An analytical introduction to Descriptive Geometry

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Preface

This is a book for people who want to understand Descriptive Geometry while they learn its methods and it is a book for those who have difficulties in directly visualizing spatial relationships, but can grasp them by analytical reasoning. The book is also meant for those who want to know on what Engineering Drawing is based. In addition, the book can serve as a bridge between traditional, manual methods and computer graphics. Therefore, the audience can include students of Engineering, Architecture, Design and Graphics.

We live in a three-dimensional world and we often want to represent this world in a drawing, a painting or a photography. Now, drawings, paintings and photos are two-dimensional. So are, indeed, the images formed in each of our eyes. Seen from two different points, those images are different. It is our brain that restores a three-dimensional view from the two images formed in our eyes. In technical drawing we must use another procedure. The aim of Descriptive Geometry is to describe the three-dimensional objects by two-dimensional drawings so as to allow to reconstitute their original forms and
their correct measures.

Let us explain in plain words what we mean above by the term *dimension*. We may consider, for example, a parallelepipedic box. We can measure its length, its breadth and its height, in total three dimensions. We can draw this object on a sheet of paper that has only two dimensions, a breadth and a height. So do photos, and TV and computer screens.

When representing a three-dimensional object on the two-dimensional surface of our retina, of a camera film or array of sensors, of a paper sheet, or of a TV or computer screen, the number of dimensions is reduced from three to two. The general process of reducing the number of dimensions of a given object is called *projection*. The type of projection that produces the image in our eye, on the film of a conventional camera or on the array of electronic sensors of a digital camera is called *central projection*. The images produced by central projection convey the sensation of depth, or, in other words, of the actual space. This is the only projection that produces the real *perspective*. However, drawing in perspective is tedious. Moreover, reading the correct measures in a central projection requires auxiliary constructions or calculations. These matters are taught in courses on *Perspective*, such as given in schools of Art or Architecture. The understanding of perspective is indispensable in Computer Vision and, as such, important in Robotics. Descriptive Geometry is based on another type of projection, specifically *parallel projection*, in most cases *parallel, orthogonal projection*. The result is realistic for small objects placed at large distances.
from the viewer. Even if the result is not realistic for other objects, the pro-
cesses of projection and of measuring in the resulting drawing are simplified
to such an extent that the method imposed itself in Engineering. In fact,
Descriptive Geometry, as developed by Monge, requires no calculations, but
only graphical constructions. This advantage may not be appreciated today
by students who learned early how to use hand-held calculators and desk
computers and do not realize that this technology is very recent. Although
we can use today computers and software for drawing, the theoretical basis
of technical drawing still derives from Descriptive Geometry. In practical
engineering there are instances in which hand drawings and Descriptive Ge-
ometry can be used to check a solution much faster than by digitizing the
data and reverting to a computer. Not to talk about the situations in which
we are not close to a computer. Moreover, research shows that Descriptive
Geometry helps in developing spatial abilities, qualities essential in many
branches of engineering, and especially in Design. But not only engineers
have to understand spatial relationships and be able to interpret ‘flat’ images.
Orthopedists have learned a long time ago how to locate fractures in two x-
ray images perpendicular one to the other. Dentists too use x-ray pictures
in their work. And what is Computer Tomography if not a very advanced
application of several geometric techniques? Descriptive geometry teaches
when it is possible to measure a length or an angle in a drawing, or how to
recover that information when direct measuring is not possible. As another
example, descriptive geometry teaches how to find out if a given object lies
on a surface, or on which side of that surface. An engineer who does not
know how to do this can place a piece of equipment outside a vehicle, and
an Interior Designer who lacks this skill can block a door by wrongly placing
an object. During their practice, two of the authors encountered more than
one example of such cases.

Descriptive geometry continues to live even when the use of it is not
openly acknowledged. Thus, the excellent book of Chirone and Tornincasa (2002) does not mention the term Descriptive Geometry, but pages
89–105 and 137–153 in it do belong to this discipline. Similarly, Simmons
and Maguire (2003) do not use the explicit term, but chapter 11–13 of their
book treat, in fact, subjects of descriptive geometry. As a counterexample,
Bertoline and Wiebe (2003) have no problem to give the title Descriptive
Geometry to Part Three of their monumental book. This part runs from
page 571 to 636 and it begins with the sentence, ‘Descriptive Geometry is
the mathematical foundation of engineering graphics.’

Many, probably most books on Descriptive Geometry are, in fact, ‘cook-
books’. The various procedures are given as ‘recipes’ without any serious
justification. Descriptive Geometry, however, is a branch of geometry, in
particular, and of mathematics, in general. There ‘is reason behind’ any
procedure and it is possible to give proofs for all procedures. Obviously, the
basis is elementary geometry, plane and solid. For many procedures one can
find a one-to-one relationship to Linear Algebra. Also, not a few procedures
can be reduced to calculations in Analytic Geometry and, sometimes, in Dif-
ferential Geometry. More advanced results belong to Projective Geometry, that branch of Mathematics that evolved from the study of perspective.

Historically, Geometry is the first branch of science that used deductive logic and rigorous formalism. More than one philosopher used the qualifier ‘geometrical’ or one derived from it to show that his reasoning is logical, rigorous. To cite only one famous example, Spinoza’s most important work bears the title *Ethica, more geometrico demonstrata* (Ethics, demonstrated in a geometrical manner). A report issued in 1997 by the US Department of Education, and known as the ‘Riley report’, shows that high-school students who took a course in geometry succeeded better in college than their colleagues who did not take such a course. ‘The report did not say anything about getting a good grade in those mathematics courses or even about passing. Simply taking the courses led to benefits. What’s more, students got the same benefits regardless of the subjects that they pursued at college.’ (Devlin, 2000, p. 262). Descriptive Geometry is a branch of Geometry and as such, if properly taught, it can help in improving the thinking skills of the students. Therefore, we consider that it’s wrong to write DG textbooks styled as cookbooks and we try to follow quite another course in our book. Truly said, there are a few German DG textbooks that try to explain the procedures and, to do so, they use also other branches of mathematics. See, for example, Reutter (1972).

It is possible to establish a one-to-one correspondence between chapters of geometry, among them descriptive geometry, and linear algebra. In Italy, for
example, university courses that cover Linear Algebra are called *Geometria* and treat geometry and linear algebra as one subject. For example, the book of Abate (1996) describes the main concept of descriptive geometry, *projection*, in pure terms of linear algebra. The links between linear algebra and geometry are also shown by Banchoff and Wermer (1983) who give their book the significant title *Linear algebra through geometry*.

In this book we try to take into account the above considerations. The book does not cover all topics found in extended treaties on the subject. We preferred to restrict ourselves to basics. Students interested in more complex problems can continue by reading traditional textbooks. Alternatively, having understood the basics, they can solve their problems by using one of the available CAD packages. We cannot refrain from pointing out that the indiscriminate use of a CAD program can produce wrong results. One must know how to prepare the input and how to interpret the output. While the acronym ‘CAD’ is meant to stay for ‘Computer-Aided Design’, it may easily become ‘Computer-Aided Disasters’.

Further, in this book we try to justify most procedures. To do so we use Analytic Geometry, Linear Algebra, Vector Calculus and Differential Geometry, and we exemplify the relationships between these disciplines. As some of our students testified, by proceeding in this way the course helped them to better understand courses in Calculus and possibly also others. By explaining the procedures, we are also trying to help students in approaching books and courses in advanced graphics, especially computer graphics. Sim-
ply said, a course based on this book may have an *Added Value*. On the other hand, to fully profit of this book, the reader must have some knowledge in the mathematical disciplines mentioned above. The more advanced graphic programming is, the more mathematics it needs.

In our book we added a few complements not found in traditional DG textbooks. Those additions provide the link between descriptive geometry and other disciplines such as geometric modelling, computer graphics, maps, or stereo vision.

To enhance readability we illustrate the book by coloured figures. Almost all figures are produced in MATLAB. The companion software of the book can be retrieved from the site provided by the editors, and from the site provided by The MathWorks. Figures called in the MATLAB environment can be rotated and this may contribute to their visualization. Other files found in the companion software can be run to see animations of some procedures described in the book. These ‘movies’ can greatly contribute to the understanding of the text.

Throughout this book we use the **first angle projection**, as usual in most European countries. In addition to traditional terminology, we are trying to introduce in this book the newest ISO terms.

Besides having its own value, the book can be used as an auxiliary text to books on Engineering Graphics and help in understanding them better and extend the material exposed in them. An example of such a book is the excellent manual of Simmons and Maguire (2003).
We summarize here the notations used in this book. Alternative notations may be found in the literature of specialty. Where we think it is important, we are trying to explain our choice as compared to other notations.
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<td>upper-case Latin letters</td>
<td>$A, B, \ldots P$</td>
</tr>
<tr>
<td>Straight line</td>
<td>lower-case Latin letters</td>
<td>$l, g$</td>
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<tr>
<td>Plane</td>
<td>upper-case Greek letter</td>
<td>$\Delta, \Gamma$</td>
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<td>$\pi_1$ for the horizontal plane</td>
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Chapter 1

Geometric constructions

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Chapter 2

Introduction

2.1 How we see an object

Before learning how to represent an object, we must understand how we see it. In Figure 2.1 the object is the letter $V$; it lies in the object plane. Rays of light coming from the object are concentrated by the lens of the eye and projected on the retina, forming there the image of the object. This image is inverted. The retina is not plane, but, for the small angles of ray bundles involved, we can consider it to be planar.

Images in photographic cameras are formed in the same way as in the eye, but in this case the image is, indeed, plane. In traditional photographic cameras the image is formed on films. In digital cameras the image is formed on arrays of electronic sensors that may be of the CCD (charged-coupled devices) or the CMOS (complementary metal oxide
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semiconductor) type. In photographic cameras too the image is inverted, and the image plane is known as the focal plane. In camera obscura, a device known for at least 2000 years, light is admitted through a narrow hole into a dark chamber. An inverted image is formed on the opposing wall.

To simplify the geometrical reasoning, it is convenient to consider an image plane situated between the lens and the object, at the same distance from the lens as the true image plane. In Figure 2.2 we see both the true image plane and the one introduced here. It is obvious that the size of the image is the same, but the position on the new image plane is the same as on the object plane and not inverted. Therefore, from here on we shall consider only such situations (see Figure 2.3).

2.2 Central projection

2.2.1 Definition

In the preceding section we considered an object and its image. We say that the image is the projection of the object on the plane of the retina, film or array of electronic sensors. All the lines that connect one point of the object with its projection (that is its image) pass through the same point that we call projection centre. This kind of projection is called central projection. In French we find the term perspective centrale (see, for example, Flocon and Taton, 1984), and in German Zentralprojektion
2.2. CENTRAL PROJECTION

Definitions regarding the central projection can be read in ISO 5456-4:1996(E).

2.2.2 Properties

Figure 2.3 allows us to discover a first property of the central projection. If the object and the image planes are parallel, the image is geometrically similar to the object.
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Figure 2.2: How we see the world

We examine now other situations. In Figure 2.4 the point $P$ is the object and $E$ the eye. We consider a system of coordinates in which the $xy$–plane is horizontal and contains the point $P$, the $yz$–plane is the image plane, and the eye lies in the $zx$–plane. The image, or projection of the point $P$ is the point $P'$ in which the line of sight $PE$ pierces the image plane.

Theoretically, we can consider object points situated before or behind the image plane.

In Figure 2.4 we can see a graphic method for finding the projection $P'$. If the method is not obvious at this stage, we invite the reader to accept it
now as such and to return to it after reading this chapter and the following one. We draw through $E$ a perpendicular on the horizontal, $xy$—plane and find the point $F$ in which it intersects the $x$—axis. The line $FP$ intersects the $y$—axis in $P_h$. We raise the vertical from $P_h$ and intersect the line $EP$ in $P'$. By construction the point $P'$ belongs to the image plane; it is the central projection of the point $P$.

The central projection of the projection centre, $E$, is not defined. Neither can we find the projection of the point $F$. In general, we cannot construct the projection of a point that lies in the plane that contains the projection.
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In practice, this fact may not appear as a disadvantage, but it is a nuisance for mathematicians. To build a consistent theory it is necessary to include the points at infinity into the space under consideration. These additional points are called ideal.

Let us turn now to Figure 2.5. The segment $EE_p$ is perpendicular to the image plane and $E_p$ belongs to this plane. We cannot find in the plane $xy$ a point whose image is $E_p$; however, we can think that if we consider a point $P$ on the axis $x$ and its image, when the point $P$ moves to infinity (in either
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Figure 2.5: A point that is not the image of any point direction) its image approaches $E_p$. To understand the geometrical significance of the point $E_p$ we must see how straight lines are projected under a central projection.

In Figure 2.6 we consider the line $d$ belonging to the horizontal plane $xy$, and perpendicular to the image plane. The point $P_1$ belongs both to the line $d$ and to the image plane. Therefore, this point coincides with its projection. We can choose two other points, $P_2$ and $P_3$, of the line $d$ and look for their projections. The line $d$ and the point $E$ define a plane. Let us
call it \( \Pi \). The plane \( \Pi \) contains all the projection rays of points on \( d \); it is the plane that projects \( d \) on the image plane, with \( E \) as the projection centre. It follows that the projection of \( d \) is the intersection of the plane \( \Pi \) with the image plane. Then, the projections of the points \( P_1, P_2 \) and \( P_3 \), and, in general, the projections of all points of \( d \) lie on the line that passes through the points \( P_1 \) and \( E_p \). Remember, the point \( E_p \) was defined in Figure 2.5. As \( EE_p \) is perpendicular to the image plane, it is parallel to \( d \) and belongs to the plane \( \Pi \). Let us think now that the point \( P_2 \) moves towards infinity. Then, its projection moves towards \( E_p \); this is the image of
the ideal point of the line $d$.

In the above reasoning we made no assumption as to the position of the line $d$; all we said is that it is perpendicular to the image plane. We can conclude that the projections of all lines perpendicular to the image plane end in the point $E_p$. In other words, $E_p$ is the projection of the ideal points of all lines perpendicular to the image plane. In fact, in *Projective Geometry* — the branch of geometry that evolved from the study of the central projection — all lines parallel to a given direction have the same ideal point. An important consequence, illustrated in Figure 2.7, is that the projections of parallel lines (not parallel to the image plane) converge in one point, the image of their ideal point.

The reader may try the above considerations on a line that is not perpendicular to the image plane, but makes any angle with it. We are going to examine the special case in which the line is parallel to the image plane. Thus, in Figure 2.8 we consider two lines, $d_1$ and $d_2$, parallel to the image plane. Their projections, $d'_1$ and $d'_2$, are parallel to the $y$–axis and, thus, are parallel one to the other.

The analysis of Figures 2.7 and 2.8 leads us to the conclusion that *in central projection parallel lines are projected as parallel if and only if they are parallel to the image plane*. In mathematical terms we like to say that the parallelism of lines parallel to the image plane is an *invariant* of central projection. In general, however, parallelism is not conserved under central projection.
Figure 2.7: Central projection, lines not parallel to the image plane

Figure 2.8 reveals us another property of central projection. The segments of the lines $d_1$ and $d_2$ shown there are equal, but the projection of the segment of $d_2$ is shorter than the projection of the segment of $d_1$. The farther a segment is from the image plane, the shorter is its projection. This phenomenon is known as foreshortening and it contributes to the realistic sensation of depth conveyed by central projection. We shall talk more about it in the section on vanishing points.

There is another property that is not invariant under central projection. In Figure 2.9 the point $O$ is the centre of projection, $\overline{AB}$ is a line segment in
the object plane, and $\overline{AB}$, its central projection on the image plane. As one can easily check, $M'$ is the midpoint of the segment $\overline{A'B'}$, while $M$ is not the midpoint of the segment $\overline{AB}$. This simple experiment shows that central projection does not conserve the simple ratio of three collinear points. Formerly we can write that, in general,

$$\frac{AM'}{M'B'} \neq \frac{AM}{MB}$$

An exception is the case in which the object and the image pane are parallel.

Central projection, however, conserves a more complex quantity, the
Central projection, variation of the simple ratio

**Figure 2.9:** Central projection, the simple ratio is not invariant

**cross-ratio** of four points. In Figure 2.10 we consider the projection centre, $O$, and three collinear points, $A$, $B$, $C$, of the object plane. The three points lie on the line $d$ that together with the projection centre defines a projecting plane. Its intersection with the image plane is the line $d'$. The projection of the point $A$ is $A'$, that of $B$, $B'$, and that of $C$ is $C'$. Following Reutter (1972) we draw through $O$ a line parallel to the object line, $d$. It intersects $d'$ in $F$. We also draw through $C'$ a line parallel to $d$. Its intersections with the projection rays are $A_1$, $B_1$ and $C_1$, the latter being identical, by construction, with $C'$. 
2.2. CENTRAL PROJECTION

Applying the theorem of Thales in the triangle $ACO$ we write

$$\frac{AC}{BC} = \frac{A_1C_1}{B_1C_1} \quad (2.1)$$

The triangles $A_1A'C_1$ and $OA'F$ are similar. Therefore, we can write

$$\frac{A'C'}{A'F} = \frac{A_1C_1}{OF} \quad (2.2)$$

The triangles $B_1B'C_1$ and $OB'F$ are also similar. Therefore, we can write

$$\frac{B'C'}{B'F} = \frac{B_1C_1}{OF} \quad (2.3)$$

Dividing Equation 2.2 by Equation 2.3 and using Equation 2.1 we obtain

$$\frac{A'C'}{A'F} \cdot \frac{B'F}{B'C'} = \frac{A_1C_1}{OF} \cdot \frac{OF}{B_1C_1}$$

$$= \frac{A_1C_1}{B_1C_1} = \frac{AC}{BC} \quad (2.4)$$
Summing up, we obtained

\[
\frac{AC}{BC} = \frac{AC'}{B'C'} \cdot \frac{B'F}{AF} \tag{2.5}
\]

Taking instead of \( C \) a point \( D \), we can write

\[
\frac{AD}{BD} = \frac{AD'}{B'D'} \cdot \frac{B'F}{AF} \tag{2.6}
\]

It remains to divide Equation 2.5 by Equation 2.6 and write

\[
\frac{AC}{BD} : \frac{AD}{BD} = \frac{AC'}{B'C'} : \frac{AD'}{B'D'} \tag{2.7}
\]

The quantity at the left of the equal sign is the \textbf{cross-ratio} of the points \( A, B, C, D \) and is noted by \((ABCD)\). The quantity at the right of the equal sign is the cross-ratio of the points \( A', B', C', D' \) and is noted by \((A'B'C'D')\). We have proved that the \textit{cross-ratio of four collinear points is an invariant of central projection}. This property has important theoretical applications, but it is also used in practice, for instance in the interpretation of aerial photos. In French, the term cross-ratio is translated as \textit{birapport}, and in German as \textit{Doppeltverhältnis}. More about the subject can be read, for example, in Brannan, Esplen and Gray (1999), or in Delachet (1964).

In Figure 2.11 we discover another property of the central projection. The object segments \( AB \) and \( CD \) are of different lengths, but their images, \( A'B' \) and \( C'D' \) have the same length. In concise notation,

\[
AB \neq CD, \quad A'B' = C'D'
\]
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This ambiguity of the central projections is used with success in cinematography, but only complicates the interpretation of engineering images.

2.2.3 Vanishing points

Let us list the properties revealed by studying the projections of various lines.

- Parallel lines parallel to the image plane are projected as parallels.
- Parallel lines not parallel to the image plane are projected as
converging lines.

- All lines parallel to a given direction are projected as lines converging to the same point.

The above properties determine the character of images formed under central projection. Thus, Figure 2.12 shows a room defined by lines parallel or perpendicular to the image plane. While the former are projected as parallel lines, the latter appear as lines converging to one point, as seen in the centre of Figure 2.13. The point of convergence is called **vanishing point**. In Figures 2.12 and 2.13 there is only one vanishing point.
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Therefore, such a projection is known as **one-point perspective**.

In a more general situation, the object can be defined by a set of lines parallel to the image plane, namely vertical, and sets of lines parallel to two directions oblique to the image plane. We obtain then a **two-point perspective**, as in Figure 2.14.

In the most general situations there are no lines parallel to the image plane. If we can identify sets of lines parallel to three directions, we can construct a **three-point perspective**, as in Figures 2.15 and 2.16. Such perspectives are used, for example, to obtain *bird views* of tall buildings.

When Renaissance artists discovered the laws of perspective, they liked to exercise with it and draw spectacular images. One particularly popular item was the *pavement* (in Italian *pavimento*) which, as exemplified in Figures 2.12 and 2.13, easily contributed to the perception of space. To cite only two names, more recently the Dutch graphic artist Maurits Cornelis
Two-point perspective

Figure 2.14: Two-point perspective

Escher (1898-1972) and the Spanish Surrealist painter Salvador Dali (1904-1989) exploited in remarkable manners the possibilities of drawing perspectives.

2.2.4 Conclusions

The central projection produces the true perspective, that is images as we see them, and as photographic cameras (film or digital) and TV cameras see.

As we saw, central projection conserves the parallelism of lines parallel to the image plane, the geometric similarity of objects in planes parallel to the image plane, and the cross-ratio. On the other hand, the central projection does not conserve the parallelism of lines that have any direction oblique to
2.2. CENTRAL PROJECTION

Figure 2.15: Three-point perspective

the image plane, does not conserve the simple ratio, and the foreshortening
effect can lead to ambiguity of size. As a result, to draw in central
projection one must use elaborate techniques, and Figure 2.13 is just a very
simple example. Still worse, to pick up measures from a central-projection
drawing one has to use special calculations. Architects and graphic artist
have to draw in central projection in order to obtain realistic effects. The
interpretation of aerial photos, and that of images seen by a robot require
the use of elaborate techniques and not-so-simple mathematics. In
engineering drawing, however, we are interested in using the simplest
techniques, both for drawing and for reading a drawing. Therefore, we must
accept an assumption that makes our life simpler, even if we have to pay
some cost. These matters are described in the next sections. We think,
however, that the preceding introduction was worth reading because
1. the user of projections described in continuation must know to what extent his work departs from reality and why he has to accept this fact;

2. the student who wants to reach higher levels or continue to the study of computer vision, must know in which direction to aim.
2.3 Parallel projection

2.3.1 Definition

We make now a new, strong assumption: \textit{the projection centre is sent to infinity}. Obviously, all projections line become parallel. What we obtain is called \textbf{parallel projection}. An example is shown in Figure 2.17. As we shall see in the next subsection, by letting the projection rays be parallel we add a few properties that greatly simplify the execution and the reading of drawings.

In real life we cannot think about looking to objects from infinity. The farther we go from an object, the smaller its image is. From a certain distance we practically see nothing. We ignore this fact and, as said, we
assume that we do see the world from infinity. In fact, in Engineering Drawing we assume many other conventions that do not correspond to reality, but greatly simplify the work.

2.3.2 A few properties

![Diagram](image)

Figure 2.18: Parallel projection — The invariance of the simple ratio

In Figure 2.18 the line $d$ belongs to the object plane and its parallel projection is $d'$. The projection of the point $A$ is $A'$, that of $B$ is $B'$, and that of $C$ is $C'$. The projection lines, $AA'$, $BB'$, $CC'$, are parallel. They can make any angle with the image plane, except being parallel to it. Let
us draw though the point \( C \) the line \( d_1 \) parallel to \( d' \). By construction

\[ \overline{A_1C_1} = \overline{AC'}, \overline{B_1C_1} = \overline{BC'} \]  
\[ (2.8) \]

Applying the theorem of Thales in the triangle \( ACA_1 \) yields

\[ \frac{BC}{AC} = \frac{B_1C_1}{A_1C_1} \]  
\[ (2.9) \]

Substituting the values given by Equations 2.8 in Equations 2.9 we obtain

\[ \frac{BC}{AC} = \frac{B'C'}{A'C'} \]  
\[ (2.10) \]

We conclude that the simple ratio of three collinear points is an invariant of parallel projection.

Another property of parallel projection can be deduced from Figure 2.17: parallel lines are projected as parallel.

Finally, Figure 2.19 shows that when the object plane is parallel to the image plane, the lengths of segments and the sizes of angles are invariant.

### 2.3.3 The concept of scale

We know now that in parallel projection the length of the projection of a segment parallel to the image plane is equal to the real length of that segment. This property greatly simplifies the process of drawing.

Apparently it would be sufficient to arrange the object so as to have as many segments parallel to the image plane and then draw their projections in real size. In practice, however, this is not always possible. Many objects
Figure 2.19: Parallel projection — The invariance of lengths and angles are larger than the paper sheets we use. Other objects are too small and drawing them in real size would result in an unintelligible image. To solve the problem we must multiply all dimensions by a number such that the resulting image can be well accommodated in the drawing sheet and can be easily understood.

The international standard ISO 5455-1979 defines scale as the ‘Ratio of the linear dimension of an element of an object as represented in the original drawing to the real linear dimension of the same element of the object itself. Drawing at full size means drawing at the scale 1:1, enlargement means a
2.4. ORTHOGRAPHIC PROJECTION

drawing made at a scale larger than 1:1, and reduction one made at a scale smaller than 1:1. For example, if a length is 5 m, that is 5000 mm, when we use the scale 1:20 we draw it as 250 mm. On the other hand, a rectangle 2 mm by 5 mm drawn at the scale 2:1 appears as a rectangle with the sides 4 and 10 mm. Table 2.1 shows the scales recommended in ISO 5455-1979.

Table 2.1: The scales recommended by ISO 545-1979

<table>
<thead>
<tr>
<th>Enlargement</th>
<th>2:1 5:1 10:1 20:1 50:1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reduction</td>
<td>1:2 1:5 1:10 1:20 1:50</td>
</tr>
<tr>
<td></td>
<td>1:100 1:200 1:500 1:1000 1:2000</td>
</tr>
<tr>
<td></td>
<td>1:5000 1:10000</td>
</tr>
</tbody>
</table>

2.4 Orthographic projection

2.4.1 Definition

Parallel projections are used for special applications in Engineering Graphics and, to a greater extent, in general graphics. In Descriptive Geometry, however, we go one step farther and we assume that the projection rays are perpendicular to the image plane. One simple example is shown in Figure 2.20. We obtain thus the orthogonal projection. This is a particular case of the parallel projection; therefore, it inherits all the
properties of the parallel projection. In addition, the assumption of projection rays perpendicular to the image plane yields a new property that is described in the next subsection.

Figure 2.21 shows a more interesting example of orthographic projection; it can be compared to Figure 2.17.

In technical drawing, when using several projections to define one object, we talk about **orthographic projections**. Some standard definitions related to orthographic representations can be read in ISO 5456-2:1996(E).

### 2.4.2 The projection of a right angle

Figure 2.22 shows two line segments, $BA$ and $BC$, and two image planes, $\pi_1$ and $\pi_2$, perpendicular one to another. As shown in Section 2.5, these are the *projection planes* adopted in Descriptive Geometry.
In Figure 2.22 the angle $\angle ABC$ is right and the side $BC$ is horizontal, that is parallel to the projection plane $\pi_1$. The figure shows also the horizontal and the frontal projections of this angle. By horizontal projection we mean the projection on the horizontal plane $\pi_1$, and by frontal projection, the projection on the frontal plane $\pi_2$. The following theorem states an important property of orthogonal projections.

\[ \text{If one of the sides of a right angle is parallel to one of the projection planes, the (orthogonal) projection of the angle on that plane is also a right angle} \]
To prove this theorem, let the coordinates of the points $A$, $B$, $C$, be

\[
A = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}, \quad B = \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}, \quad C = \begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix}
\]

Figure 2.22: Projecting a right angle

The horizontal projections of the three points have the coordinates

\[
A' = \begin{bmatrix} A_x \\ A_y \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_x \\ B_y \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_x \\ C_y \\ 0 \end{bmatrix}
\]
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The angle $\hat{ABC}$ has as sides the vectors

$$
\overrightarrow{BA} = (A_x - B_x)i + (A_y - B_y)j + (A_z - B_z)k
$$

$$
\overrightarrow{BC} = (C_x - B_x)i + (C_y - B_y)j + (C_z - B_z)k
$$

(2.11)

If the angle $\hat{ABC}$ is right, the scalar product of the above vectors is zero

$$(A_x - B_x)(C_x - B_x) + (A_y - B_y)(C_y - B_y) + (A_z - B_z)(C_z - B_z) = 0$$

(2.12)

The scalar product of the horizontal projections of the vectors $\overrightarrow{BA}$, $\overrightarrow{BC}$ is

$$
\overrightarrow{BA}' \cdot \overrightarrow{BC}' = (A_x - B_x)(C_x - B_x) + (A_y - B_y)(C_y - B_y)
$$

(2.13)

If the projected angle is right, the scalar product $\overrightarrow{BA} \cdot \overrightarrow{BC}$ must be zero and this happens only if

$$(A_z - B_z)(C_z - B_z) = 0$$

that is, if either $A_z = B_z$, or $C_z = B_z$. In other words, the right angle $\hat{ABC}$ is projected as a right angle, $\hat{A'B'C'}$, if $\overrightarrow{BA}$ or $\overrightarrow{BC}$ is parallel to the horizontal projection plane, $\pi_1$. This proves that the theorem states a necessary condition. We can see immediately that this condition is also sufficient. Indeed, if in Equation 2.12 either $A_z = B_z$, or $C_z = B_z$, then the scalar product in Equation 2.13 is zero. In Figure 2.22 the side $\overrightarrow{BC}$ is parallel to the horizontal projection plane, $\pi_1$. The extension of the proof to
cases in which one side of the right angle is parallel to another projection plane is straightforward.

Is this theorem valid only for orthogonal projections? Doesn’t it hold also for an oblique parallel projection? Without loss of generality, let us check the case of a parallel, oblique projection in which the projecting rays are parallel to the frontal projection plane, \( \pi_2 \), but make the angle \( \alpha \) with the horizontal projection plane, \( \pi_1 \). In Figure 2.23 we consider the point \( B \) of Figure 2.22. The orthogonal projection of the point \( B \) would be \( B'_O \) with the \( x \)-coordinate equal to \( B_x \). We consider, however, the oblique projection with \( B'_O B'_P B = \alpha \). The projection of \( B \) is now the point \( B'_P \) with the
2.4. ORTHOGRAPHIC PROJECTION

$x$–coordinate

\[ B_x - \frac{B_z}{\tan \alpha} \]

Then, the coordinates of the oblique projections would be

\[
A_P' = \begin{bmatrix} A_x - \frac{A_z}{\tan \alpha} \\ A_y \\ 0 \end{bmatrix}, \quad B_P' = \begin{bmatrix} B_x - \frac{B_z}{\tan \alpha} \\ B_y \\ 0 \end{bmatrix}, \quad C_P' = \begin{bmatrix} C_x - \frac{C_z}{\tan \alpha} \\ C_y \\ 0 \end{bmatrix}
\]

and the scalar product of the projections is

\[
\overrightarrow{B_P' A_P'} \cdot \overrightarrow{B_P' C_P'} = \left( A_x - B_X - \frac{A_z - B_z}{\tan \alpha} \right) \left( C_x - B_x - \frac{C_z - B_z}{\tan \alpha} \right) + (A_y - B_y)(C_y - B_y)
= (A_x - B_x)(C_x - B_x) + (A_y - B_y)(C_y - B_y) - \\
\left( A_x - B_x \right) \frac{C_z - B_z}{\tan \alpha} - \left( C_x - B_x \right) \frac{A_z - B_z}{\tan \alpha} \quad (2.14)
\]

As we assumed that the angle \( \hat{ABC} \) is right and that one of its sides is parallel to the horizontal projection plane, the sum of the first two terms in Equation 2.14 is zero. The sum of the other two terms is zero if, and only if we have both \( C_z = B_z \) and \( A_z = B_z \), that is if both sides of the angle \( \hat{ABC} \) are parallel to the horizontal projection plane. But, in this case the angle \( \hat{ABC} \) lies in a horizontal plane, that is in a plane parallel to one of the projection planes and we already know that in parallel projections the size of angles is invariant if the object and image planes are parallel.
To the right-angle theorem

There is an experimental way of proving the right-angle theorem. We invite the reader to take a cardboard box and a square set. First, locate the square set in the corner of the box, and parallel to the bottom, as in Figure 2.24. The square set is shown in yellow, and its horizontal projection in a darker, brown hue. This projection is right-angled.

Let now the side $b$ of the square set remain horizontal, while we rotate the side $a$ as in Figure 2.25. The square set still fits well in the corner of the box and the horizontal projection remains right-angled. Alternatively, keep the side $a$ horizontal and rotate the side $b$ as in Figure 2.26. The square set
Figure 2.25: Experimental proof of the right-angle theorem

again fits well in the corner of the box and the horizontal projection remains right-angled.

Finally, try to rotate both sides $a$ and $b$ of the square set so that none of them is horizontal, or, in other words, none of them is parallel to the bottom. It is impossible to do this without changing the angle of the box corner. In fact it must be increased. This means that the horizontal projection is no more right-angled.
2.5 The two-sheet method of Monge

It seems that the great German painter and printmaker Albrecht Dürer (1471-1528) was the first to show how three-dimensional objects can be represented by projections on two or more planes (Coolidge, 1963, pp. 109–12, Bertoline and Wiebe, 2003, pp. 394–5). However, the first to organize the methods for doing so in a coherent system was the French mathematician Gaspard Monge (1746-1818). Therefore, he is considered to be the founder of Descriptive Geometry. Monge was one of the scientists...
2.5. THE TWO-SHEET METHOD OF MONGE

who accompanied Bonaparte in the Egypt campaign and was one of the founders of the École Polytechnique in Paris.

Figure 2.27: Preparing the two-sheet projection of Monge

Figure 2.27 shows a point $P$ and a system of cartesian coordinates. The basic idea of Descriptive Geometry is to use two projection planes, a horizontal one, defined by the axes $x$ and $y$, and a frontal one defined by the axes $x$ and $z$. We note the horizontal plane by $\pi_1$, and the frontal one by $\pi_2$. These projection planes are shown in Figure 2.28.

The next step is to project orthogonally the point $P$ on $\pi_1$ and $\pi_2$. The horizontal projection, $P'$, is the point in which a ray emitted from $P$, perpendicularly to $\pi_1$, pierces the plane $\pi_1$. Similarly, the frontal projection, $P''$, is the point in which a ray emitted from $P$, perpendicularly to $\pi_2$, 
pierces the plane $\pi_2$. These projections are shown in Figure 2.29.

Finally, as shown in Figure 2.30, we rotate the plane $\pi_2$ until it is coplanar with $\pi_1$. We obtain thus what we call the sketch of the point $P$. It is shown in Figure 2.31. This figure shows that the two projections contain all the information about all the coordinates of a given point.

As an example of projection on two sheets we show in Figure 2.32 the sketch of the right angle $\triangle ABC$ introduced in Figure 2.22.
To define the central projection we assumed that all projection rays pass through one point, the projection centre. There is no condition on the position of the projection centre. We saw then that the only properties that remain invariant between an object and its image are:

1. the parallelism of line parallel to the image plane;

2. the cross ratio of four collinear points.
Figure 2.30: Preparing the two-sheet projection of Monge

To define the parallel projection we accept an additional assumption: the projection centre is sent to infinity. Then, the projection rays become parallel and we obtain three new invariants:

1. sizes of objects (segment lengths and angles) parallel to the image plane;

2. parallelism of lines;

3. the simple ratio of three collinear points.
The coordinates of the point P in the Monge sketch

\[ P \]

\[ p' \]

\[ x_p \]

\[ z_p \]

\[ y_p \]

\[ \pi_1 \]

\[ \pi_2 \]

Figure 2.31: The Monge sketch contains all coordinates

The parallel projection inherits the invariants of the central projection. By adding a new assumption we restricted the number of possible projections. The parallel projection is a particular case of the central projection. On the other hand, the parallel projection has more useful properties than the central projection; it is easier to draw an object in parallel projection than in central projection and it is easier to measure lengths in parallel projection than in central projection.

Let us further assume that the projection rays are perpendicular to the image plane. We are dealing now with the orthogonal or orthographic
CHAPTER 2. INTRODUCTION

The projections of a right angle

\[ \pi_1 \]

\[ \pi_2 \]

Figure 2.32: The sketch of the projections of a right angle

projection and obtain a further invariant:

1. the size of a right angle with a side parallel to the image plane.

The orthographic projection inherits the invariants of the parallel projection. The orthographic projection is a particular case of the parallel projection. By restricting further the number of possible projections we obtained an additional, useful property; it is easier to draw and to measure in orthographic projection than in an oblique parallel projection.

Figure 2.33 summarizes the hypotheses and invariants of the various projection methods and their relations of inclusion. In ISO 5456-1:1996,
Part 1, we find a summary of the properties of different projection methods. A technical vocabulary of terms relating to projection methods in English, French, German, Italian and Swedish can be found in ISO 10209-2:1993(E/F).

We said that when we draw in parallel projection we accept the seemingly unnatural assumption that we look from infinity. Nature, however, provides us with a real example of parallel projection: shadows cast by objects illuminated by the sun. The distance between us and the sun is so large that its rays are practically parallel. And if we are talking about shadows, let us say that shadows of objects illuminated by a small source of light are approximations of central projections.

2.7 Examples

Example 2.1 — Dividing a segment

In this example we learn a widely used application of one of the properties of parallel projection, specifically the invariance of the simple ratio. In Figure 2.34 we see a segment $P'_0P'_7$ defined on the $x-$ axis. Let us suppose that we are asked to divide it into seven equal segments. The trivial solution seems to consists of:

1. measuring the length $P'_0P'_7$;
2. dividing the above length by 7;
3. applying the resulting length, $\frac{P_0P_7}{7}$, seven times along the $x$–axis.

This procedure may be convenient in particular cases. In general however, the calculated length, $\frac{P_0P_7}{7}$, does not fit exactly the gradations of our rulers or square sets. Then, measuring seven times the calculated length may result in unequal segments and an accumulated error towards the end.

A good solution is to draw an arbitrary segment, inclined with respect to the given segment. In Figure 2.34 we draw thus the segment $P_0P_7$, through a point outside the given segment. In practice however, we can draw the arbitrary segment through one of the ends of the given segment, for example from the point $P'_0$. On the oblique segment we measure an exact multiple of 7, for example 70 mm. Let the measured segment be $P_0P_7$. We mark on it seven equal segments

$$P_0P_1 = P_1P_2 = \ldots = P_6P_7$$

We connect $P_7$ to $P'_7$ and we draw through the divisions of $P_0P_7$ ‘projection rays’ parallel to $P_7P'_7$:

$$P_0P_0' \parallel P_1P_1' \parallel \ldots \parallel P_7P_7'$$

To prove that we divided the given segment into equal parts, we draw through $P_0$ a line parallel to the $x$–axis and note by $Q_1$, $Q_2$, $\ldots$, $Q_7$ its intersections with the parallel lines $P_0P'_0$, $\ldots$, $P_7P'_7$. By Thales’ theorem

$$\frac{P_0P_1}{P_0P_7} = \frac{P_0Q_1}{P_0Q_7} = \ldots = \frac{P_0P_7}{P_0P_7} = \frac{Q_6Q_7}{P_0Q_7}$$
As
\[ P_0Q_1 = P_0'P_1', \ldots, Q_6Q_7 = P_6'P_7' \]
by construction, we conclude that
\[ \frac{P_0P_1'}{P_0'P_7'} = \frac{P_0P_1}{P_0P_7} \]
and so on.

**Example 2.2 — Decomposing a vector**

The term *projection* appears also in other contexts than Descriptive Geometry. The reader may be familiar with this use, for instance in Vector Geometry or Mechanics. For example, figure 2.35 shows a vector, \( \mathbf{F} \), and a system of cartesian coordinates. The horizontal axis is \( Ox \), and the vertical axis is \( Oy \). We decompose the vector \( \mathbf{F} \) into a horizontal component, \( \mathbf{F}_x \) and a vertical component, \( \mathbf{F}_y \). Using an alternative terminology we can say that \( \mathbf{F}_x \) is the *horizontal projection* and \( \mathbf{F}_y \) is the *vertical projection* of the given vector.

If \( \alpha \) is the angle made by the vector \( \mathbf{F} \) with the \( x \)-axis, then
\[ |\mathbf{F}_x| = |\mathbf{F}| \cos \alpha, \ |\mathbf{F}_y| = |\mathbf{F}| \sin \alpha \]

Using the concept of *scalar product* we can, however, write a more interesting expression for these projections. Let \( F_x \) be the length of \( \mathbf{F}_x \) and \( F_y \) that of \( \mathbf{F}_y \). As usual, we note by regular letters scalar quantities, and by bold-face letters, vectors. Then
\[ \mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} \]
where \( \mathbf{i} \) is the unit vector along the \( x \)-axis, and \( \mathbf{j} \) the unit vector along the \( y \)-axis. Using the dot, \( \cdot \), to mark the scalar product (therefore known also as dot product), the horizontal projection of \( \mathbf{F} \) is given by

\[
\mathbf{F}_x = ((\mathbf{F}) \cdot \mathbf{i})\mathbf{i} = F_x \mathbf{i}
\]

Similarly, the vertical projection of \( \mathbf{F} \) is given by

\[
\mathbf{F}_y = ((\mathbf{F}) \cdot \mathbf{j})\mathbf{j} = F_y \mathbf{j}
\]

### 2.8 Exercises

In Figures 2.22 and 2.32 the coordinates of the three points \( A, B, C \) are

\[
\begin{align*}
A &= \begin{bmatrix} 3 \\ -7.5 \\ 4.5 \end{bmatrix}, & B &= \begin{bmatrix} 5.5 \\ -2.5 \\ 5.5 \end{bmatrix}, & C &= \begin{bmatrix} 10.5 \\ -5 \\ 5.5 \end{bmatrix}
\end{align*}
\]

Use the scalar product to show that the angles \( \widehat{ABC} \) and \( \widehat{A'B'C'} \) are right.
2.8. EXERCISES

CENTRAL PROJECTION

Hypothesis: all projection rays pass through one centre

Invariants:
1) Parallelism of lines parallel to the image plane
2) The cross ratio of four collinear points

PARALLEL PROJECTION

Additional hypothesis: projection centre at infinity, that is, parallel projection rays

Invariants:
1) Sizes of objects in planes parallel to the image plane
2) Parallelism of lines
3) Simple ratio of three collinear points

ORTHOGRAPHIC PROJECTION

Additional hypothesis: projection rays perpendicular to projection planes

Invariants:
Right angle if one side is parallel to a projection plane

Figure 2.33: Projection methods and their invariants
Parallel projection – Applying the invariance of the simple ratio

Figure 2.34: An application of the simple-ratio invariance
Figure 2.35: The components, or projections of a vector
Chapter 3

The point

3.1 The projections of a point

3.2 Points in four quadrants

3.3 The first angle view

3.4 The third angle view

3.5 Exercises
Chapter 4

The straight line

4.1 The projections of a straight line

4.2 Special straight lines

4.2.1 Parallel lines

4.2.2 Horizontal lines

4.2.3 Frontal lines

4.2.4 Profile lines

4.2.5 Lines perpendicular on $\pi_2$

4.2.6 Vertical lines

4.2.7 Lines perpendicular on $\pi_3$

4.3 When a third projection can help

4.4 Intersecting lines

4.5 The true length of a straight-line segment

4.6 The traces of a given straight line

4.7 Point on given line

4.8 The straight line as a one-dimensional manifold

4.9 Exercises
Chapter 5

The plane

5.1 Defining a plane

5.2 The traces of a given plane

5.3 The traces of special planes

5.3.1 Definitions

5.3.2 How to visualize a plane

5.3.3 Horizontal planes

5.3.4 Frontal planes

5.3.5 Planes parallel to $\pi_3$

5.3.6 Planes perpendicular on $\pi_1$

5.3.7 Planes perpendicular on $\pi_3$

5.3.8 Parallel planes

5.4 Special straight lines of a given plane
Chapter 6

Axonometry I

6.1 The need for axonometry

In Figure 6.1 we see the orthographic projections of a simplified house model. In Descriptive Geometry we call the projection planes $\pi_1$, $\pi_2$ and $\pi_3$. In Engineering Drawing it is usual to consider the projection on $\pi_2$ as the main projection and call it front elevation. The projection on $\pi_1$ is called top view and the projection on $\pi_3$, side elevation.

It is obvious that none of the three projections fully defines the shape of the house. Moreover, while the overall breadth, $B_1$, can be measured in the front elevation and the top view, the breadth $B_2$ can be measured only in the front elevation. Similarly, while the overall length, $L_1$, can be measured in the top view and the side elevation, the length $L_2$ can be measured only in the side elevation. Finally, the overall height, $H_1$, and the height $H_2$ can
be measured in the front and side elevations, but not in the top view. We conclude that to fully define this simple house form and to measure all its dimensions we need three projections. One question may naturally arise, 'Isn’t it really possible to find a single projection that fully defines the object and allows us to dimension it completely?’ In some simple cases the answer is, 'Yes’, and one example of a corresponding drawing is shown in Figure 6.2. We call this representation of the house **axonometric projection**. Webster’s Ninth New Collegiate Dictionary shows that the first appearance of the term *axonometric* dates from 1908 and that it is
composed of the Greek word *axon*, meaning ‘axis’, and the English word *metric*.

\[\text{Axonometry of simple house model}\]

\[\text{Figure 6.2: The axonometric projection of a simplified house model}\]

\[\text{6.2 Definition of orthographic axonometry}\]

We assume that the object to be drawn is placed in a cartesian coordinate system. For orthographic projections we place the object so as to have as many plane faces as possible parallel to the planes \(xy\), \(xz\), \(yz\). We noted these planes by \(\pi_1\), \(\pi_2\), \(\pi_3\), respectively. Next, we project the object on those three planes. For axonometric projections we still consider the object
as placed in the same coordinate system, but we proceed differently. There are two ways to define axonometric projections. The first possibility is to add a new plane, $\pi$, that is oblique with respect with the three planes $\pi_1$, $\pi_2$, $\pi_3$. We project now our object on the plane $\pi$. We shall follow in continuation this definition because it allows us to draw an important conclusion. The other possibility is to rotate the object so that its plane faces are oblique with respect to the projection planes and project now the object on one of the planes $\pi_1$, $\pi_2$, $\pi_3$.

Figure 6.3: The axonometric projections of the coordinate axes
6.3 The law of scales

In Figure 6.3 a) we consider a system of cartesian coordinates with the origin in \( O \) and the axes \( O_x, O_y, O_z \). In part b) of the figure we choose a projection plane, \( \pi \), whose traces are

\[
\begin{align*}
X_0Y_0 & \text{ on } \pi_1 \\
X_0Z_0 & \text{ on } \pi_2 \\
Y_0Z_0 & \text{ on } \pi_3
\end{align*}
\]

Let us project orthogonally the three coordinate axes on the projection plane \( \pi \), as shown in part c) of the figure. In part d) the projection of the origin is \( O_p \), that of the \( x \)-axis is \( O_px_0 \), and so on.

In the right-angled triangle \( OX_0O_p \) let the coordinate along the side \( \overline{OX}_0 \) be \( x \), and the coordinate along the side \( \overline{O_pX}_0 \) be \( x' \). In other words, while \( x \) is the coordinate in the actual world system, \( x' \) is the coordinate in the system that we see in the axonometric projection. We can write

\[
x' = x \cos \hat{O_pX}_0O = x \sin \hat{O_pOX}_0 \tag{6.1}
\]

The ratio \( x'/x \) is the scale at which lengths parallel to the \( x \)-axis are projected on the plane \( \Delta \). Let us note it by \( S_x \). Similarly to Equation 6.1 we can write from the right-angled triangles \( OO_yO_0 \) and \( OO_zO_0 \)

\[
\begin{align*}
y' &= y \cos \hat{O_pY}_0O = y \sin \hat{O_pOY}_0 \\
z' &= z \cos \hat{O_pZ}_0O = z \sin \hat{O_pOZ}_0
\end{align*}
\]
We also define the scales at which lengths parallel to the \( y \)-axis and the \( z \)-axis are projected on the \( \Delta \) plane as

\[
\frac{y'}{y} = S_y, \quad \frac{z'}{z} = S_z
\]

The sum of the squares of the three scales is

\[
S_x^2 + S_y^2 + S_z^2 = \sin^2 \hat{O}_pOX_0 + \sin^2 \hat{O}_pOY_0 + \sin^2 \hat{O}_pOZ_0 \quad (6.2)
\]

\[
= 3 - (\cos^2 \hat{O}_pOX_0 + \cos^2 \hat{O}_pOY_0 + \cos^2 \hat{O}_pOZ_0)
\]

Now, \( \hat{O}_pOX, \hat{O}_pOY \) and \( \hat{O}_pOZ \) are the direction angles of \( OO_p \), and their cosines are the direction cosines of the line that contains \( OO_p \) (see, for example, Fuller and Tarwater, 1992, pp. 330-1). The sum of the squares of the direction cosines of a given line equals 1. This result is easy to prove.

For example, in our particular case, if we consider the projections \( OO_x, OO_y \) and \( OO_z \) of the segment \( OO_p \) on the coordinate planes

\[
\overline{OO_x} = \overline{OO_p} \cos \hat{O}_pOX_0 \quad (6.3)
\]

\[
\overline{OO_y} = \overline{OO_p} \cos \hat{O}_pOY_0
\]

\[
\overline{OO_z} = \overline{OO_p} \cos \hat{O}_pOZ_0
\]

Squaring both sides of the above equations and adding them side by side we obtain

\[
\overline{OO_x}^2 + \overline{OO_y}^2 + \overline{OO_z}^2 = \overline{OO_p}^2 (\cos^2 \hat{O}_pOX_0 + \cos^2 \hat{O}_pOY_0 + \cos^2 \hat{O}_pOZ_0) \quad (6.4)
\]
6.4 Isometry

On the other hand, by Pitagora’s theorem in three dimensions

\[
\overline{OO}_x^2 + \overline{OO}_y^2 + \overline{OO}_z^2 = \overline{OP}_p^2
\]  

(6.5)

By combining Equations 6.3, 6.4 and 6.5 we obtain

\[
S_x^2 + S_y^2 + S_z^2 = 2
\]  

(6.6)

6.4 Isometry

Equation 6.6 allows us an infinity of choices. The simples one is

\[
S_x = S_y = S_z = \sqrt{2/3}
\]  

(6.7)

We call this case isometric projection.

6.4.1 More about the definition

6.4.2 Practical rules

6.5 Dimetry

6.5.1 Definition

In the dimetric projection two scales are equal. In the usual choices one of these two scales is \( S_z \).
6.5.2 When is dimetry preferable to isometry

Figure 6.4 shows an isometric view of a parallelepiped whose breadth and height are equal, but the length is ten times larger. Looking at this figure we can have the sensation that the lines that run in the direction of the larger dimension are diverging towards right. Checking the figure with more attention reveals that those lines are parallel. The cause of our illusion resides in our visual experience. As explained in Chapter 2, the images we see are produced by central projection. In this projection, although the edges $P_0P_1$, $P_1P_2$, $P_2P_3$, $P_3P_0$, $P_3P_5$ and $P_5P_6$ are all equal, the projections of the edges $P_4P_5$ and $P_5P_6$ are shorter. This is a
6.5. **DIMETRY**

Consequence of the *foreshortening effect*. As compared to the central projection, in isometry all the above-mentioned edges appear as equal. This contradicts our expectations and we have the feeling that the edge $P_3P_5$ is longer than the edge $P_0P_1$, and the edge $P_5P_6$ is longer than the edge $P_1P_2$. We may generalize and conclude that the isometric projection of objects having one dimension much larger than the others can produce an unpleasant optical illusion. To correct this effect we must use another axonometric projection and the dimetric projection is an adequate alternative.

![Figure 6.5: The dimetry of a slender body](image-url)
6.6 Trimetry

In trimetric projections all three scales are different, that is

\[ S_x \neq S_y \neq S_z \]  \hspace{1cm} (6.8)

6.7 Other axonometries

6.8 Examples

Example 1 - Sketch corresponding to an axonometric projection

Figure 6.6 shows the traces of the projection plane, \( \Delta \), that appears in Figure 6.3. Thus, \( YX \) is the horizontal projection of the trace of \( \Delta \) on \( \pi_1 \), and \( XZ \) is the frontal projection of the trace of \( \Delta \) on \( \pi_2 \). Obviously, the frontal projection of the trace on \( \pi_1 \) and the horizontal projection of the trace on \( \pi_2 \) coincide with the ground line that contains the segment \( OX \). In Figure 6.6 \( O_p \) is the projection of the origin, \( O \), on \( \Delta \). Let us find the projections of this point in the sketch shown in Figure 6.6. To do so, we first find the projections of the perpendicular on \( \Delta \) that passes through the origin, \( O \). Next, we must find the point in which this perpendicular pierces the projection plane \( \Delta \).

To draw the perpendicular from \( O \) on \( \Delta \) we use the theorem that says that a right angle is projected on a plane as a right angle if one of its sides is parallel to that plane. The trace \( XY \) is parallel to the projection plane \( \pi_1 \).
because it belongs to it. Thus, the horizontal projection $\overline{OH'}$ of the perpendicular from $O$ on $\Delta$ is perpendicular to the trace $XY$. Similarly, the trace $XZ$ is parallel to the projection plane $\pi_2$ because it belongs to it. Then, the frontal projection $\overline{OF''}$ of the perpendicular from $O$ on $\Delta$ is perpendicular to the trace $XZ$.

It remains now to find the point $O_p$ in which the perpendicular from $O$ on $\Delta$ intersects this plane. Let us consider the plane $\Gamma$ that projects $O_p$ on $\pi_1$. This plane intersects the trace $XY$ in the point $H'$, whose frontal projection is $H''$. As the line $XY$ belongs to $\Delta$, the point $H$ also belongs to $\Delta$. But, by construction, the point $H$ also belongs to $\Gamma$. The point $Z$ belongs to $\Delta$ because it lies on a trace of its plane. But the point $Z$ belongs also to $\Gamma$ because the line $OZ$ is perpendicular to $\pi_1$, while the plane $\Gamma$ is also perpendicular to $\pi_1$ and contains the origin $O$. We conclude that the line $HZ$ is the intersection of the planes $\Delta$ and $\Gamma$ and, therefore, any point of it belongs to $\Delta$. On the other hand, as both lines $OO_p$ and $OH$ belong to the same plane $\Gamma$ they intersect. Thus we find the projection $O'_p$ as intersection of $H''Z$ and $\overline{OF''}$, and afterwards the projection $O''_p$ by drawing through $O''_p$ the perpendicular on the ground line and intersecting it with $\overline{OH'}$.

6.9 Summary

Definitions regarding axonometric representations can be found in ISO 5456-3:1996(E). A splendidly illustrated book on axonometry is that of
Figure 6.6: Sketch corresponding to an axonometric projection

Pavanelli, Miliani and Marchesini (1999).

6.10 Exercises

Exercise 1 - Vector normal to the projection plane

Referring to the Appendix and to Figure 6.3, show that the vector product \( \vec{Y}_0 \vec{Z}_0 \times \vec{Z}_0 \vec{X}_0 \) yields the same vector normal to the projection plane as Equation 6.12.

Exercise 2 - The sketch of the perpendicular on the projection plane
6.11 Appendix - Origin of coordinates in axonometric projection

In Figure 6.3 the projection plane intercepts the $x$–axis at $X$, the $y$–axis at $Y$, and the $z$–axis at $Z$. It easy to verify that the equation of the projection plane is

$$\frac{x}{X} + \frac{y}{Y} + \frac{z}{Z} = 1 \quad (6.9)$$

Let us find now the equation of a vector normal to the projection plane. To do this we need the expressions of two vectors that belong to the projection plane. Thus, from

$$\overrightarrow{OX_0} + \overrightarrow{X_0Y_0} = \overrightarrow{OY_0} \quad (6.10)$$
$$\overrightarrow{OY_0} + \overrightarrow{Y_0Z_0} = \overrightarrow{OZ_0}$$

we deduce

$$\overrightarrow{X_0Y_0} = -Xi + Yj \quad (6.11)$$
The vector normal to the plane is obtained from the vector product

\[ N = \overrightarrow{X_0Y_0} \times \overrightarrow{Y_0Z_0} = i \begin{vmatrix} j & k \\ -X & Y \\ 0 & 0 \\ 0 & -Y & Z \end{vmatrix} = YZi + XZj + XYk \] (6.12)

We want now to find the equations of the line that passes through the origin, \( O(0,0,0) \), and is parallel to the vector \( N \). This is the line that projects the origin, \( O \), to \( O_p \), the origin on the projection plane. Given a point \( P \) on the line we are seeking, there is a scalar \( t \) such that

\[ \overrightarrow{OP} = tN \]

or

\[ xi + yj + zk = YZti + XZtj + XYtk \] (6.13)

Equating the coefficients of \( i \), \( j \), and \( k \) extracting \( t \) and equating the three expressions that yield it, we obtain

\[ \frac{x}{YZ} = \frac{y}{XZ} = \frac{z}{XY} \] (6.14)

These are the symmetric equations of the line (see Fuller and Tarwater, 1992, pp. 327-8). In fact, we have here three equations that define three
planes perpendicular to the coordinate planes. Their intersection is the line that passes through the origin and is normal to the projection plane, $\Delta$.

The intersection of this line with the projection plane defined by Equation 6.9 is the projection, $O_p$, of the origin on the projection plane, $\Delta$.

Solving the system defined by Equations 6.9 and 6.14 we obtain the coordinates of the point $O_p$

\[
x = \frac{1}{\frac{x}{x} + \frac{y}{y^2} + \frac{z}{z^2}} \\
y = \frac{1}{\frac{y}{x^2} + \frac{1}{x} + \frac{y}{y^2}} \\
z = \frac{1}{\frac{z}{x^2} + \frac{z}{y^2} + \frac{1}{z}}
\]
Chapter 7

Cubes and prisms

7.1 The cube

7.2 Euler’s formula

7.3 The projections of a cube
Figure 7.2: The same cube shaded
The projections of a cube

Figure 7.3: The projections of a cube
The projections of a cube, second link to $\pi_3$.

Figure 7.4: The projections of a cube, second link to $\pi_3$. 
A cube section parallel to one of its faces

Figure 7.5: Cube section parallel to one of its faces
Figure 7.6: The development of a cube
Developing the surface of a cube, another solution

Figure 7.7: The development of a cube, another solution
Chapter 8

The pyramid

8.1 Definition

Figure 8.1: Defining a pyramid
CHAPTER 8. THE PYRAMID

8.2 Projections

8.3 Sections

8.3.1 Section parallel to basis

8.3.2 Oblique sections

8.4 Developing a pyramid

8.4.1 Regular pyramid

8.4.2 Truncated pyramid

8.5 Exercises
Pyramid sections parallel to their bases

Figure 8.3: Pyramid, sections parallel to base

Pyramid truncated parallel to their bases

Figure 8.4: Pyramid, sections parallel to base
A regular pyramid and its development

Figure 8.5: Developing a pyramid
Figure 8.6: Finding the true size of a face
Figure 8.7: Finding the true size of a face, second solution
Figure 8.8: Finding the true size of a face, second solution
CHAPTER 8. THE PYRAMID

Figure 8.9: Finding the true size of a face, third solution
Figure 8.10: Finding the true size of a face, fourth solution
Pyramid development, the fourth solution

Figure 8.11: Finding the true size of a face, fourth solution
Pyramid development, the fourth solution

Figure 8.12: Finding the true size of a face, fourth solution
Chapter 9

The cylinder

9.1 Definition

9.2 Point on the surface of a circular, right cylinder

9.2.1 The geometric aspect
Chapter 10

The helix

10.1 Introduction

Figure 10.1: The shortest path on a cylinder surface
Figure 10.1 shows two projections of a cylinder and of two points, \( P_1 \) and \( P_2 \), that lie on the cylinder surface. What is the curve along which we can travel the shortest distance between the two points? To solve the problem we can cut the cylinder, for example along the generator which passes through \( O \), and develop the surface. The developed surface is shown in the right-hand part of Figure 10.1. On the developed surface the points \( P_1 \) and \( P_2 \) appear as \( P_1d \) and \( P_2d \). The obvious solution is obtained by joining \( P_1d \) and \( P_2d \) by a straight line and bending back the developed surface into the given cylinder. The shortest path between \( P_1 \) and \( P_2 \) appears in in the frontal projection as the red curve that joins \( P_1'' \) and \( P_2'' \). The horizontal projection coincides with the arc of the base circle limited by the points \( P_1' \) and \( P_2' \). The curve we are looking for is a helix.

The helix has important applications in technology and science. The surfaces of screw threads and of many springs are generated by points lying on helices. So are parts of the surfaces of screw propellers and fans. In biology, the DNA model is based on two helices.

### 10.2 The definition of the helix

For the definition of the helix we refer to Figure 10.2. Given the axis \( Oz \), the helix is the curve described by a point, \( P \), that rotates around the axis \( Oz \) at constant radius, \( r \), and for each complete rotation advances in the direction \( Oz \) by a constant distance, \( p \), called pitch. From the above
10.2. THE DEFINITION OF THE HELIX

Figure 10.2: Defining a helix

definition we deduce the parametric equations of the helix

\[ x = r \cos 2\pi t \quad (10.1) \]
\[ y = r \sin 2\pi t \]
\[ z = tp \]

where \( t \) is a parameter that for one complete rotation runs from 0 to 1. In Figure 10.2 we see various stages of the generation of the helix.

If the generating point rotates clockwise and advances in the positive \( z \)
direction, we say that the helix is right-handed, otherwise we say that the helix is left-handed. Most screws used in technology are right-handed.

A few translations of the terms introduced in this section are

<table>
<thead>
<tr>
<th></th>
<th>helix</th>
<th>pitch</th>
<th>right-handed</th>
<th>left-handed</th>
</tr>
</thead>
<tbody>
<tr>
<td>French</td>
<td>hélice</td>
<td>pas</td>
<td>à droite</td>
<td>à gauche</td>
</tr>
<tr>
<td>Italian</td>
<td>elica</td>
<td>passo</td>
<td>destrorso</td>
<td>sinistrorso</td>
</tr>
<tr>
<td>German</td>
<td>Schraubenlinie</td>
<td>Ganghöhe</td>
<td>rechtsgewundene</td>
<td>linksgewundene</td>
</tr>
</tbody>
</table>

### 10.3 The projections of the helix

To draw the projections of a helix we proceed as in Figure 10.3. We first sketch the projections of the cylindrical surface on which the helix is defined. Next, in the horizontal projection (that is, on $\pi_1$) we divide the base circle into a number of equal arcs, in this example twelve. In the frontal projection (that is, on $\pi_2$) and side projection (on $\pi_3$) we divide the pitch height into the same number of equal segments. For good order we number the divisions. In this example the divisions are numbered from 0 to 12, turning on the base circle in the direct trigonometric sense. In Figure 10.3 corresponding points of the three projections are connected by blue, pointed lines.

Our helix starts from the first division marked 0. Therefore, in the frontal projection the first point is found at the intersection of the vertical, blue line rising from 0 in the $\pi_1$–plane and the horizontal, blue line starting
10.3. THE PROJECTIONS OF THE HELIX

from the first division (also marked 0) of the pitch in the $\pi_2$-plane. From here on, as the point rotates one twelfth of a circle, it also progresses in the $z$ direction one twelfth of the pitch height. Therefore, the second point in the frontal projection lies at the intersection of the vertical, blue line rising from 1 in the $\pi_1$ plane and the horizontal, blue line corresponding to the second pitch division (also marked 1). We proceed so until the thirteenth point, corresponding to the division marked 12. The side projection (that is, on the $\pi_3$ plane) is obtained in the obvious way from the horizontal and the frontal projections.

Connecting the points found above we obtain curves that must seem familiar to us: the graphs of a cosine and of a sine function. Below we show that this is, indeed, the case.

Figure 10.3: The projections of a helix

Figure 10.4 shows three projections of the helix defined by Equations 10.2
for $t = 0...3$. The horizontal projection is given by

$$
\begin{align*}
  x &= r \cos 2\pi t \\
  y &= r \sin 2\pi t
\end{align*}
$$

Dividing both sides of the first equation by $\cos 2\pi t$, both sides of the second equation by $\sin 2\pi t$, raising the resulting equations to the second power and adding them we obtain

$$
  x^2 + y^2 = r^2
$$
10.3. *THE PROJECTIONS OF THE HELIX*

This is the equation of a circle with the centre in the origin of the coordinates and the radius $r$.

The frontal projection, that is the projection on $\pi_2$, is given by the equations

\begin{align}
x & = r \cos 2\pi t \\
z & = tp
\end{align} \hspace{1cm} (10.5)

Extracting the value of $t$ from the second equation and substituting into the first equation yields

\begin{equation}
x = r \cos \frac{2\pi}{p} z
\end{equation} \hspace{1cm} (10.6)

As $r$ is a constant, and $2\pi/p$ another one, we deduce that the frontal projection of the helix is a cosine line.

The side projection, that is the projection on $\pi_3$, is given by the equations

\begin{align}
y & = r \sin 2\pi t \\
z & = tp
\end{align} \hspace{1cm} (10.7)

From the above equations we obtain

\begin{equation}
y = r \sin \frac{2\pi}{p} z
\end{equation} \hspace{1cm} (10.8)

We conclude that the side projection is a sine line.
In Figure 10.4 we see only the helix. Figure 10.5 shows a cylinder with a helix on it. The cylinder is opaque so that part of the helix is visible and part hidden. It is easy to check that the helix is right-handed.

### 10.4 Developing the helix

To develop the helix we begin by developing the cylindrical surface on which the helix lies. Thus, in Figure 10.6, the developed cylindrical surface is shown at the right of the sketch of the cylinder with base radius $r$, and height equal to the pitch $p$. Next, we divide the perimeter of the base, in the horizontal projection, into a number of equal arcs, in this example 12. We divide the length of the development into the same number of segments. Further, we divide the height, both in the frontal projection and in the
Preparing to develop a helix

developed surface, into the same number of segments. For good order we assign numbers to the various divisions. In this example we assign the numbers 0, 1, \ldots, 12 progressing in the direct trigonometric sense. If we consider the external surface of the surface, the numbers of the divisions of developed length are as shown in Figure 10.6.

The starting point corresponds to the division marked 0. Then, the first point of the developed helix is defined by the intersection of the horizontal, blue line marked 0 in the frontal projection, and the vertical, blue line marked 0 on the developed surface. As the point advances one division in the horizontal direction, it also rises one division in the vertical direction.
Therefore, the second point of the developed helix is defined by the
intersection of the horizontal, blue line marked 1 in the frontal projection,
and the vertical, blue line marked 1 on the developed surface. Continuing
in the same way we find the sequence of points shown in Figure 10.7. It is
easy to check that all points lie on a straight line.

We can prove analytically that the developed helix is, indeed, a straight
line. To do this we assume that the projections of the helix are described in
the same system of coordinates as in Figure 10.2 and in Equations 10.2,
while the developed helix is represented in a system in which the horizontal
coordinate is $\xi$, and the vertical coordinate is $\eta$. We immediately see that

$$\eta = z = tp$$  \hspace{1cm} (10.9)$$

The horizontal coordinate of a point on the developed helix is equal to the developed length of the arc of the horizontal projection of the same point. For a rotation angle equal to $2\pi t$, the arc length is $2\pi rt$, and

$$\xi = 2\pi rt$$  \hspace{1cm} (10.10)$$

Combining Equations 10.9 and 10.10 we obtain

$$\eta = \frac{p}{2\pi r} \xi$$  \hspace{1cm} (10.11)$$

that is, the equation of a straight line with the constant slope

$$\frac{d\eta}{d\xi} = \frac{p}{2\pi r}$$  \hspace{1cm} (10.12)$$

The knowledge of this slope is important in applications such as power screws and worm gears and according to its value the mechanism can be reversible or not. Moreover, the efficiency of the mechanism depends on the slope value.
10.5 The tangent to the helix

10.6 Technical applications

Appendix A

The MATLAB code that produced the fourth plot in Figure 10.2 is

```matlab
r = 1; % cylinder radius
p = 1; % helix pitch

subplot(2, 2, 4)
plot3([ 0 0 ], [ 0, 0 ], [ -0.5*p, 3.5*p ], 'k-.')
hold on
t = 0: 0.01: 3;
x = r*cos(2*pi*t);
y = r*sin(2*pi*t);
z = t*p;
plot3(x, y, z, 'k-')
axis([-3*r 3*r -3*r 3*r -p 4*p])
axis equal, axis off
Ht = title('t = 3');
set(Ht, 'FontSize', 14)
plot3([ r 2.8*r ], [ 0 0 ], [ 0 0 ], 'r-')
plot3([ r 2.8*r ], [ 0 0 ], [ 3*p 3*p ], 'r-')
```
The arrows used in dimensioning are added by using the Double Arrow facility of the figure toolbar.

10.7 Exercises

Exercise 1 - Developed helix

Develop the helix shown in Figure 10.4, cut the developed rectangle leaving an area for gluing and bend it into a cylinder. Turn the cylinder around to experiment the sensation of an advancing screw thread. Check that this thread is right-handed.
Chapter 11

The cone and the conic sections

11.1 Definition

Figure 11.1: Generating a circular, right cone
11.2 The projections of a cone

11.3 Point on conic section

11.3.1 Finding the horizontal projection using a section perpendicular to the cone axis

11.3.2 Finding the horizontal projection using a generator

11.4 Conic sections

11.4.1 The point

11.4.2 The circle

11.4.3 The ellipse

Figure 11.2: An elliptic, right cone
11.5. DEVELOPING A CONIC SURFACE

11.4.4 The parabola

11.4.5 The hyperbola

11.4.6 Straight-line sections

11.4.7 Dandelin’s spheres

11.5 Developing a conic surface

11.5.1 The circular, right cone

11.5.2 The truncated circular, right cone

11.6 The cone as a two-dimensional manifold

11.7 The isometric view of the cone

11.8 Quadrics

11.9 Appendix A - The parabola
Example of truncated cone, defining the problem

\[ V \]

\[ h \]

\( f \)

\[ h', f' \]

\[ \text{Figure 11.4: Truncating a circular, right cone} \]

%VACUUM_BALLISTICS PLOTS TRAJECTORY IN VACUUM

% define parameters

\( V_0 = 200; \) % initial velocity, m/s
\( g = 9.81; \) % acceleration of gravity, m/s^2
\( \alpha = \pi/4; \) % initial angle
\( t_{\text{max}} = 2*V_0*\sin(\alpha)/g; \) % time until falling on horizon
\( \text{a}_{\text{deg}} = 180*\alpha/\pi; \)
\( t = 0: 0.25: t_{\text{max}}; \)
Example of truncated cone, horizontal projection of intersection line

Figure 11.5: Horizontal projection of the intersection line

```matlab
x = V0*cos(alpha)*t;
y = V0*sin(alpha)*t - g*t.^2/2;
plot(x, y, 'k-') axis equal, grid
```

```matlab
title(['Vacuum ballistics, V_0 = ' num2str(V0) ' ms^{-1}, \alpha = ' num2str(adeg) '^o'])
xlabel('Horizontal distance, m')
ylabel('Height, m')
```
11.9.4 How to draw a parabola

11.10 Appendix B - The hyperbola

11.10.1 Equations

11.10.2 A few properties

11.10.3 Scientific and technical applications

11.10.4 How to draw a hyperbola
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The sphere

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(Rebatting)

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