,dm + \sum_{i=1}^{N} \mathbf{p}_i \times \mathbf{F}_i + \sum_{i=1}^{M} \mathbf{M}_i] \bullet \boldsymbol{\omega}$$

$$= m\,\dot{\bar{\mathbf{v}}} \bullet \bar{\mathbf{v}} + \dot{\bar{\mathbf{H}}} \bullet \boldsymbol{\omega}\ , \tag{34.9}$$

where $\mathbf{p}_i$ is the vector measured from the center of mass to the point of application of the force $\mathbf{F}_i$. Using the properties of the scalar triple product and the fact that $\bar{\mathbf{v}}$ and $\boldsymbol{\omega}$ are functions of time only it is possible to rearrange this equation into the form

$$\int \mathbf{b} \bullet (\bar{\mathbf{v}} + \boldsymbol{\omega} \times \mathbf{p})\, dm + \sum_{i=1}^{N} \mathbf{F}_i \bullet (\bar{\mathbf{v}} + \boldsymbol{\omega} \times \mathbf{p}_i) + \sum_{i=1}^{M} \mathbf{M}_i \bullet \boldsymbol{\omega} = m\, \dot{\bar{\mathbf{v}}} \bullet \bar{\mathbf{v}} + \dot{\bar{\mathbf{H}}} \bullet \boldsymbol{\omega} \quad . \tag{34.10}$$

However, since $\mathbf{p}$ and $\mathbf{p}_i$ are each vectors that connect two material points on the rigid body it follows that the absolute velocity $\mathbf{v}$ of an arbitrary material point and the absolute velocity $\mathbf{v}_i$ of the point of application of the force $\mathbf{F}_i$ are given by

$$\mathbf{v} = \bar{\mathbf{v}} + \boldsymbol{\omega} \times \mathbf{p} \;, \quad \mathbf{v}_i = \bar{\mathbf{v}} + \boldsymbol{\omega} \times \mathbf{p}_i \;, \tag{34.11a,b}$$

so that (34.10) reduces to

$$\int \mathbf{b} \bullet \mathbf{v}\, dm + \sum_{i=1}^{N} \mathbf{F}_i \bullet \mathbf{v}_i + \sum_{i=1}^{M} \mathbf{M}_i \bullet \boldsymbol{\omega} = m\, \dot{\bar{\mathbf{v}}} \bullet \bar{\mathbf{v}} + \dot{\bar{\mathbf{H}}} \bullet \boldsymbol{\omega} \;. \tag{34.12}$$

Next, using (34.7) and the fact that $\mathbf{e}_i'$ is a body coordinate system it may be shown that

$$\dot{\bar{T}} = \omega_i' \, \bar{I}_{ij}' \, \dot{\omega}_j' \quad . \tag{34.13}$$

Moreover, expressing $\bar{\mathbf{H}}$ in terms of its components relative to $\mathbf{e}_i'$ we have

$$\bar{\mathbf{H}} = \bar{I}_{ij}' \, \omega_j' \, \mathbf{e}_i' \quad , \tag{34.14a}$$

$$\dot{\bar{\mathbf{H}}} = \frac{\delta \bar{\mathbf{H}}}{\delta t} + \boldsymbol{\omega} \times \bar{\mathbf{H}} \;, \tag{34.14b}$$

$$\dot{\bar{\mathbf{H}}} \bullet \boldsymbol{\omega} = \frac{\delta \bar{\mathbf{H}}}{\delta t} \bullet \boldsymbol{\omega} = \bar{I}_{ij}' \, \dot{\omega}_j' \, \mathbf{e}_i' \bullet \boldsymbol{\omega} = \omega_i' \, \bar{I}_{ij}' \, \dot{\omega}_j' = \dot{\bar{T}} \quad . \tag{34.14c}$$

Thus, it follows that

$$m\, \dot{\bar{\mathbf{v}}} \bullet \bar{\mathbf{v}} + \dot{\bar{\mathbf{H}}} \bullet \boldsymbol{\omega} = \dot{T} \quad , \tag{34.15}$$

so the energy equation for a rigid body may be written in the usual form

$$P = \dot{T} \;, \tag{34.16}$$

where the mechanical power P is defined by

$$P = \int \mathbf{b} \bullet \mathbf{v}\, dm + \sum_{i=1}^{N} \mathbf{F}_i \bullet \mathbf{v}_i + \sum_{i=1}^{M} \mathbf{M}_i \bullet \boldsymbol{\omega} \quad . \tag{34.17}$$

For a uniform (independent of position) body force **b**, the rate of work of the body force can be expressed in the simple form

$$\int \mathbf{b} \bullet \mathbf{v}\, dm = m\, \mathbf{b} \bullet \bar{\mathbf{v}} \ , \tag{34.18}$$

which is equivalent to an external force (m **b**) acting at the center of mass of the body. Then, (34.17) reduces to the simpler form

$$P = m\mathbf{b} \bullet \bar{\mathbf{v}} + \sum_{i=1}^{N} \mathbf{F}_i \bullet \mathbf{v}_i + \sum_{i=1}^{M} \mathbf{M}_i \bullet \boldsymbol{\omega} \ . \tag{34.19}$$

Here it is important to emphasize that the external forces $\mathbf{F}_i$ are multiplied by the absolute velocities $\mathbf{v}_i$ of the material points where they are applied and that all the external moments $\mathbf{M}_i$ are multiplied by the <u>same</u> absolute angular velocity $\boldsymbol{\omega}$ of the rigid body.

Using the form (34.19) of the mechanical power we may integrate the energy equation (34.16) over the time interval $[t_1,t_2]$ and write an energy equation in the form

$$\bar{U}_{2/1} = (T_2 - T_1) + (V_{g2} - V_{g1}) + (V_{e2} - V_{e1}) \ , \tag{34.20}$$

where $\bar{U}_{2/1}$ is the work done by all external forces which don't contribute to gravitational potential energy V_g or elastic potential energy V_e and the work done by all external moments.

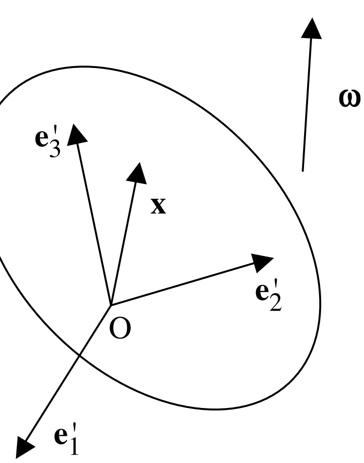

Fig. 34.1

35. Angular Momentum And Transformation Relations

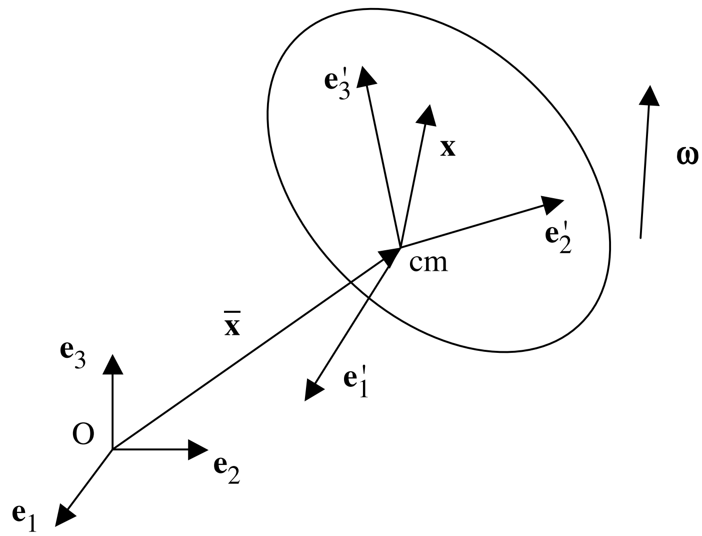

Fig. 35.1

Consider a rigid body that is moving and rotating with absolute angular velocity $\boldsymbol{\omega}$ relative to a fixed coordinate system $\mathbf{e}_i$ with origin O. Let $\mathbf{e}_i'$ be a body coordinate system rotating with angular velocity $\boldsymbol{\omega}$ and let $\mathbf{e}_i''$ be another coordinate system rotating with angular velocity $\boldsymbol{\Omega}$ such that

$$\dot{\mathbf{e}}_i' = \boldsymbol{\omega} \times \mathbf{e}_i' \ , \ \dot{\mathbf{e}}_i'' = \boldsymbol{\Omega} \times \mathbf{e}_i'' \ . \tag{35.1a,b}$$

Recall from (29.25) that the angular momentum $\mathbf{H}_o$ of a body about the fixed point O is the angular momentum of the center of mass plus the angular momentum $\bar{\mathbf{H}}$ about the center of mass

$$\mathbf{H}_o = \bar{\mathbf{x}} \times \mathrm{m}\, \bar{\mathbf{v}} + \bar{\mathbf{H}} \ , \tag{35.2}$$

where $\bar{\mathbf{H}}$ may be expressed in terms of the components $\bar{I}_{ij}'$ of the inertia tensor $\bar{\mathbf{I}}$ relative to the basis $\mathbf{e}_i'$ as

$$\bar{\mathbf{H}} = \bar{I}_{ij}' \, \omega_j' \, \mathbf{e}_i' \ . \tag{35.3}$$

Note that since $\bar{\mathbf{H}}$, $\bar{\mathbf{I}}$, $\boldsymbol{\omega}$ are tensors we can express them in terms of any basis which we choose. In particular we can use the basis $\mathbf{e}_i''$ and write

$$\bar{\mathbf{H}} = \bar{I}''_{ij}\, \omega''_j\, \mathbf{e}''_i \; . \tag{35.4}$$

To prove this we recall the properties (30.33a,c),(30.34a) of the transformation tensor A_{ij} between the two bases $\mathbf{e}''_i$ and $\mathbf{e}'_i$

$$A_{ij} = \mathbf{e}''_i \bullet \mathbf{e}'_j \;,\; \mathbf{e}'_i = A_{mi}\, \mathbf{e}''_m \;,\; A_{im}A_{jm} = \delta_{ij} \;, \tag{35.5a,b,c}$$

and use (30.33b,c) to derive the transformation relations between the components ω_i' and ω''_i of the angular velocity vector $\boldsymbol{\omega}$

$$\omega''_i = A_{ij}\, \omega'_j \;,\quad \begin{pmatrix} \omega''_1 \\ \omega''_2 \\ \omega''_3 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} \omega'_1 \\ \omega'_2 \\ \omega'_3 \end{pmatrix} , \tag{35.6a,b}$$

$$\omega'_i = A_{ji}\, \omega''_j = A^T_{ij}\, \omega''_j \;,\quad \begin{pmatrix} \omega'_1 \\ \omega'_2 \\ \omega'_3 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} \begin{pmatrix} \omega''_1 \\ \omega''_2 \\ \omega''_3 \end{pmatrix} . \tag{35.6a,b}$$

Then, with the help of (35.5), (35.6) and the transformation relations (30.37b) it may be shown that

$$\bar{\mathbf{H}} = \bar{I}'_{ij}\, \omega'_j\, \mathbf{e}'_i = (A_{mi}\, A_{nj}\, \bar{I}''_{mn})\, (A_{rj}\, \omega''_r)\, (A_{si}\, \mathbf{e}''_s) \;,$$

$$\bar{\mathbf{H}} = (A_{mi}\, A_{si})\, (A_{nj}\, A_{rj})\, (\bar{I}''_{mn}\, \omega''_r\, \mathbf{e}''_s) = \delta_{ms}\, \delta_{nr}\, (\bar{I}''_{mn}\, \omega''_r\, \mathbf{e}''_s) \;,$$

$$\bar{\mathbf{H}} = \bar{I}''_{sr}\, \omega''_r\, \mathbf{e}''_s = \bar{I}''_{ij}\, \omega''_j\, \mathbf{e}''_i \; . \tag{35.7}$$

Using the expressions (35.3) and (35.4) and recalling the associated angular velocities of the base vectors $\mathbf{e}'_i$ and $\mathbf{e}''_i$ we may calculate the rate of change of angular momentum in the equivalent forms

$$\dot{\bar{\mathbf{H}}} = \frac{d}{dt}(\bar{I}'_{ij}\, \omega'_j)\, \mathbf{e}'_i + \boldsymbol{\omega} \times \bar{\mathbf{H}} \;, \tag{35.8a}$$

$$\dot{\bar{\mathbf{H}}} = \frac{d}{dt}(\bar{I}''_{ij}\, \omega''_j)\, \mathbf{e}''_i + \boldsymbol{\Omega} \times \bar{\mathbf{H}} \; . \tag{35.8b}$$

It is important to note that although the components $\bar{I}'_{ij}$ are independent of time because $\mathbf{e}'_i$ is a body coordinate system the components $\bar{I}''_{ij}$ may depend on time.

For example, let us consider the simple case when $\mathbf{e}'_i$ are parallel to the principal axes of inertia so that

$$\bar{I}'_{ij} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \quad . \tag{35.9}$$

Also, let $\mathbf{e}'_i$ be related to $\mathbf{e}''_i$ such that

$$\mathbf{e}'_1 = \cos\phi\, \mathbf{e}''_1 + \sin\phi\, \mathbf{e}''_2 \ \ , \ \ \mathbf{e}'_2 = -\sin\phi\, \mathbf{e}''_1 + \cos\phi\, \mathbf{e}''_2 \ , \tag{35.10a,b}$$

$$\mathbf{e}'_3 = \mathbf{e}''_3 \ \ , \ \ \phi = \mathrm{pt} \ \ , \tag{35.10c,d}$$

For this case the rigid body rotates with angular velocity p $\mathbf{e}''_3$ relative to the $\mathbf{e}''_i$ coordinate system. It follows from the definition (35.5a) that

$$A_{ij} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \ \ , \tag{35.11}$$

so that the transformation relations (30.36c) yield

$$\begin{pmatrix} \bar{I}''_{11} & \bar{I}''_{12} & \bar{I}''_{13} \\ \bar{I}''_{12} & \bar{I}''_{22} & \bar{I}''_{23} \\ \bar{I}''_{13} & \bar{I}''_{23} & \bar{I}''_{33} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \bar{I}''_{11} & \bar{I}''_{12} & \bar{I}''_{13} \\ \bar{I}''_{12} & \bar{I}''_{22} & \bar{I}''_{23} \\ \bar{I}''_{13} & \bar{I}''_{23} & \bar{I}''_{33} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_1\cos\phi & I_1\sin\phi & 0 \\ -I_2\sin\phi & I_2\cos\phi & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

$$\begin{pmatrix} \bar{I}''_{11} & \bar{I}''_{12} & \bar{I}''_{13} \\ \bar{I}''_{12} & \bar{I}''_{22} & \bar{I}''_{23} \\ \bar{I}''_{13} & \bar{I}''_{23} & \bar{I}''_{33} \end{pmatrix} = \begin{pmatrix} I_1\cos^2\phi + I_2\sin^2\phi & \{I_1 - I_2\}\sin\phi\cos\phi & 0 \\ \{I_1 - I_2\}\sin\phi\cos\phi & I_1\sin^2\phi + I_2\cos^2\phi & 0 \\ 0 & 0 & I_3 \end{pmatrix} . \tag{35.12}$$

Notice from (35.12) that in general the components $\bar{I}''_{ij}$ are functions of time and that $\bar{I}''_{ij}$ is not a diagonal tensor. However, if the components of inertia I_1 and I_2 of the body along the axes perpendicular to the axis of relative rotation $\mathbf{e}''_3$ are equal

$$I_1 = I_2 \ , \tag{35.13}$$

then (35.12) simplifies to give

$$\overline{I}''_{ij} = \overline{I}'_{ij} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \ . \tag{35.14}$$

Physically, this means that when two principal values of inertia are equal then the body possesses rotational symmetry about any axis in the plane characterized by the associated principal axes of inertia. Consequently, the components of the tensor of inertia are insensitive to rotations of the coordinate axes about the axis perpendicular to this plane.

36. Point Masses, Massless Links, And A System Of Rigid Bodies

A point mass is an idealized rigid body which has finite mass m but zero volume so that its mass is concentrated at a point instead of distributed over a region of space. It follows from the definition (30.20a) of the inertia tensor about a point B that the inertia tensor $\bar{\mathbf{I}}$ about the center of mass of a mass point vanishes

$$\bar{\mathbf{I}} = 0 \ . \tag{36.1}$$

Thus, the angular momentum $\bar{\mathbf{H}}$ about the center of mass of the mass point vanishes

$$\bar{\mathbf{H}} = 0 \ , \tag{36.2}$$

so the balance of angular momentum requires the resultant moment $\bar{\mathbf{M}}$ about the center of mass to also vanish

$$\bar{\mathbf{M}} = 0 \ , \tag{36.3}$$

These results indicate that a mass point can only be subjected to a force which passes through its center of mass.

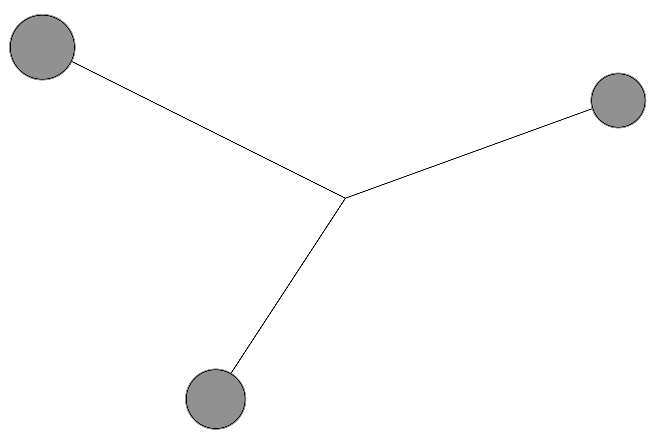

Fig. 36.1

Sometimes a rigid body may be idealized as a collection of point masses connected by massless rigid links (see Fig. 36.1). It follows that if the rigid body is idealized as massless then both the linear momentum $\mathbf{G}$ and the angular momentum $\bar{\mathbf{H}}$ vanish so the balances of linear momentum and angular momentum require the resultant force $\mathbf{F}$ and moment $\bar{\mathbf{M}}$ to vanish

$$\mathbf{F} = 0 \;\; , \;\; \bar{\mathbf{M}} = 0 \;\; . \tag{36.4}$$

Furthermore since the mass vanishes it follows from the balance of angular momentum (29.18) about an arbitrary moving point B that the resultant moment $\mathbf{M}_B$ vanishes

$$\mathbf{M}_B = 0 \;\; . \tag{36.5}$$

This means that the resultant moment about any point vanishes.

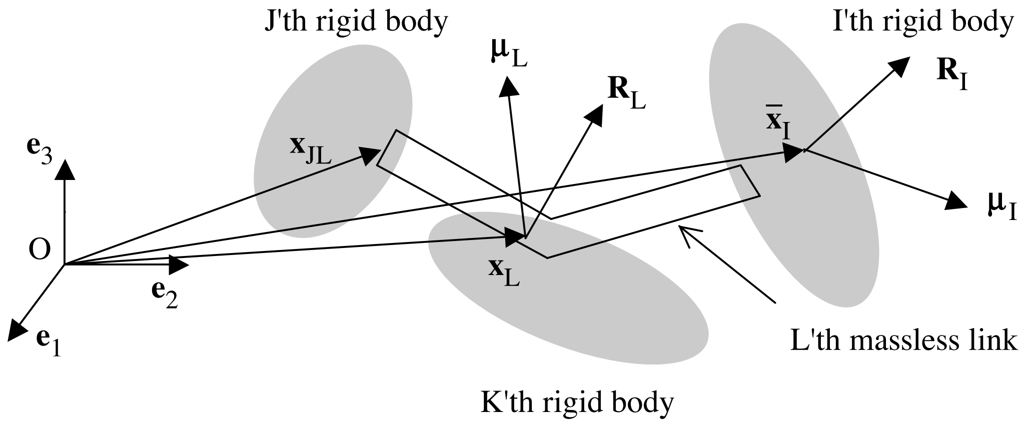

Fig. 36.2

Here we consider a system of N rigid bodies connected by M *massless* links. For example, consider a typical case where the L'th massless link connects the I'th, J'th and K'th rigid bodies (see Fig. 36.2). Let $\mathbf{R}_I$ be the resultant external force applied to the I'th rigid body and let $\boldsymbol{\mu}_I$ be the resultant external moment applied to the I'th rigid body about its center of mass $\bar{\mathbf{x}}_I$. Also, let $\mathbf{R}_L$ and $\boldsymbol{\mu}_L$ be the resultant external force and moment applied to the L'th link at the point $\mathbf{x}_L$. Furthermore, let $\mathbf{f}_{IL}$ be the force and $\mathbf{m}_{IL}$ be the moment, both applied by the L'th link on the I'th rigid body at their point of contact $\mathbf{x}_{IL}$. Since the force and moment applied by the I'th rigid body on the L'th link are equal in magnitude and opposite in direction to the $\mathbf{f}_{IL}$ and $\mathbf{m}_{IL}$, respectively, the free-body diagrams of the I'th rigid body and the L'th link are given by Fig. 36.3.

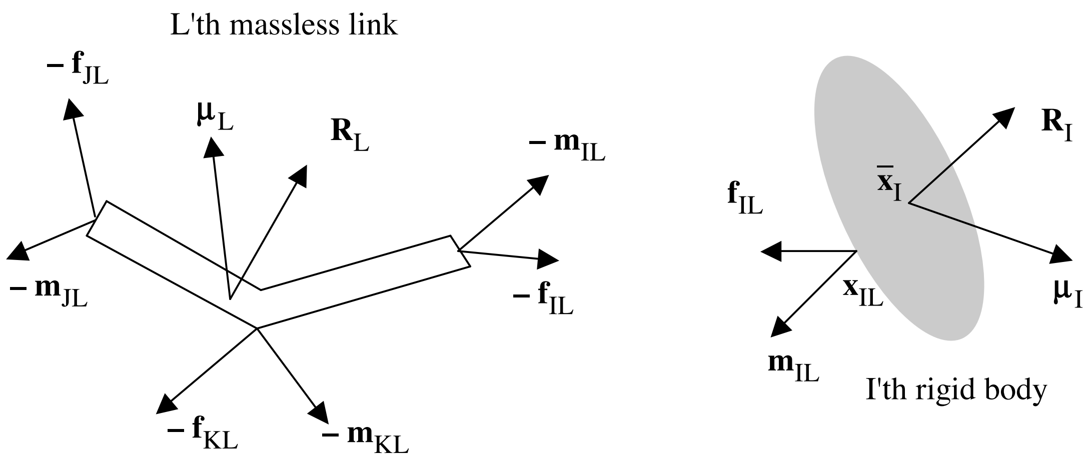

Fig. 36.3

It follows from the free body diagram that the balances of linear and angular momentum of the I'th rigid body may be written in the forms

$$\dot{\mathbf{G}}_I = \mathbf{R}_I + \mathbf{f}_{IL} \ , \tag{36.6a}$$

$$\dot{\mathbf{H}}_{Io} = \bar{\mathbf{x}}_I \times \mathbf{R}_I + \boldsymbol{\mu}_I + \mathbf{x}_{IL} \times \mathbf{f}_{IL} + \mathbf{m}_{IL} \ , \tag{36.6b}$$

where $\mathbf{G}_I$ is the linear momentum of the I'th rigid body and $\mathbf{H}_{Io}$ is the angular momentum of the I'th rigid body about the fixed origin O.

For our present purposes we allow the massless link to be a joint which enforces kinematical constraints on the relative motion of the connected body or even a motorized joint which can control the relative motion between the connected bodies. However, since the link is considered to be massless we assume that like a massless rigid body, the resultant force and moment applied to it both vanish. In particular, with reference to the L'th link in Fig. 36.3 this assumption requires

$$\mathbf{R}_L + (-\mathbf{f}_{IL}) + (-\mathbf{f}_{JL}) + (-\mathbf{f}_{KL}) = 0 \ , \tag{36.7a}$$

$$\mathbf{x}_L \times \mathbf{R}_L + \boldsymbol{\mu}_L + \Big[\, \mathbf{x}_{IL} \times (-\mathbf{f}_{IL}) + \mathbf{x}_{JL} \times (-\mathbf{f}_{JL}) + \mathbf{x}_{KL} \times (-\mathbf{f}_{KL})$$

$$+ (-\mathbf{m}_{IL}) + (-\mathbf{m}_{JL}) + (-\mathbf{m}_{KL})\Big] = 0 \ . \tag{36.7b}$$

Now, if we sum the balances of linear and angular momentum of the system of three rigid bodies in Fig. 36.2 we may deduce that

$$\dot{\mathbf{G}}_I + \dot{\mathbf{G}}_J + \dot{\mathbf{G}}_K = (\mathbf{R}_I + \mathbf{f}_{IL}) + (\mathbf{R}_J + \mathbf{f}_{JL}) + (\mathbf{R}_K + \mathbf{f}_{KL}) \ ,$$

$$\dot{\mathbf{G}}_I + \dot{\mathbf{G}}_J + \dot{\mathbf{G}}_K = (\mathbf{R}_I + \mathbf{R}_J + \mathbf{R}_K) + (\mathbf{f}_{IL} + \mathbf{f}_{JL} + \mathbf{f}_{KL}) \ , \tag{36.8a}$$

$$\begin{aligned}\dot{\mathbf{H}}_{Io} + \dot{\mathbf{H}}_{Jo} + \dot{\mathbf{H}}_{Ko} = {} & (\bar{\mathbf{x}}_I \times \mathbf{R}_I + \boldsymbol{\mu}_I + \mathbf{x}_{IL} \times \mathbf{f}_{IL} + \mathbf{m}_{IL}) \\ & + (\bar{\mathbf{x}}_J \times \mathbf{R}_J + \boldsymbol{\mu}_J + \mathbf{x}_{JL} \times \mathbf{f}_{JL} + \mathbf{m}_{JL}) \\ & + (\bar{\mathbf{x}}_K \times \mathbf{R}_K + \boldsymbol{\mu}_K + \mathbf{x}_{KL} \times \mathbf{f}_{KL} + \mathbf{m}_{KL}) \ ,\end{aligned}$$

$$\begin{aligned}\dot{\mathbf{H}}_{Io} + \dot{\mathbf{H}}_{Jo} + \dot{\mathbf{H}}_{Ko} = {} & (\bar{\mathbf{x}}_I \times \mathbf{R}_I + \boldsymbol{\mu}_I + \bar{\mathbf{x}}_J \times \mathbf{R}_J + \boldsymbol{\mu}_J + \bar{\mathbf{x}}_K \times \mathbf{R}_K + \boldsymbol{\mu}_K) \\ & + \left[\mathbf{x}_{IL} \times \mathbf{f}_{IL} + \mathbf{m}_{IL} + \mathbf{x}_{JL} \times \mathbf{f}_{JL} + \mathbf{m}_{JL} + \mathbf{x}_{KL} \times \mathbf{f}_{KL} + \mathbf{m}_{KL}\right] .\end{aligned} \tag{36.8b}$$

With the help of the equations of motion (36.7) of the massless link equations (36.8) may be rewritten in the forms

$$\dot{\mathbf{G}}_I + \dot{\mathbf{G}}_J + \dot{\mathbf{G}}_K = (\mathbf{R}_I + \mathbf{R}_J + \mathbf{R}_K) + \mathbf{R}_L \ , \tag{36.9a}$$

$$\begin{aligned}\dot{\mathbf{H}}_{Io} + \dot{\mathbf{H}}_{Jo} + \dot{\mathbf{H}}_{Ko} = {} & (\bar{\mathbf{x}}_I \times \mathbf{R}_I + \boldsymbol{\mu}_I + \bar{\mathbf{x}}_J \times \mathbf{R}_J + \boldsymbol{\mu}_J + \bar{\mathbf{x}}_K \times \mathbf{R}_K + \boldsymbol{\mu}_K) \\ & + \mathbf{x}_L \times \mathbf{R}_L + \boldsymbol{\mu}_L \ .\end{aligned} \tag{36.9b}$$

These equations state that the rate of change of the linear momentum of the system is equal to the total resultant <u>external</u> force applied to the system

$$\dot{\mathbf{G}} = \mathbf{F} \ , \tag{36.10}$$

and that the rate of change of angular momentum of the system about the fixed point O is equal to the total resultant <u>external</u> moment applied to the system about O

$$\dot{\mathbf{H}}_o = \mathbf{M}_o \ . \tag{36.11}$$

It is important to emphasize that in calculating the resultant external force applied to the system we must include both the external forces and moments applied directly to the rigid bodies as well as those applied directly to the link.

Obviously, this analysis of a system of three rigid bodies connected by a single massless link can be generalized to a system of any number of rigid bodies connected by any number of massless links. The end result of such and analysis is the statement of the balances of linear momentum and angular momentum of the system in the forms (36.10) and (36.11), respectively.

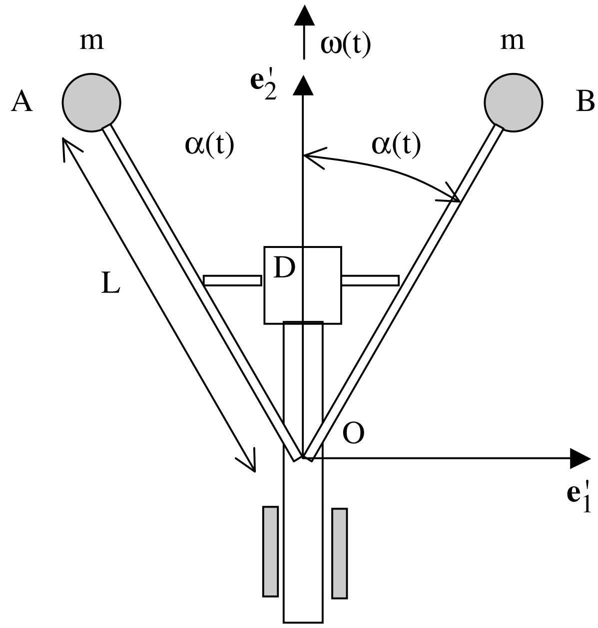

Fig. 36.4

Consider the example shown in Fig. 36.4 of two point masses, each of mass m, which are connected to a massless link AOBD which is allowed to rotate freely about the fixed $\mathbf{e}'_2$ axis. Letting $\mathbf{e}'_i$ be a rotating set of base vectors defined so that the two masses remain in the $\mathbf{e}'_1$–$\mathbf{e}'_2$ plane we have

$$\dot{\mathbf{e}}'_i = \boldsymbol{\omega} \times \mathbf{e}'_i \ , \quad \boldsymbol{\omega} = \omega\, \mathbf{e}'_2 \ . \tag{36.12a,b}$$

Furthermore, the link has a motor D (also idealized as massless!) which controls the angle α between the bars OA and OB and the $\mathbf{e}'_2$ direction. Initially the system has constant angular velocity ω_0 and constant angle α_0 so that

$$\omega(0) = \omega_0 \ , \ \alpha(0) = \alpha_0 \ , \ \dot{\alpha}(0) = 0 \ . \tag{36.13a,b,c}$$

The motor D is then operated to change the angle α as a function of time. The objective is to determine the value of $\omega(t)$ caused by this change in α.

	$\mathbf{e}'_1$	$\mathbf{e}'_2$	$\mathbf{e}'_3$
$\boldsymbol{\omega}$	0	ω	0
$\mathbf{x}_A$	$-L\sin\alpha$	$L\cos\alpha$	0
$\delta\mathbf{x}_A/\delta t$	$-L\dot{\alpha}\cos\alpha$	$-L\dot{\alpha}\sin\alpha$	0
$\boldsymbol{\omega}\times\mathbf{x}_A$	0	0	$\omega L\sin\alpha$
$\mathbf{v}_A$	$-L\dot{\alpha}\cos\alpha$	$-L\dot{\alpha}\sin\alpha$	$\omega L\sin\alpha$
$\mathbf{x}_A\times m\,\mathbf{v}_A$	$m\,\omega L^2\sin\alpha\cos\alpha$	$m\,\omega L^2\sin^2\alpha$	$mL^2\dot{\alpha}$
$\mathbf{x}_B$	$L\sin\alpha$	$L\cos\alpha$	0
$\delta\mathbf{x}_B/\delta t$	$L\dot{\alpha}\cos\alpha$	$-L\dot{\alpha}\sin\alpha$	0
$\boldsymbol{\omega}\times\mathbf{x}_B$	0	0	$-\omega L\sin\alpha$
$\mathbf{v}_B$	$L\dot{\alpha}\cos\alpha$	$-L\dot{\alpha}\sin\alpha$	$-\omega L\sin\alpha$
$\mathbf{x}_B\times m\,\mathbf{v}_B$	$-m\,\omega L^2\sin\alpha\cos\alpha$	$m\,\omega L^2\sin^2\alpha$	$-m\,L^2\dot{\alpha}$
$\mathbf{G}$	0	$-2\,m\,L\dot{\alpha}\sin\alpha$	0
$\delta\mathbf{G}/\delta\mathbf{t}$	0	$-2\,m\,L\frac{d}{dt}(\dot{\alpha}\sin\alpha)$	0
$\boldsymbol{\omega}\times\mathbf{G}$	0	0	0
$\dot{\mathbf{G}}$	0	$-2\,m\,L\frac{d}{dt}(\dot{\alpha}\sin\alpha)$	0
$\mathbf{H}_o$	0	$2\,m\,\omega L^2\sin^2\alpha$	0
$\delta\mathbf{H}_o/\delta t$	0	$2\,m\,L^2\frac{d}{dt}(\omega\sin^2\alpha)$	0
$\boldsymbol{\omega}\times\mathbf{H}_o$	0	0	0
$\dot{\mathbf{H}}_o$	0	$2\,m\,L^2\frac{d}{dt}(\omega\sin^2\alpha)$	0

Table 36.1

Since the link is presumed to be massless the balance laws (36.10) and (36.11) hold for the system under consideration where

$$\mathbf{G} = m\,\mathbf{v}_A + m\,\mathbf{v}_B \quad , \quad \mathbf{H}_o = \mathbf{x}_A \times m\,\mathbf{v}_A + \mathbf{x}_B \times m\,\mathbf{v}_B \ . \tag{36.14a,b}$$

In particular, with the help of Table 36.1 we have

$$\mathbf{F} = -\,2\,m\,L\,\frac{d}{dt}(\dot{\alpha}\,\sin\alpha)\ \mathbf{e}_2' \ , \tag{36.15a}$$

$$\mathbf{M}_o = 2\,m\,L^2\,\frac{d}{dt}(\omega\,\sin^2\alpha)\,\mathbf{e}_2' \ . \tag{36.15b}$$

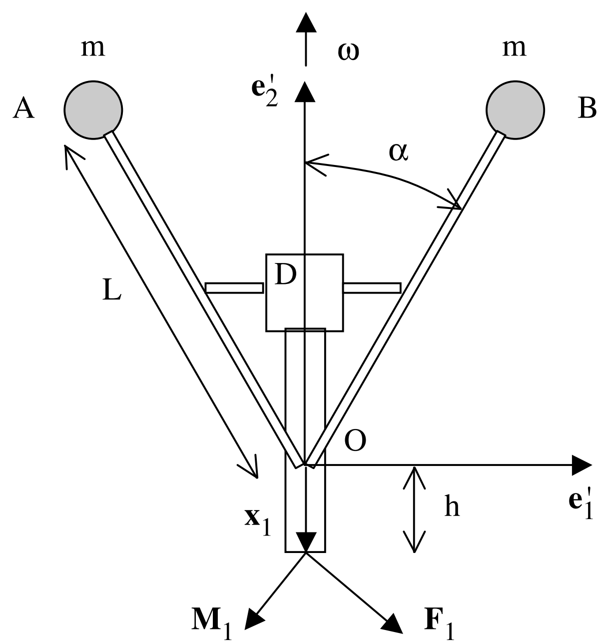

Fig. 36.5

Replacing the bearing with a force $\mathbf{F}_1$ and moment $\mathbf{M}_1$, each associated with the point $\mathbf{x}_1$ of application of the force, and neglecting the force of gravity we obtain the free body diagram Fig. 36.5 and expressions for $\mathbf{F}$ and $\mathbf{M}_o$ of the form

$$\mathbf{F} = \mathbf{F}_1 \ , \ \mathbf{M}_o = \mathbf{x}_1 \times \mathbf{F}_1 + \mathbf{M}_1 \ . \tag{36.16a,b}$$

However since $\mathbf{x}_1$ is parallel to the $\mathbf{e}_2'$ axis with

$$\mathbf{x}_1 = -\,h\,\mathbf{e}_2' \ , \tag{36.17}$$

it follows from (36.15a) that (36.16b) may be written in the simpler form

$$\mathbf{M}_o = \mathbf{M}_1 \ . \tag{36.18}$$

Furthermore, since the system is free to rotate about the $\mathbf{e}_2'$ axis the component of the external moment $\mathbf{M}_1$ in the $\mathbf{e}_2'$ direction vanishes

$$\mathbf{M}_1 \bullet \mathbf{e}_2' = 0 \ . \tag{36.19}$$

Thus, in view of the result (36.15b) and (36.18) we may conclude that $\mathbf{M}_o$ vanishes

$$\mathbf{M}_o = 0 \ , \tag{36.20}$$

so the angular momentum $\mathbf{H}_o$ is constant

$$\mathbf{H}_o = 2\, m\, \omega\, L^2 \sin^2\alpha \ \ \mathbf{e}_2' = \text{constant} \ . \tag{36.21}$$

With the help of the initial conditions (36.13) we may conclude that

$$\omega = \left(\frac{\sin\alpha_0}{\sin\alpha}\right)^2 \omega_0 \ , \tag{36.22}$$

which is the desired result. Notice from (36.15a) and (36.22) that the external force applied by the bearings on the system is in the $\mathbf{e}_2'$ direction only and that the angular velocity ω increases as the angle α decreases.

As another example consider the system of two rigid disks A and B connected by a massless link that includes a motor D (see Fig. 36.6). The system is free to rotate about the fixed $\mathbf{e}_2''$ axis and the angle α between the horizontal plane and the shaft of the motor remains constant. The system is initially at rest. Then the motor D is turned on and it starts rotating body A relative to body B until the shaft of motor D attains a constant angular speed p. At this time the system reaches steady state with body B rotating with angular speed Ω in the $\mathbf{e}_2''$ direction. The objective is to determine the angular velocity of the body B.

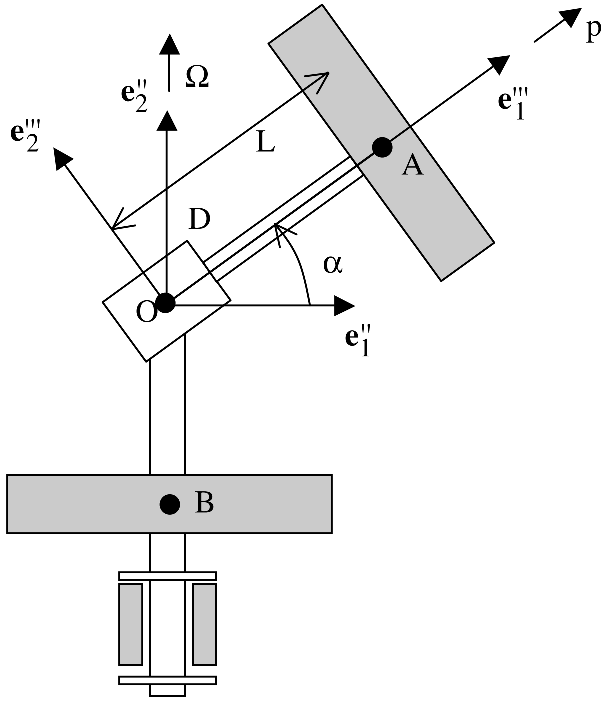

Fig. 36.6

To this end, let $\mathbf{e}_i''$ be base vectors defining a body coordinate system which rotates with body B such that the angular velocity $\mathbf{\Omega}$ is given by

$$\dot{\mathbf{e}}_i'' = \mathbf{\Omega} \times \mathbf{e}_i'' \ , \ \ \mathbf{\Omega} = \Omega \, \mathbf{e}_2'' \ , \tag{36.23a,b}$$

and the shaft of motor D remains in the $\mathbf{e}_1''$–$\mathbf{e}_2''$ plane. Furthermore, let $\mathbf{e}_i'''$ be another set of rotating base vectors defined by taking $\mathbf{e}_3'''$ parallel to $\mathbf{e}_3''$ and $\mathbf{e}_1'''$ parallel to the shaft of the motor so that

$$\mathbf{e}_1''' = \cos\alpha \, \mathbf{e}_1'' + \sin\alpha \, \mathbf{e}_2'' \ , \tag{36.24a}$$

$$\mathbf{e}_2''' = -\sin\alpha \, \mathbf{e}_1'' + \cos\alpha \, \mathbf{e}_2'' \ , \tag{36.24b}$$

$$\mathbf{e}_3''' = \mathbf{e}_3'' \ , \tag{36.24c}$$

$$\dot{\mathbf{e}}_i''' = \mathbf{\Omega} \times \mathbf{e}_i''' \ . \tag{36.24d}$$

Also, since the body A rotates relative to $\mathbf{e}_i''$ with angular speed p in the direction of the shaft of motor D it has angular velocity $\boldsymbol{\omega}$ given by

$$\boldsymbol{\omega} = \mathbf{\Omega} + p \, \mathbf{e}_1''' = \Omega \, (\sin\alpha \, \mathbf{e}_1''' + \cos\alpha \, \mathbf{e}_2''') + p \, \mathbf{e}_1''' \ . \tag{36.25}$$

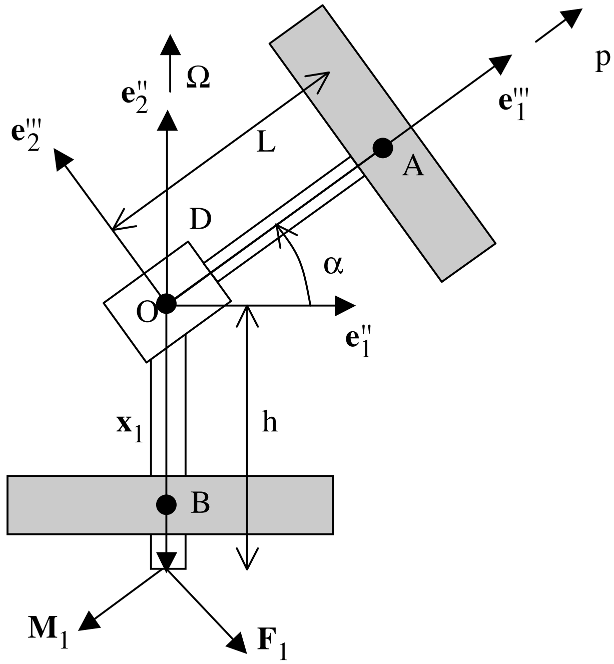

Fig. 36.7

Replacing the bearing with a force $\mathbf{F}_1$ and moment $\mathbf{M}_1$, each associated with the point $\mathbf{x}_1$ of application of the force, and neglecting the force of gravity we obtain the free body diagram Fig. 36.7 and expressions for resultant external force $\mathbf{F}$ and moment $\mathbf{M}_o$ of the form

$$\mathbf{F} = \mathbf{F}_1 \ , \ \mathbf{M}_o = \mathbf{x}_1 \times \mathbf{F}_1 + \mathbf{M}_1 \ , \tag{36.26a,b}$$

where $\mathbf{x}_1$ is parallel to the $\mathbf{e}''_2$ and is given by

$$\mathbf{x}_1 = -\, h\, \mathbf{e}''_2 \ . \tag{36.27}$$

Since the system is free to rotate about the $\mathbf{e}''_2$ axis, the component of the external moment $\mathbf{M}_1$ in the $\mathbf{e}''_2$ direction vanishes

$$\mathbf{M}_1 \bullet \mathbf{e}''_2 = 0 \ . \tag{36.28}$$

It then follows from (36.26b)–(36.28) that no matter what the value of $\mathbf{F}_1$ we have

$$\mathbf{M}_o \bullet \mathbf{e}''_2 = 0 \ , \tag{36.29}$$

so that the balance of angular momentum of the system yields the result that the component of angular momentum $\mathbf{H}_o$ in the constant $\mathbf{e}''_2$ direction remains constant

$$\mathbf{H}_o \bullet \mathbf{e}''_2 = \text{constant} \ . \tag{36.30}$$

However, since the system initially is at rest the value of the constant is zero so that

$$\mathbf{H}_o \bullet \mathbf{e}_2'' = 0 \ . \tag{36.31}$$

In order to derive an expression for $\mathbf{H}_o$ of the system for the steady state situation we denote the mass of body A by m_A and its inertia tensor $\bar{I}'''_{Aij}$ about its center of mass and relative to the $\mathbf{e}_i'''$ system by

$$\bar{I}'''_{Aij} = \begin{pmatrix} I_{A1} & 0 & 0 \\ 0 & I_{A2} & 0 \\ 0 & 0 & I_{A2} \end{pmatrix} \ . \tag{36.32}$$

This indicates that body A has rotational symmetry about the $\mathbf{e}_1'''$ axis. Similarly, we denote the mass of body B by m_B and its inertia tensor $\bar{I}''_{Bij}$ about its center of mass and relative to the $\mathbf{e}_i''$ system by

$$\bar{I}''_{Bij} = \begin{pmatrix} I_{B1} & 0 & 0 \\ 0 & I_{B2} & 0 \\ 0 & 0 & I_{B1} \end{pmatrix} \ . \tag{36.33}$$

This indicates that body B has rotational symmetry about the $\mathbf{e}_2''$ axis.

Now, the angular momentum $\mathbf{H}_o$ of the system may be represented in the form

$$\mathbf{H}_o = \left[\bar{\mathbf{x}}_A \times m_A \, \bar{\mathbf{v}}_A + \bar{\mathbf{H}}_A\right] + \left[\bar{\mathbf{x}}_B \times m_B \, \bar{\mathbf{v}}_B + \bar{\mathbf{H}}_B\right] \ . \tag{36.34}$$

However, for body A we have

$$\bar{\mathbf{x}}_A = L \, \mathbf{e}_1''' \ , \quad \bar{\mathbf{v}}_A = \mathbf{\Omega} \times \bar{\mathbf{x}}_A = -\, \Omega \, L \cos\alpha \, \mathbf{e}_3''' \ , \tag{36.35a,b}$$

$$\bar{\mathbf{x}}_A \times m_A \, \bar{\mathbf{v}}_A = m_A \, \Omega \, L^2 \cos\alpha \, \mathbf{e}_2''' \ , \tag{36.35c}$$

$$\bar{\mathbf{H}}_A = \bar{\mathbf{I}}_A \, \boldsymbol{\omega} = I_{A1} \, (\Omega \sin\alpha + p) \, \mathbf{e}_1''' + I_{A2} \, (\Omega \cos\alpha) \, \mathbf{e}_2''' \ , \tag{36.35d}$$

and for body B we have

$$\bar{\mathbf{v}}_B = 0 \ , \ \bar{\mathbf{H}}_B = \bar{\mathbf{H}}_B \, \mathbf{\Omega} \ = I_{B2} \, \Omega \, \mathbf{e}_2'' \ . \tag{36.36a,b}$$

Combining these results we have

$$\mathbf{H}_o = I_{A1} \, (\Omega \sin\alpha + p) \, \mathbf{e}_1''' \ + (\Omega \cos\alpha) \, (m_A \, L^2 + I_{A2}) \, \mathbf{e}_2''' + I_{B2} \, \Omega \, \mathbf{e}_2'' \ . \tag{36.37}$$

Here, it is important to emphasize that we have calculated each of the vectors with respect to the base vectors which yield the simplest expressions and that we have added these results in vectorial form. It follows from (36.31) that Ω and p are related by the expression

$$I_{A1}\,(\Omega \sin\alpha + p)\,(\mathbf{e}_1''' \bullet \mathbf{e}_2'') + (\Omega \cos\alpha)\,(m_A L^2 + I_{A2})\,(\mathbf{e}_2''' \bullet \mathbf{e}_2'') + I_{B2}\,\Omega = 0 \ .$$

Next, with the help of (36.24) we may deduce that

$$I_{A1}\,(\Omega \sin\alpha + p)\sin\alpha + (\Omega \cos\alpha)\,(m_A L^2 + I_{A2})\cos\alpha + I_{B2}\,\Omega = 0 \ , \tag{36.39}$$

so that

$$\Omega = -\frac{I_{A1}\,p \sin\alpha}{I_{A1}\sin^2\alpha + (m_A L^2 + I_{A2})\cos^2\alpha + I_{B2}} \ , \tag{36.40}$$

which is the desired result. Notice that the denominator is positive and that for α between 0 and $\pi/2$ and the sign of Ω is opposite to that of p.

37. Gyroscopic Effects

In order to begin to understand gyroscopic effects let us first consider the simple case of a gyroscope which is designed with three perpendicular frictionless gimbals that allow the gyroscope to rotate freely without applying any moment about its center of mass

$$\bar{\mathbf{M}} = 0 \ . \tag{37.1}$$

It follows from the balance of angular momentum that (37.1) causes the angular momentum $\bar{\mathbf{H}}$ about the center of mass to be constant

$$\bar{\mathbf{H}} = \text{constant} \ . \tag{37.2}$$

Now if we start the gyroscope spinning with angular velocity $\boldsymbol{\omega}$ parallel to one of its principal axis of inertia we have

$$\bar{\mathbf{H}} = \bar{\mathbf{I}}\,\boldsymbol{\omega} = \bar{\mathrm{I}}\,\boldsymbol{\omega} \ , \tag{37.3}$$

where $\bar{\mathrm{I}}$ is the principal value of inertia associated with this axis of rotation. Since $\bar{\mathrm{I}}$ is constant it follows from (37.2) and (37.3) that angular velocity $\boldsymbol{\omega}$ remains constant

$$\boldsymbol{\omega} = \text{constant} \ . \tag{37.4}$$

This means that the gyroscope will continue to rotate with constant rotational speed $|\boldsymbol{\omega}|$ about a fixed direction in space. For this reason the gyroscope tends to point in a constant direction and can be used for navigational purposes. Of course, frictional effects cause small moments to be applied to the gyroscope which must be corrected.

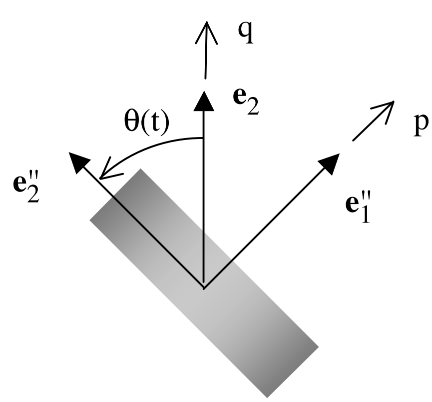

Fig. 37.1

Now that we know what happens to a gyroscope which is free of moment let us consider the case when moments are applied that cause constant angular speed about two axes. With reference to Fig. 37.1, let $\mathbf{e}_1''$ be parallel to a principal axis of a body which is tilted by an angle $\theta(t)$ relative to the horizontal plane. Furthermore, let the $\mathbf{e}_i''$ coordinate system rotate with angular velocity $\mathbf{\Omega}$ so that $\mathbf{e}_3''$ always remains in the horizontal plane and the vertical $\mathbf{e}_1''$–$\mathbf{e}_2''$ plane rotates about the vertical axis $\mathbf{e}_2$ with constant angular speed q so that

$$\dot{\mathbf{e}}_i'' = \mathbf{\Omega} \times \mathbf{e}_i'' \ , \tag{37.5a}$$

$$\mathbf{\Omega} = q\,\mathbf{e}_2 + \dot{\theta}\,\mathbf{e}_3'' = q\,(\sin\theta\,\mathbf{e}_1'' + \cos\theta\,\mathbf{e}_2'') + \dot{\theta}\,\mathbf{e}_3'' \ . \tag{37.5b}$$

Also, let the body rotate relative to the $\mathbf{e}_i''$ coordinate system with constant angular speed p about the $\mathbf{e}_1''$ axis so that the absolute angular velocity $\boldsymbol{\omega}$ of the body becomes

$$\boldsymbol{\omega} = \mathbf{\Omega} + p\,\mathbf{e}_1'' = (q\sin\theta + p)\,\mathbf{e}_1'' + q\cos\theta\,\mathbf{e}_2'' + \dot{\theta}\ \mathbf{e}_3'' \ . \tag{37.6}$$

In this example we calculate the moments applied to the body about its center of mass O and we calculate the rotation $\theta(t)$ caused when the body is assumed to rotate freely about the $\mathbf{e}_3''$ axis.

Assuming that the body is a homogeneous body of revolution with respect to the $\mathbf{e}_1''$ axis, the components $\bar{I}_{ij}'$ of the inertia tensor relative to its center of mass and relative to a body coordinate system $\mathbf{e}_i'$ parallel to the principal axes of inertia may be written in the form

$$\bar{I}_{ij}' = \begin{pmatrix} \bar{I}_1 & 0 & 0 \\ 0 & \bar{I}_2 & 0 \\ 0 & 0 & \bar{I}_2 \end{pmatrix} \ . \tag{37.7}$$

Since $\bar{I}_{22}'$ and $\bar{I}_{33}'$ are equal it follows that the components of inertia $\bar{I}_{ij}''$ relative to $\mathbf{e}_i''$ are unaffected by the rotation of $\mathbf{e}_i'$ relative to $\mathbf{e}_i''$ so that

$$\bar{I}_{ij}'' = \bar{I}_{ij}' \ . \tag{37.8}$$

Thus, the angular momentum $\bar{\mathbf{H}}$ about the center of mass becomes

$$\bar{\mathbf{H}} = \bar{I}''_{ij}\,\omega''_j\,\mathbf{e}''_j = \bar{I}_1\,(q\sin\theta + p)\,\mathbf{e}''_1 + \bar{I}_2\,(q\cos\theta)\,\mathbf{e}''_2 + \bar{I}_2\,\dot{\theta}\,\mathbf{e}''_3 \quad . \tag{37.9}$$

From the balance of angular momentum we may calculate the moment $\bar{\mathbf{M}}$ applied to the center of mass

$$\bar{\mathbf{M}} = \dot{\bar{\mathbf{H}}} \quad . \tag{37.10}$$

The results of this calculation are summarized in Table 37.1. Since the body is free to rotate about the $\mathbf{e}_3''$ axis the component of the moment about this axis vanishes

$$\bar{\mathbf{M}} \bullet \mathbf{e}''_3 = 0 \quad . \tag{37.11}$$

This gives an equation for determining $\theta(t)$ of the form

$$\bar{I}_2\,\ddot{\theta} + (\bar{I}_2 - \bar{I}_1)\,q^2 \sin\theta\cos\theta - \bar{I}_1\,p\,q\cos\theta = 0 \quad . \tag{37.12}$$

Multiplying (37.12) by $\dot{\theta}$ and integrating we obtain

$$\frac{1}{2}\,\bar{I}_2\,\dot{\theta}^2 + \frac{1}{2}\,(\bar{I}_2 - \bar{I}_1)\,q^2 \sin^2\theta - \bar{I}_1\,p\,q\sin\theta = C \quad , \tag{37.13}$$

where C is a constant of integration which is determined by the initial conditions. For example if the initial values of θ and $\dot{\theta}$ are given by

$$\theta(0) = 0 \quad , \quad \dot{\theta}(0) = 0 \quad , \tag{37.14a,b}$$

then the constant C vanishes and (37.12) and (37.13) may be written in the forms

$$\ddot{\theta} = \frac{1}{\bar{I}_2}\left[\,(\bar{I}_1 - \bar{I}_2)\,q^2 \sin\theta + \bar{I}_1\,p\,q\,\right]\cos\theta \,, \tag{37.15a}$$

$$\dot{\theta}^2 = \frac{1}{\bar{I}_2}\left[\,(\bar{I}_1 - \bar{I}_2)\,q^2 \sin\theta + 2\,\bar{I}_1\,p\,q\,\right]\sin\theta \quad . \tag{37.15b}$$

Notice that initially we have

$$\dot{\theta}(0) = 0 \quad , \quad \ddot{\theta}(0) = \frac{\bar{I}_1}{\bar{I}_2}\,p\,q \quad , \tag{37.16a,b}$$

so that if $pq > 0$ then $\ddot{\theta}(0) > 0$ and the axis $\mathbf{e}_1''$ tends to tilt up whereas if $pq < 0$ then $\ddot{\theta}(0) < 0$ and the axis $\mathbf{e}_1''$ tends to tilt down. Also, notice that when $\theta = \pm\pi$ we have

$$\dot{\theta}(\pm\pi) = 0 \;\; , \;\; \ddot{\theta}(\pm\pi) = -\frac{\bar{I}_1}{\bar{I}_2}\, p\, q \quad . \tag{37.17a,b}$$

This means that the body will oscillate between $\theta=0$ and $\theta=\pi$ for $pq > 0$ whereas it will oscillate between $\theta=0$ and $\theta=-\pi$ for $pq < 0$.

	$\mathbf{e}_1''$	$\mathbf{e}_2''$	$\mathbf{e}_3''$
$\mathbf{\Omega}$	$q \sin\theta$	$q \cos\theta$	$\dot{\theta}$
$\bar{\mathbf{H}}$	$\bar{I}_1 (q \sin\theta + p)$	$\bar{I}_2 (q \cos\theta)$	$\bar{I}_2 \dot{\theta}$
$\delta\bar{\mathbf{H}}/\delta t$	$\bar{I}_1 q \dot{\theta} \cos\theta$	$-\bar{I}_2 q \dot{\theta} \sin\theta$	$\bar{I}_2 \ddot{\theta}$
$\mathbf{\Omega} \times \bar{\mathbf{H}}$	$q \cos\theta \bar{I}_2 \dot{\theta}$ $-q \cos\theta \bar{I}_2 \dot{\theta}$	$-q \sin\theta \bar{I}_2 \dot{\theta}$ $+\bar{I}_1 (q \sin\theta + p) \dot{\theta}$	$q \sin\theta \bar{I}_2 (q \cos\theta)$ $-\bar{I}_1 (q \sin\theta + p) q \cos\theta$
$\dot{\bar{\mathbf{H}}}$	$\bar{I}_1 q \dot{\theta} \cos\theta$	$-2 \bar{I}_2 q \dot{\theta} \sin\theta$ $+\bar{I}_1 (q \sin\theta + p) \dot{\theta}$	$\bar{I}_2 \ddot{\theta}$ $+(\bar{I}_2 - \bar{I}_1) q^2 \sin\theta\cos\theta$ $-\bar{I}_1 p q \cos\theta$

Table 37.1

38. Euler Angles And A Spinning Top

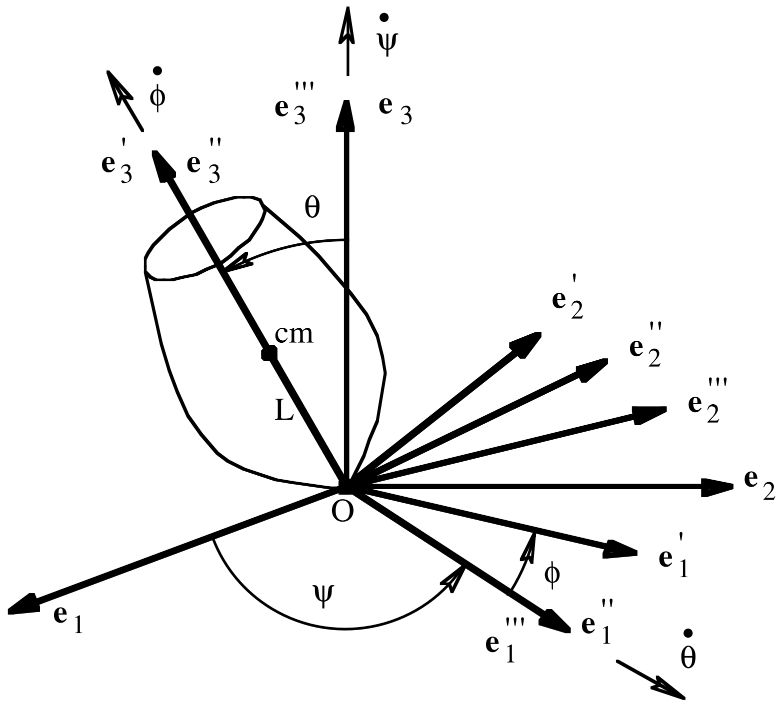

Fig. 38.1

In order to describe Euler angles it is convenient to consider the physical problem of a spinning top whose tip O is fixed in space (see Fig. 38.1). In this problem we introduce four coordinate systems and consider the transformation relations between them

$$\mathbf{e}_i \rightarrow \mathbf{e}_i''' \rightarrow \mathbf{e}_i'' \rightarrow \mathbf{e}_i' \quad . \tag{38.1}$$

The base vectors $\mathbf{e}_i$ are fixed in space with

$$\dot{\mathbf{e}}_i = 0 \quad , \tag{38.2}$$

whereas the base vectors $\mathbf{e}_i'''$, $\mathbf{e}_i''$, $\mathbf{e}_i'$ rotate with angular velocities which are functions of the rates of change of the Euler angles $\{\psi, \theta, \phi\}$ which define the orientation of $\mathbf{e}_i'$ relative to $\mathbf{e}_i$.

<u>Precession Angle ψ</u>: The base vectors $\mathbf{e}_i'''$ are related to $\mathbf{e}_i$ by a rotation about the $\mathbf{e}_3$ axis through the angle ψ, which is called the precession angle, and the transformation relations are given by

$$\mathbf{e}_1^{'''} = \cos\psi\, \mathbf{e}_1 + \sin\psi\, \mathbf{e}_2 \ , \tag{38.3a}$$

$$\mathbf{e}_2^{'''} = -\sin\psi\, \mathbf{e}_1 + \cos\psi\, \mathbf{e}_2 \ , \tag{38.3b}$$

$$\mathbf{e}_3^{'''} = \mathbf{e}_3 \ . \tag{38.3c}$$

<u>Nutation Angle θ</u>: The base vectors $\mathbf{e}_i^{''}$ are related to $\mathbf{e}_i^{'''}$ by a rotation about the $\mathbf{e}_1^{'''}$ axis through the angle θ, which is called the nutation angle, and the transformation relations are given by

$$\mathbf{e}_1^{''} = \mathbf{e}_1^{'''} \ . \tag{38.4a}$$

$$\mathbf{e}_2^{''} = \cos\theta\, \mathbf{e}_2^{'''} + \sin\theta\, \mathbf{e}_3^{'''} \ , \tag{38.4a}$$

$$\mathbf{e}_3^{''} = -\sin\theta\, \mathbf{e}_2^{'''} + \cos\theta\, \mathbf{e}_3^{'''} \ . \tag{38.4b}$$

<u>Spin Angle ϕ</u>: The base vectors $\mathbf{e}_i^{'}$ are associated with a body coordinates system which rotates with the top. These base vectors are related to $\mathbf{e}_i^{''}$ by a rotation about the $\mathbf{e}_3^{''}$ axis through the angle ϕ, which is called the spin angle, and the transformation relations are given by

$$\mathbf{e}_1^{'} = \cos\phi\, \mathbf{e}_1^{''} + \sin\phi\, \mathbf{e}_2^{''} \ , \tag{38.5a}$$

$$\mathbf{e}_2^{'} = -\sin\phi\, \mathbf{e}_1^{''} + \cos\phi\, \mathbf{e}_2^{''} \ , \tag{38.5b}$$

$$\mathbf{e}_3^{'} = \mathbf{e}_3^{''} \ . \tag{38.5c}$$

It is important to note that even though the relationship between $\mathbf{e}_i^{'}$ and $\mathbf{e}_i$ is quite complicated it can be described by three consecutive simple rotations, each about a single axis (see Fig. 38.2).

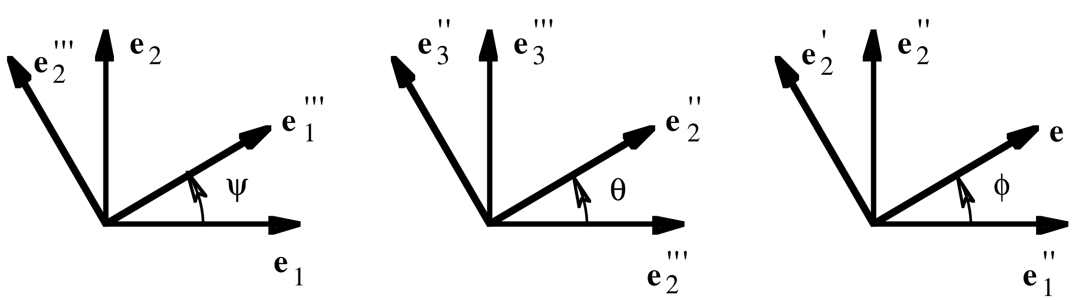

Fig. 38.2

Before proceeding with the solution of the spinning top problem it is important to note that the transformations (38.3)–(38.5) can be written in a convenient matrix forms

$$\begin{pmatrix} \mathbf{e}_1''' \\ \mathbf{e}_2''' \\ \mathbf{e}_3''' \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} , \tag{38.6a}$$

$$\begin{pmatrix} \mathbf{e}_1'' \\ \mathbf{e}_2'' \\ \mathbf{e}_3'' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{e}_1''' \\ \mathbf{e}_2''' \\ \mathbf{e}_3''' \end{pmatrix} , \tag{38.6b}$$

$$\begin{pmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1'' \\ \mathbf{e}_2'' \\ \mathbf{e}_3'' \end{pmatrix} . \tag{38.6c}$$

Furthermore, since the matrices in (38.6) are orthogonal their inverses are equal to their transposes so that the inverse transformations can be written in the matrix forms

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1''' \\ \mathbf{e}_2''' \\ \mathbf{e}_3''' \end{pmatrix} , \tag{38.7a}$$

$$\begin{pmatrix} \mathbf{e}_1''' \\ \mathbf{e}_2''' \\ \mathbf{e}_3''' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{e}_1'' \\ \mathbf{e}_2'' \\ \mathbf{e}_3'' \end{pmatrix} , \tag{38.7b}$$

$$\begin{pmatrix} \mathbf{e}_1'' \\ \mathbf{e}_2'' \\ \mathbf{e}_3'' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{pmatrix} . \tag{38.7c}$$

Now the angular velocity $\mathbf{\Omega}$ of the system $\mathbf{e}_i''$ is given by

$$\dot{\mathbf{e}}_i'' = \mathbf{\Omega} \times \mathbf{e}_i'' \ , \ \mathbf{\Omega} = \dot{\psi} \ \mathbf{e}_3 + \dot{\theta} \ \mathbf{e}_1'' \ . \tag{38.8a,b}$$

However from the geometry of Fig. 38.1 we may write $\mathbf{e}_3$ in terms of $\mathbf{e}_i''$ in the form

$$\mathbf{e}_3 = \sin\theta \, \mathbf{e}_2'' + \cos\theta \, \mathbf{e}_3'' \ , \tag{38.9}$$

so that $\mathbf{\Omega}$ becomes

$$\mathbf{\Omega} = \dot{\theta} \ \mathbf{e}_1'' + \dot{\psi} \, (\sin\theta \, \mathbf{e}_2'' + \cos\theta \, \mathbf{e}_3'') \ . \tag{38.10}$$

Also, the angular velocity $\boldsymbol{\omega}$ of the system $\mathbf{e}_i'$ is given by

$$\dot{\mathbf{e}}'_i = \boldsymbol{\omega} \times \mathbf{e}'_i \ , \tag{38.11a}$$

$$\boldsymbol{\omega} = \boldsymbol{\Omega} + \dot{\phi}\,\mathbf{e}''_3 = \dot{\theta}\ \mathbf{e}''_1 + (\dot{\psi}\ \sin\theta\,)\,\mathbf{e}''_2 + (\dot{\psi}\ \cos\theta + \dot{\phi})\,\mathbf{e}''_3 \ . \tag{38.11b}$$

Furthermore, let the body base vectors $\mathbf{e}'_i$ be parallel to the principal axes of inertia and let the top have rotational symmetry about the $\mathbf{e}'_1$ and $\mathbf{e}'_2$ axes so that the components I'_{oij} of the inertia tensor about the point O and relative to the basis $\mathbf{e}'_i$ are

$$I'_{oij} = \begin{pmatrix} I_{o11} & 0 & 0 \\ 0 & I_{o11} & 0 \\ 0 & 0 & I_{o33} \end{pmatrix} \ . \tag{38.11}$$

However since $I'_{o11} = I'_{o22}$ it follows that the components I''_{oij} of the inertia tensor relative to $\mathbf{e}''_i$ are unchanged by the spin $\dot{\phi}$ so that

$$I''_{oij} = I'_{oij} \ . \tag{38.12}$$

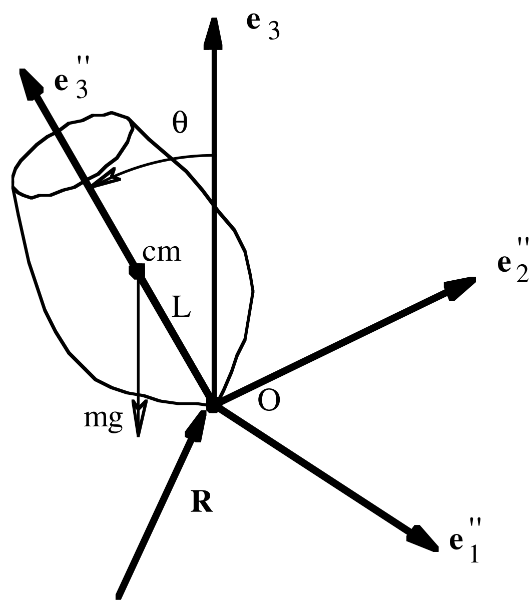

Fig. 38.3

From the free-body diagram in Fig. 38.3 we see that the only forces acting on the top are the force of gravity mg which acts through the center of mass in the negative $\mathbf{e}_3$ direction and the reaction $\mathbf{R}$ applied at the point O by the floor on the top so that

$$\mathbf{F} = \mathbf{R} - mg\,\mathbf{e}_3 = R''_i\,\mathbf{e}''_i - mg\,(\sin\theta\,\mathbf{e}''_2 + \cos\theta\,\mathbf{e}''_3) \ , \tag{38.13a}$$

$$\mathbf{F} = (R''_1)\,\mathbf{e}''_1 + (R''_2 - mg\sin\theta)\,\mathbf{e}''_2 + (R''_3 - mg\cos\theta)\,\mathbf{e}''_3 \ . \tag{38.13b}$$

In order to calculate the value of $\mathbf{R}$ and the motion $\{\psi(t), \theta(t), \phi(t)\}$ we must consider the balances of linear and angular momentum. To this end we note that the vector $\bar{\mathbf{x}}$ from O to the center of mass is given by

$$\bar{\mathbf{x}} = L\,\mathbf{e}''_3 \ , \tag{38.14}$$

where L is a constant. Also, the angular momentum $\mathbf{H}_o$ about the fixed point O becomes

$$\mathbf{H}_o = \mathbf{I}_o\boldsymbol{\omega} = I''_{oij}\,\omega''_j\,\mathbf{e}''_i \ , \tag{38.15a}$$

$$\mathbf{H}_o = (I_{o11}\,\dot{\theta}\,)\,\mathbf{e}''_1 + (I_{o11}\,\dot{\psi}\,\sin\theta)\,\mathbf{e}''_2 + [\,I_{o33}\,(\dot{\psi}\,\cos\theta + \dot{\phi}\,)\,]\,\mathbf{e}''_3 \ . \tag{38.15b}$$

Now the acceleration $\bar{\mathbf{a}}$ of the center of mass and the rate of change of angular momentum are calculated in Table 38.1.

It follows from the balance of linear momentum that

$$\mathbf{R} = mg\,\mathbf{e}_3 + m\,\bar{\mathbf{a}} = mg\,(\sin\theta\,\mathbf{e}''_2 + \cos\theta\,\mathbf{e}''_3) + m\,\bar{\mathbf{a}} \ , \tag{38.16a}$$

$$R''_1 = m\,[L\,\ddot{\psi}\,\sin\theta + 2\,L\,\dot{\psi}\,\dot{\theta}\,\cos\theta\,] \ , \tag{38.16b}$$

$$R''_2 = m\,[\,g\sin\theta - L\,\ddot{\theta} + L\,\dot{\psi}^2\sin\theta\cos\theta\,] \ , \tag{38.16c}$$

$$R''_3 = m\,[\,g\cos\theta - L\,\dot{\theta}^2 - L\,\dot{\psi}^2\sin^2\theta\,] \ . \tag{38.16d}$$

Furthermore, the moment $\mathbf{M}_o$ about the point O becomes

$$\mathbf{M}_o = (L\,\mathbf{e}''_3) \times [\,-mg\,\mathbf{e}_3\,] \ , \tag{38.17a}$$

$$\mathbf{M}_o = (L\,\mathbf{e}''_3) \times [\,-mg\,(\sin\theta\,\mathbf{e}''_2 + \cos\theta\,\mathbf{e}''_3)\,] \ , \tag{38.17b}$$

$$\mathbf{M}_o = mg\,L\sin\theta\,\mathbf{e}''_1 \ , \tag{38.17c}$$

so the balance of angular momentum yields the equations

$$I_{o11}\,[\,\ddot{\theta} - \dot{\psi}^2\sin\theta\cos\theta\,] + I_{o33}\,\dot{\psi}\sin\theta\,(\dot{\psi}\cos\theta + \dot{\phi}) = mg\,L\sin\theta , \tag{38.18a}$$

$$I_{o11}\,[\,\ddot{\psi}\sin\theta + 2\,\dot{\psi}\,\dot{\theta}\cos\theta\,] - I_{o33}\,(\dot{\psi}\cos\theta + \dot{\phi})\,\dot{\theta} = 0 \ , \tag{38.18b}$$

$$I_{o33}\,\frac{d}{dt}\,(\dot{\psi}\cos\theta + \dot{\phi}) = 0 \ . \tag{38.18c}$$

In general the solution of (38.18) is quite complicated, however, for the special case of steady precession we have

$$\dot{\psi} = \Omega \ , \ \ddot{\psi} = 0 \ , \tag{38.19a,b}$$

$$\dot{\theta} = 0 \ , \ \ddot{\theta} = 0 \ , \tag{38.19c,d}$$

$$\dot{\phi} = p \ , \ \ddot{\phi} = 0 \ , \tag{38.19e,f}$$

so that the equations (38.16) for the forces R_i'' reduce to

$$R_1'' = 0 \ , \tag{38.20a}$$

$$R_2'' = m \, [\, g \sin\theta + L \, \Omega^2 \sin\theta \cos\theta \,] \ , \tag{38.20b}$$

$$R_3'' = m \, [\, g \cos\theta - L \, \Omega^2 \sin^2\theta \,] \ , \tag{38.20c}$$

and the angular momentum equations (38.18) reduce to the single equation

$$- I_{o11} \, \Omega^2 \sin\theta \cos\theta + I_{o33} \, \Omega \sin\theta \, (\Omega \cos\theta + p) = mgL \sin\theta . \tag{38.21}$$

Rewriting (38.21 we have

$$\Big[\, \{ \, (I_{o11} - I_{033}) \cos\theta \, \} \, \Omega^2 - (I_{o33} \, p) \, \Omega + mgL \Big] \sin\theta = 0 \ . \tag{38.22}$$

Trivial solutions of (38.22) correspond to $\sin\theta = 0$ for which $\theta = 0$ or $\theta = \pm\pi$ and the top is vertical. If we discard these solutions then the term in square brackets must vanish so we get a quadratic equation for the rate of precession Ω in terms of rate of spin p which gives the solutions

$$\Omega = \frac{I_{o33} \, p \pm \sqrt{(I_{o33} \, p)^2 - 4 \, mgL \, (I_{o11} - I_{o33}) \cos\theta}}{2 \, (I_{o11} - I_{o33}) \cos\theta} \ . \tag{38.23}$$

Since we require Ω to be real, the term under the square root sign must be nonnegative so that

$$p^2 \geq \frac{4 \, mgL \, (I_{o11} - I_{o33}) \cos\theta}{(I_{o33})^2} \ . \tag{38.24}$$

If $I_{o11} \leq I_{o33}$ then (38.24) place no restriction on the magnitude of the spin, whereas if $I_{o11} > I_{o33}$ then (38.24) means that the spin p must be greater than a minimum value for steady precession to exist.

At this point it is reasonable to ask which one of the solutions (38.23) is the one most observed? In order to get a simple estimate for the difference in magnitude of these two solution let us consider the reasonable approximation that the spin p is quite large. It follows that (38.23) may be rewritten in the equivalent, but alternative, form

$$\Omega = \frac{I_{o33}\, p}{2\,(I_{o11} - I_{o33})\cos\theta}\left[\, 1 \pm \sqrt{1 - \frac{4\, mgL\,(I_{o11} - I_{o33})\cos\theta}{(I_{o33}\, p)^2}}\,\right] . \qquad (38.25)$$

Now, for large values of the spin p the second term in the square root function is small so by using a Taylor series expansion we can approximate (38.25) by

$$\Omega = \frac{I_{o33}\, p}{2\,(I_{o11} - I_{o33})\cos\theta}\left[\, 1 \pm \left\{ 1 - \frac{2\, mgL\,(I_{o11} - I_{o33})\cos\theta}{(I_{o33}\, p)^2} \right\} \right] , \qquad (38.26)$$

which yields the two solutions

$$\Omega \approx \Omega_1 = \frac{mgL}{I_{o33}\, p} \ , \qquad (38.27a)$$

$$\Omega \approx \Omega_2 = \frac{I_{o33}\, p}{(I_{o11} - I_{o33})\cos\theta} \ . \qquad (38.27b)$$

Notice that for large values of the spin p, the value of Ω_1 is much smaller than Ω_2. Since tops which have large spin are usually observed to have slow precession rates we may conclude that the solution (38.26) with the negative sign is the one usually observed. However, it is important to emphasize that the above observation does not replace theoretical analysis of the stability of the solutions (38.26). For example, if it could be shown that the solution with a positive sign is unstable to perturbations then we could conclude that it is unlikely to be observed during steady precession of the top.

	$\mathbf{e}''_1$	$\mathbf{e}''_2$	$\mathbf{e}''_3$
$\boldsymbol{\Omega}$	$\dot{\theta}$	$\dot{\psi}\sin\theta$	$\dot{\psi}\cos\theta$
$\bar{\mathbf{x}}$	0	0	L
$\bar{\mathbf{v}} = \boldsymbol{\Omega} \times \bar{\mathbf{x}}$	$L\,\dot{\psi}\sin\theta$	$-L\,\dot{\theta}$	0
$\delta\bar{\mathbf{v}}/\delta t$	$L\,\ddot{\psi}\sin\theta + L\,\dot{\psi}\,\dot{\theta}\cos\theta$	$-L\,\ddot{\theta}$	0
$\boldsymbol{\Omega} \times \bar{\mathbf{v}}$	$L\,\dot{\psi}\,\dot{\theta}\cos\theta$	$L\,\dot{\psi}^2\sin\theta\cos\theta$	$-L\,\dot{\theta}^2 - L\,\dot{\psi}^2\sin^2\theta$
$\bar{\mathbf{a}}$	$L\,\ddot{\psi}\sin\theta$ $+ 2L\,\dot{\psi}\,\dot{\theta}\cos\theta$	$-L\,\ddot{\theta}$ $+ L\,\dot{\psi}^2\sin\theta\cos\theta$	$-L\,\dot{\theta}^2 - L\,\dot{\psi}^2\sin^2\theta$
$\mathbf{H}_o$	$I_{o11}\,\dot{\theta}$	$I_{o11}\,\dot{\psi}\sin\theta$	$I_{o33}\,(\dot{\psi}\cos\theta+\dot{\phi})$
$\delta\mathbf{H}_o/\delta t$	$I_{o11}\,\ddot{\theta}$	$I_{o11}\,\ddot{\psi}\sin\theta$ $+I_{o11}\,\dot{\psi}\,\dot{\theta}\cos\theta$	$I_{o33}\,(\ddot{\psi}\cos\theta$ $-\dot{\psi}\,\dot{\theta}\sin\theta+\ddot{\phi})$
$\boldsymbol{\Omega} \times \mathbf{H}_o$	$I_{o33}(\dot{\psi}\cos\theta+\dot{\phi})\dot{\psi}\sin\theta$ $-I_{o11}\,\dot{\psi}^2\sin\theta\cos\theta$	$-I_{o33}\,(\dot{\psi}\cos\theta+\dot{\phi})\,\dot{\theta}$ $+I_{o11}\,\dot{\theta}\,\dot{\psi}\cos\theta$	$I_{o11}\,\dot{\psi}\,\dot{\theta}\sin\theta$ $-I_{o11}\,\dot{\psi}\,\dot{\theta}\sin\theta$
$\dot{\mathbf{H}}_o$	$I_{o11}\,\ddot{\theta}$ $-I_{o11}\,\dot{\psi}^2\sin\theta\cos\theta$ $+I_{o33}\,\dot{\psi}^2\sin\theta\cos\theta$ $+I_{o33}\,\dot{\phi}\,\dot{\psi}\sin\theta$	$I_{o11}\,\ddot{\psi}\sin\theta$ $+2I_{o11}\,\dot{\psi}\,\dot{\theta}\cos\theta$ $-I_{o33}\,(\dot{\psi}\cos\theta+\dot{\phi})\,\dot{\theta}$	$I_{o33}\,(\ddot{\psi}\cos\theta$ $-\dot{\psi}\,\dot{\theta}\sin\theta+\ddot{\phi})$

Table 38.1

39. Euler Equations Of Motion

Recall that the balances of linear momentum and angular momentum are vector equations which can be referred to any coordinate system. However, if we refer these equations to base vectors $\mathbf{e}'_i$ which remain parallel to the principal axes of inertia of the body then the component equations of angular momentum simplify considerably. These simplified equations are called the Euler equations of motion of a rigid body. In particular, let the body rotate with angular velocity $\boldsymbol{\omega}$ so that the base vectors $\mathbf{e}'_i$ attached to the body also rotate with angular velocity $\boldsymbol{\omega}$

$$\dot{\mathbf{e}}'_i = \boldsymbol{\omega} \times \mathbf{e}'_i \quad . \tag{39.1}$$

Referring the balance of linear momentum to the base vectors $\mathbf{e}'_i$ we may write

$$m\,\bar{a}'_1 = F'_1 \quad , \quad m\,\bar{a}'_2 = F'_2 \ , \ m\,\bar{a}'_3 = F'_3 \quad . \tag{39.2a,b,c}$$

Since the vectors $\mathbf{e}'_i$ are parallel to the principal directions of inertia the components $\bar{I}'_{ij}$ of the inertia tensor $\bar{\mathbf{I}}$ about the center of mass may be written in the simple diagonalized form

$$\bar{I}'_{ij} = \begin{pmatrix} \bar{I}'_{11} & 0 & 0 \\ 0 & \bar{I}'_{22} & 0 \\ 0 & 0 & \bar{I}'_{33} \end{pmatrix} \ , \tag{39.3}$$

so the angular momentum $\bar{\mathbf{H}}$ about the center of mass becomes

$$\bar{\mathbf{H}} = \bar{I}'_{ij}\,\omega'_j\,\mathbf{e}'_i = (\bar{I}'_{11}\,\omega'_1)\,\mathbf{e}'_1 + (\bar{I}'_{22}\,\omega'_2)\,\mathbf{e}'_2 + (\bar{I}'_{33}\,\omega'_3)\,\mathbf{e}'_3 \quad . \tag{39.4}$$

Thus, using Table 39.1 the balance of angular momentum yields the equations

$$\bar{M}'_1 = \bar{I}'_{11}\,\dot{\omega}'_1 + (\bar{I}'_{33} - \bar{I}'_{22})\,\omega'_2\,\omega'_3 \ , \tag{39.5a}$$

$$\bar{M}'_2 = \bar{I}'_{22}\,\dot{\omega}'_2 + (\bar{I}'_{11} - \bar{I}'_{33})\,\omega'_1\,\omega'_3 \ , \tag{39.5b}$$

$$\bar{M}'_3 = \bar{I}'_{33}\,\dot{\omega}'_3 + (\bar{I}'_{22} - \bar{I}'_{11})\,\omega'_1\,\omega'_2 \ , \tag{39.5c}$$

for the components $\bar{M}'_i$ of the moment $\bar{\mathbf{M}}$ about the center of mass.

	$\mathbf{e}'_1$	$\mathbf{e}'_2$	$\mathbf{e}'_3$
$\boldsymbol{\omega}$	ω'_1	ω'_2	ω'_3
$\bar{\mathbf{H}}$	$\bar{I}'_{11}\, \omega'_1$	$\bar{I}'_{22}\, \omega'_2$	$\bar{I}'_{33}\, \omega'_3$
$\delta\bar{\mathbf{H}}/\delta t$	$\bar{I}'_{11}\, \dot{\omega}'_1$	$\bar{I}'_{22}\, \dot{\omega}'_2$	$\bar{I}'_{33}\, \dot{\omega}'_3$
$\boldsymbol{\omega} \times \bar{\mathbf{H}}$	$(\bar{I}'_{33} - \bar{I}'_{22})\, \omega'_2\, \omega'_3$	$(\bar{I}'_{11} - \bar{I}'_{33})\, \omega'_1\, \omega'_3$	$(\bar{I}'_{22} - \bar{I}'_{11})\, \omega'_1\, \omega'_2$
$\dot{\bar{\mathbf{H}}}$	$\bar{I}'_{11}\, \dot{\omega}'_1 + (\bar{I}'_{33} - \bar{I}'_{22})\, \omega'_2\, \omega'_3$	$\bar{I}'_{22}\, \dot{\omega}'_2 + (\bar{I}'_{11} - \bar{I}'_{33})\, \omega'_1\, \omega'_3$	$\bar{I}'_{33}\, \dot{\omega}'_3 + (\bar{I}'_{22} - \bar{I}'_{11})\, \omega'_1\, \omega'_2$

Table 39.1

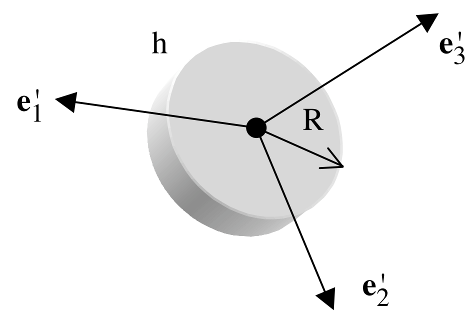

Fig. 39.1

Now consider the problem of a flipping coin which is a cylindrical disk of radius R, thickness h and mass m (see Fig. 39.1). Initially the center of mass of the coin is located at position $\bar{\mathbf{x}}_0$, it has velocity $\bar{\mathbf{v}}_0$, and the coin has angular velocity $\boldsymbol{\omega}_0$ so that

$$\bar{\mathbf{x}}(0) = \bar{\mathbf{x}}_0 \ , \quad \bar{\mathbf{v}}(0) = \bar{\mathbf{v}}_0 \ , \quad \boldsymbol{\omega}(0) = \boldsymbol{\omega}_0 \ . \tag{39.6a,b,c}$$

The coin then flips in space and its motion is influenced only by the force of gravity which acts in the negative $\mathbf{e}_3$ direction (see the free-body diagram in Fig. 39.2) so that the resultant force $\mathbf{F}$ acting on the coin is

$$\mathbf{F} = -\, mg\, \mathbf{e}_3 \ . \tag{39.7}$$

Notice that since the force of gravity acts in a fixed direction it is most convenient to solve the balance of linear momentum in terms of the fixed base vectors $\mathbf{e}_i$. It follows from (39.7) that the balance of linear momentum becomes

$$m\,\ddot{\bar{\mathbf{x}}} = -\,mg\,\mathbf{e}_3 \ . \tag{39.8}$$

Integrating (39.8) subject to the initial conditions (39.6) we deduce that

$$\bar{\mathbf{x}}(t) = \bar{\mathbf{x}}_0 + \bar{\mathbf{v}}_0 t - \frac{1}{2}\,g\,t^2\,\mathbf{e}_3 \ , \tag{39.9}$$

which shows that the center of mass of the coin moves in a plane parallel to the $\bar{\mathbf{v}}_0$–$\mathbf{e}_3$ plane.

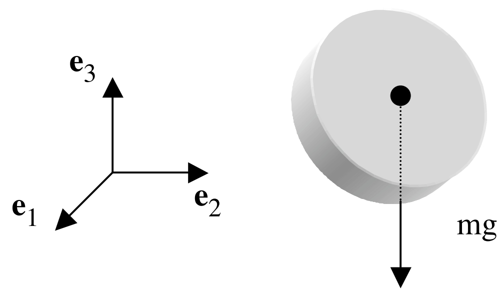

Fig. 39.2

Taking $\mathbf{e}'_3$ to be parallel to the axis of revolution of the coin, the components $\bar{I}'_{ij}$ of the inertia tensor $\bar{\mathbf{I}}$ about the center of mass of the coin are

$$\bar{I}'_{ij} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \ , \ I_1 = \frac{1}{2}\,mR^2\,[\frac{1}{2} + \frac{h^2}{6R^2}] \ , \ I_3 = \frac{1}{2}\,mR^2 \ . \tag{39.10}$$

Since the coin is free from moments ($\bar{M}'_i = 0$), the Euler equations (39.5) reduce to

$$I_1\dot{\omega}'_1 + (I_3 - I_1)\,\omega'_2\,\omega'_3 = 0 \ , \tag{39.11a}$$

$$I_1\,\dot{\omega}'_2 - (I_3 - I_1)\,\omega'_1\,\omega'_3 = 0 \ , \tag{39.11b}$$

$$\dot{\omega}'_3 = 0 \ . \tag{39.11c}$$

Integrating (39.11c) subject to the initial condition (39.6c) we have

$$\omega'_3 = \omega'_{03} = \boldsymbol{\omega}_0 \bullet \mathbf{e}'_3(0) = \text{constant} \ , \tag{39.12}$$

where we emphasize that the components ω'_{0i} are the components of the vector $\boldsymbol{\omega}_0$ in the direction of the base vectors $\mathbf{e}'_i(0)$ at time t=0 so that

$$\omega'_{0i} = \boldsymbol{\omega}_0 \bullet \mathbf{e}'_i(0) \ . \tag{39.13}$$

Now the equations (39.11a,b) may be rewritten in the simpler forms

$$\dot{\omega}'_1 + \lambda \, \omega'_2 = 0 \ , \ \dot{\omega}'_2 - \lambda \, \omega'_1 = 0 \ , \tag{39.14a,b}$$

where the constant λ is defined by

$$\lambda = \frac{(I_3 - I_1)}{I_1} \, \omega'_{03} \ . \tag{39.15}$$

Differentiating (39.14a) and substituting (39.14b) into the result we deduce that

$$\ddot{\omega}'_1 + \lambda \, \dot{\omega}'_2 = \ddot{\omega}'_1 + \lambda^2 \, \omega'_1 = 0 \ . \tag{39.16}$$

Thus, the solution of (39.16) and hence (39.14b) may be written in the forms

$$\omega'_1 = \omega'_{01} \cos(\lambda t) - \omega'_{02} \sin(\lambda t) \ , \ \ \omega'_2 = \omega'_{01} \sin(\lambda t) + \omega'_{02} \cos(\lambda t), \tag{39.17a,b}$$

where the constants have been determined by satisfying the initial conditions (39.13).

Next, it is convenient to introduce the base vectors $\mathbf{e}''_i$ such that

$$\mathbf{e}''_1 = \cos(\lambda t)\, \mathbf{e}'_1 + \sin(\lambda t)\, \mathbf{e}'_2 \ , \mathbf{e}''_2 = -\sin(\lambda t)\, \mathbf{e}'_1 + \cos(\lambda t)\, \mathbf{e}'_2 \ , \mathbf{e}''_3 = \mathbf{e}'_3 \ ,$$

$$\dot{\mathbf{e}}''_i = \boldsymbol{\lambda} \times \mathbf{e}''_i \ , \ \boldsymbol{\lambda} = \boldsymbol{\omega} + \lambda\, \mathbf{e}''_3 = \omega'_{01}\, \mathbf{e}''_1 + \omega'_{02}\, \mathbf{e}''_2 + \frac{I_3}{I_1}\, \omega'_{03}\, \mathbf{e}''_3 \ . \tag{39.18}$$

However, since the components of $\boldsymbol{\lambda}$ are constants

$$\dot{\boldsymbol{\lambda}} = \frac{\delta \boldsymbol{\lambda}}{\delta t} + \boldsymbol{\lambda} \times \boldsymbol{\lambda} = 0 \ , \tag{39.19}$$

$\boldsymbol{\lambda}$ is a constant vector which can be written in the form

$$\boldsymbol{\lambda} = \dot{\psi} \ \mathbf{e} \ , \dot{\psi} = \alpha\, |\boldsymbol{\lambda}| \ , \ \mathbf{e} = \frac{\alpha \boldsymbol{\lambda}}{|\boldsymbol{\lambda}|} \ , \ \ \alpha = 1 \text{ for } \omega'_{03} > 0 \ \text{ and } \alpha = -1 \ \text{ for } \omega'_{03} < 0 \ , \tag{39.20}$$

It then follows that the motion of the coin can be described like a spinning top with the rate of precession $\dot{\psi}$ about the fixed $\mathbf{e}$ axis, with spin rate $(-\lambda)$ (relative to the $\mathbf{e}''_i$ axes) about $\mathbf{e}''_3$, and with constant angle of nutation θ given by

$$\cos\theta = \mathbf{e}''_3 \bullet \mathbf{e} \ . \tag{39.21}$$

Moreover, using the fact that the components of $\bar{\mathbf{I}}$ relative to $\mathbf{e}'_i$ and $\mathbf{e}''_i$ are the same it follows that the angular momentum about the center of mass

$$\bar{\mathbf{H}} = \bar{I}''_{ij}\,\omega''_j\,\mathbf{e}''_i = I_1\left[\omega'_{01}\,\mathbf{e}''_1 + \omega'_{02}\,\mathbf{e}''_2\right] + I_3\left[\omega'_{03}\,\mathbf{e}''_3\right] \;,$$

$$\bar{\mathbf{H}} = I_1\left[\boldsymbol{\lambda} - \frac{I_3}{I_1}\,\omega'_{03}\,\mathbf{e}''_3\right] + I_3\left[\omega'_{03}\,\mathbf{e}''_3\right] = I_1\,\boldsymbol{\lambda}\,, \tag{39.22}$$

is a constant vector. This is consistent with the fact that $\bar{\mathbf{M}}$ vanishes.

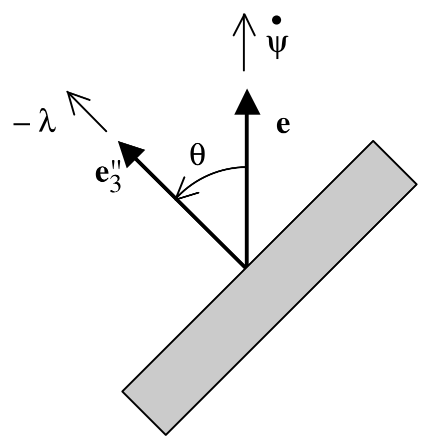

Fig. 39.3