INTRODUCTION TO CONTINUUM MECHANICS

ME 36003

Prof. M. B. Rubin

Faculty of Mechanical Engineering
Technion - Israel Institute of Technology

Fall 1991
Latest revision Spring 2006

These lecture notes are a modified version of notes developed by the late Professor P. M. Naghdi of the University of California, Berkeley.

Table Of Contents

1. Introduction

Continuum Mechanics is concerned with the fundamental equations that describe the nonlinear thermomechanical response of all deformable media. Although the theory is a phenomenological theory, which is proposed to model the macroscopic response of materials, it even is reasonably accurate for many studies of micro- and nano-mechanics where the typical length scales approach, but are still larger than, those of individual atoms. In this sense, the general thermomechanical theory provides a theoretical umbrella for most areas of study in mechanical engineering. In particular, continuum mechanics includes as special cases theories of: solids (elastic, plastic, viscoplastic, etc), fluids (compressible, incompressible, viscous) and the thermodynamics of heat conduction including dissipation due to viscous effects.

The material in this course on continuum mechanics is loosely divided into four parts. Part 1 includes sections 2-5 which develop a basic knowledge of tensor analysis using both indicial notation and direct notation. Although tensor operations in general curvilinear coordinates are needed to express spatial derivatives like those in the gradient and divergence operators, these special operations required to translate quantities in direct notation to component forms in special coordinate systems are merely mathematical in nature. Moreover, general curvilinear tensor analysis unnecessarily complicates the presentation of the fundamental physical issues in continuum mechanics. Consequently, here attention is restricted to tensors expressed in terms of constant rectangular Cartesian base vectors in order to simplify the discussion of spatial derivatives and concentrate on the main physical issues.

Part 2 includes sections 6-13 which develop tools to analyze nonlinear deformation and motion of continua. Specifically, measures of deformation and their rates are introduced. Also, the group of superposed rigid body motions (SRBM) is introduced for later fundamental analysis of invariance under SRBM.

Part 3 includes sections 14-23 which develop the balance laws that are applicable for general continua. The notion of the stress tensor and its relationship to the traction vector is developed. Local forms of the equations of motion are derived from the global forms of the balance laws. Referential forms of the equations of motion are discussed and the relationships between different stress measures are developed. Also, invariance under

SRBM of the balance laws and the kinetic quantities are discussed. Although attention is focused on the purely mechanical theory, the first law of thermodynamics is introduced to show the intimate relationship between the balance laws and invariance under SRBM.

Part 4 includes sections 24-29 which present an introduction to constitutive theory. Although there is general consensus on the kinematics of continua, the notion of constitutive equations for special materials remains an active area of research in continuum mechanics. Specifically, in these sections the theoretical structure of constitutive equations for nonlinear elastic solids, isotropic elastic solids, viscous and inviscid fluids and elastic-plastic solids are discussed.

2. Indicial Notation

In continuum mechanics it is necessary to use tensors and manipulate tensor equations. To this end it is desirable to use a language called indicial notation which develops simple rules governing these tensor manipulations. For the purposes of describing this language we introduce a set of right-handed orthonormal base vectors denoted by $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$. Although it is not our purpose here to review in detail the subject of linear vector spaces, we recall that vectors satisfy certain laws of addition and multiplication by a scalar. Specifically, if $\mathbf{a},\mathbf{b}$ are vectors then the quantity

$$\mathbf{c} = \mathbf{a} + \mathbf{b} \tag{2.1}$$

is a vector defined by the parallelogram law of addition. Furthermore, we recall that the operations

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \quad \text{(commutative law)} \ , \tag{2.2a}$$

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) \quad \text{(associative law)} \ , \tag{2.2b}$$

$$\alpha\, \mathbf{a} = \mathbf{a}\, \alpha \ \text{(multiplication by a real number)} \ , \tag{2.2c}$$

$$\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a} \quad \text{(commutative law)} \ , \tag{2.2d}$$

$$\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c} \quad \text{(distributive law)} \ , \tag{2.2e}$$

$$\alpha\, (\mathbf{a} \bullet \mathbf{b}) = (\alpha\, \mathbf{a}) \bullet \mathbf{b} \quad \text{(associative law)} \ , \tag{2.2f}$$

$$\mathbf{a} \times \mathbf{b} = -\, \mathbf{b} \times \mathbf{a} \quad \text{(lack of commutativity)} \ , \tag{2.2g}$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \quad \text{(distributive law)} \ , \tag{2.2h}$$

$$\alpha\, (\mathbf{a} \times \mathbf{b}) = (\alpha\, \mathbf{a}) \times \mathbf{b} \quad \text{(associative law)} \ , \tag{2.2i}$$

are satisfied for all vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ and all real numbers α, where $\mathbf{a} \bullet \mathbf{b}$ denotes the scalar product (or dot product) and $\mathbf{a} \times \mathbf{b}$ denotes the vector product (or cross product) between the vectors $\mathbf{a}$ and $\mathbf{b}$.

Quantities written in indicial notation will have a finite number of indices attached to them. Since the number of indices can be zero a quantity with no index can also be considered to be written in index notation. The language of index notation is quite simple because only two types of indices may appear in any term. Either the index is a free index or it is a repeated index. Also, we will define a simple summation convention which applies only to repeated indices. These two types of indices and the summation convention are defined as follows.

Free Indices: Indices that appear only once in a given term are known as free indices. For our purposes each of these free indices will take the values (1,2,3). For example, i is a free index in each of the following expressions

$$(x_1 , x_2 , x_3) = x_i \ \ (i=1,2,3) \ , \tag{2.3a}$$

$$(\mathbf{e}_1 , \mathbf{e}_2 , \mathbf{e}_3) = \mathbf{e}_i \ \ (i=1,2,3) \ . \tag{2.3b}$$

Repeated Indices: Indices that appear twice in a given term are known as repeated indices. For example i and j are free indices and m and n are repeated indices in the following expressions

$$a_i \, b_j \, c_m \, T_{mn} \, d_n \ \ , \ A_{immjnn} \ \ , \ A_{imn} \, B_{jmn} \ . \tag{2.4a,b,c}$$

It is important to emphasize that in the language of indicial notation an index can never appear more than twice in any term.

Einstein Summation Convention: When an index appears as a repeated index in a term, that index is understood to take on the values (1,2,3) and the resulting terms are summed. Thus, for example,

$$x_i \, \mathbf{e}_i = x_1 \, \mathbf{e}_1 + x_2 \, \mathbf{e}_2 + x_3 \, \mathbf{e}_3 \ . \tag{2.5}$$

Because of this summation convention, repeated indices are also known as dummy indices since their replacement by any other letter not appearing as a free index and also not appearing as another repeated index does not change the meaning of the term in which they occur. For examples,

$$x_i \, \mathbf{e}_i = x_j \, \mathbf{e}_j \ , \ a_i \, b_m c_m = a_i \, b_j \, c_j \ . \tag{2.6a,b}$$

It is important to emphasize that the same free indices must appear in each term in an equation so that for example the free index i in (2.6b) must appear on each side of the equality.

Kronecker Delta: The Kronecker delta symbol δ_{ij} is defined by

$$\delta_{ij} = \mathbf{e}_i \bullet \mathbf{e}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad . \tag{2.7}$$

Since the Kronecker delta δ_{ij} vanishes unless i=j it exhibits the following exchange property

$$\delta_{ij}\, x_j = (\, \delta_{1j}\, x_j \,,\, \delta_{2j}\, x_j \,,\, \delta_{3j}\, x_j \,) = (\, x_1 \,,\, x_2 \,,\, x_3 \,) = x_i \quad . \tag{2.8}$$

Notice that the Kronecker symbol may be removed by replacing the repeated index j in (2.8) by the free index i.

Recalling that an arbitrary vector **a** in Euclidean 3-Space may be expressed as a linear combination of the base vectors $\mathbf{e}_i$ such that

$$\mathbf{a} = a_i\, \mathbf{e}_i \quad , \tag{2.9}$$

it follows that the components a_i of **a** can be calculated using the Kronecker delta

$$a_i = \mathbf{e}_i \bullet \mathbf{a} = \mathbf{e}_i \bullet (a_m\, \mathbf{e}_m) = (\mathbf{e}_i \bullet \mathbf{e}_m)\, a_m = \delta_{im}\, a_m = a_i \quad . \tag{2.10}$$

Notice that when the expression (2.9) for **a** was substituted into (2.10) it was necessary to change the repeated index i in (2.9) to another letter (m) because the letter i already appeared in (2.10) as a free index. It also follows that the Kronecker delta may be used to calculate the dot product between two vectors **a** and **b** with components a_i and b_i, respectively by

$$\mathbf{a} \bullet \mathbf{b} = (a_i\, \mathbf{e}_i) \bullet (b_j\, \mathbf{e}_j) = a_i\, (\mathbf{e}_i \bullet \mathbf{e}_j)\, b_j = a_i\, \delta_{ij}\, b_j = a_i\, b_i \quad . \tag{2.11}$$

Permutation symbol: The permutation symbol ε_{ijk} is defined by

$$\varepsilon_{ijk} = \mathbf{e}_i \times \mathbf{e}_j \bullet \mathbf{e}_k = \begin{cases} 1 & \text{if (i,j,k) are an even permutation of (1,2,3)} \\ -1 & \text{if (i,j,k) are an odd permutation of (1,2,3)} \\ 0 & \text{if at least two of (i,j,k) have the same value} \end{cases} \tag{2.12}$$

From the definition (2.12) it appears that the permutation symbol can be used in calculating the vector product between two vectors. To this end, let us prove that

$$\mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk}\, \mathbf{e}_k \quad . \tag{2.13}$$

Proof: Since $\mathbf{e}_i \times \mathbf{e}_j$ is a vector in Euclidean 3-Space for each choice of the values of i and j it follows that it may be represented as a linear combination of the base vectors $\mathbf{e}_k$ such that

$$\mathbf{e}_i \times \mathbf{e}_j = A_{ijk}\, \mathbf{e}_k \quad , \tag{2.14}$$

where the components A_{ijk} need to be determined. In particular, by taking the dot product of (2.14) with $\mathbf{e}_k$ and using the definition (2.12) we obtain

$$\varepsilon_{ijk} = \mathbf{e}_i \times \mathbf{e}_j \bullet \mathbf{e}_k = A_{ijm}\, \mathbf{e}_m \bullet \mathbf{e}_k = A_{ijm}\, \delta_{mk} = A_{ijk} \quad , \tag{2.15}$$

which proves the result (2.13). Now using (2.13) it follows that the vector product between the vectors **a** and **b** may be represented in the form

$$\mathbf{a} \times \mathbf{b} = (a_i\, \mathbf{e}_i) \times (b_j\, \mathbf{e}_j) = (\mathbf{e}_i \times \mathbf{e}_j)\, a_i\, b_j = \varepsilon_{ijk}\, a_i\, b_j\, \mathbf{e}_k \quad . \tag{2.16}$$

<u>Contraction</u>: Contraction is the process of identifying two free indices in a given expression together with the implied summation convention. For example we may contract on the free indices i,j in δ_{ij} to obtain

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3 \quad . \tag{2.17}$$

Note that contraction on the set of $9=3^2$ quantities T_{ij} can be performed by multiplying T_{ij} by δ_{ij} to obtain

$$T_{ij}\, \delta_{ij} = T_{ii} \quad . \tag{2.18}$$

3. Tensors and Tensor Products

A scalar is sometimes referred to as a zero order tensor and a vector is sometimes referred to as a first order tensor. Here we define higher order tensors inductively starting with the notion of a first order tensor or vector.

Tensor of Order M: The quantity **T** is called a tensor of order M (M≥2) if it is a linear operator whose domain is the space of all vectors **v** and whose range **Tv** or **vT** is a tensor of order M–1. Since **T** is a linear operator it satisfies the following rules

$$\mathbf{T}(\mathbf{v} + \mathbf{w}) = \mathbf{Tv} + \mathbf{Tw} \quad , \tag{3.1a}$$

$$\alpha(\mathbf{Tv}) = (\alpha\mathbf{T})\mathbf{v} = \mathbf{T}(\alpha\mathbf{v}) \quad , \tag{3.1b}$$

$$(\mathbf{v} + \mathbf{w})\mathbf{T} = \mathbf{vT} + \mathbf{wT} \quad , \tag{3.1c}$$

$$\alpha(\mathbf{vT}) = (\alpha\mathbf{v})\mathbf{T} = (\mathbf{vT})\alpha \quad , \tag{3.1d}$$

where **v**,**w** are arbitrary vectors and α is an arbitrary real number. Notice that the tensor **T** may operate on its right [e.g. (3.1a,b)] or on its left [e.g. (3.1c,d)] and that in general operation on the right and the left is not commutative

$$\mathbf{Tv} \neq \mathbf{vT} \quad \text{(Lack of commutativity in general)} \ . \tag{3.2}$$

Zero Tensor of Order M: The zero tensor of order M is denoted by **0**(M) and is a linear operator whose domain is the space of all vectors **v** and whose range **0**(M–1) is the zero tensor of order M–1.

$$\mathbf{0}(\mathrm{M})\ \mathbf{v} = \mathbf{v}\ \mathbf{0}(\mathrm{M}) = \mathbf{0}(\mathrm{M}-1) \ . \tag{3.3}$$

Notice that these tensors are defined inductively starting with the known properties of the real number 0 which is the zero tensor **0**(0) of order 0.

Addition and Subtraction: The usual rules of addition and subtraction of two tensors **A** and **B** apply when the two tensors have the same order. We emphasize that tensors of different orders cannot be added or subtracted.

In order to define the operations of tensor product, dot product, and juxtaposition for general tensors it is convenient to first consider the definitions of these properties for the special case of the tensor product of a string of M (M≥2) vectors ($\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\ldots,\mathbf{a}_M$). Also, we will define the left and right transpose of the tensor product of a string of vectors.

Tensor Product (Special Case): The tensor product operation is denoted by the symbol ⊗ and it is defined so that the tensor product of a string of M (M≥1) vectors $(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,...,\mathbf{a}_M)$ is a tensor of order M having the following properties

$$(\mathbf{a}_1\otimes\mathbf{a}_2\otimes\mathbf{a}_3\otimes...\otimes\mathbf{a}_{M-1}\otimes\mathbf{a}_M)\,\mathbf{v} = (\mathbf{a}_M \bullet \mathbf{v})\,(\mathbf{a}_1\otimes\mathbf{a}_2\otimes\mathbf{a}_3\otimes...\otimes\mathbf{a}_{M-1}) \;, \tag{3.4a}$$

$$\mathbf{v}\,(\mathbf{a}_1\otimes\mathbf{a}_2\otimes\mathbf{a}_3\otimes...\otimes\mathbf{a}_{M-1}\otimes\mathbf{a}_M) = (\mathbf{v} \bullet \mathbf{a}_1)\,(\mathbf{a}_2\otimes\mathbf{a}_3\otimes...\otimes\mathbf{a}_M) \;, \tag{3.4b}$$

$$\alpha(\mathbf{a}_1\otimes\mathbf{a}_2\otimes...\otimes\mathbf{a}_M) = (\alpha\mathbf{a}_1\otimes\mathbf{a}_2\otimes...\otimes\mathbf{a}_M) = (\mathbf{a}_1\otimes\alpha\mathbf{a}_2\otimes...\otimes\mathbf{a}_M)$$
$$= ... = (\mathbf{a}_1\otimes\mathbf{a}_2\otimes...\otimes\alpha\mathbf{a}_M) = (\mathbf{a}_1\otimes\mathbf{a}_2\otimes...\otimes\mathbf{a}_M)\alpha \;, \tag{3.4c}$$

$$(\mathbf{a}_1\otimes\mathbf{a}_2\otimes\mathbf{a}_3\otimes...\otimes\mathbf{a}_{K-1}\otimes\{\mathbf{a}_K + \mathbf{w}\}\otimes\mathbf{a}_{K+1}\otimes...\otimes\mathbf{a}_{M-1}\otimes\mathbf{a}_M)$$
$$= (\mathbf{a}_1\otimes\mathbf{a}_2\otimes\mathbf{a}_3\otimes...\otimes\mathbf{a}_{K-1}\otimes\mathbf{a}_K\otimes\mathbf{a}_{K+1}\otimes...\otimes\mathbf{a}_{M-1}\otimes\mathbf{a}_M)$$
$$+ (\mathbf{a}_1\otimes\mathbf{a}_2\otimes\mathbf{a}_3\otimes...\otimes\mathbf{a}_{K-1}\otimes\mathbf{w}\otimes\mathbf{a}_{K+1}\otimes...\otimes\mathbf{a}_{M-1}\otimes\mathbf{a}_M)$$
$$\text{for } 1\le K\le M \;, \tag{3.4d}$$

where $\mathbf{v}$ and $\mathbf{w}$ are arbitrary vectors, the symbol (•) in (3.4) is the usual dot product between two vectors, and α is an arbitrary real number. It is important to note from (3.4a,b) that in general the order of the operation is not commutative. As specific examples we have

$$(\mathbf{a}_1\otimes\mathbf{a}_2)\,\mathbf{v} = (\mathbf{a}_2 \bullet \mathbf{v})\,\mathbf{a}_1 \;, \quad \mathbf{v}\,(\mathbf{a}_1\otimes\mathbf{a}_2) = (\mathbf{a}_1 \bullet \mathbf{v})\,\mathbf{a}_2 \;, \tag{3.5a,b}$$

Dot Product (Special Case): The dot product operation between two vectors may be generalized to an operation between any two tensors (including higher order tensors). Specifically, the dot product of the tensor product of a string of M vectors $(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,...,\mathbf{a}_M)$ with the tensor product of another string of N vectors $(\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3,...,\mathbf{b}_N)$ is a tensor of order |M–N| which is defined by

$$(\mathbf{a}_1\otimes\mathbf{a}_2\otimes\mathbf{a}_3\otimes...\otimes\mathbf{a}_M) \bullet (\mathbf{b}_1\otimes\mathbf{b}_2\otimes\mathbf{b}_3\otimes...\otimes\mathbf{b}_N)$$

$$= (\mathbf{a}_1\otimes\mathbf{a}_2\otimes...\otimes\mathbf{a}_{M-N})\left\{\prod_{K=1}^{N} (\mathbf{a}_{M-N+K} \bullet \mathbf{b}_K)\right\} \text{ (for } M>N) \;, \tag{3.6a}$$

$$(\mathbf{a}_1\otimes\mathbf{a}_2\otimes\mathbf{a}_3\otimes...\otimes\mathbf{a}_M) \bullet (\mathbf{b}_1\otimes\mathbf{b}_2\otimes\mathbf{b}_3\otimes...\otimes\mathbf{b}_N)$$

$$= \left\{ \prod_{K=1}^{M} (\mathbf{a}_K \bullet \mathbf{b}_K) \right\} \quad \text{(for M=N)} \ , \tag{3.6b}$$

$(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3 \otimes ... \otimes \mathbf{a}_M) \bullet (\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3 \otimes ... \otimes \mathbf{b}_N)$

$$= \left\{ \prod_{K=1}^{M} (\mathbf{a}_K \bullet \mathbf{b}_K) \right\} (\mathbf{b}_{M+1} \otimes \mathbf{b}_{M+2} \otimes ... \otimes \mathbf{b}_N) \ \text{(for M<N)} \ , \tag{3.6c}$$

where Π is the usual product operator indicating the product of the series of quantities defined by the values of K

$$\left\{ \prod_{K=1}^{N} (\mathbf{a}_K \bullet \mathbf{b}_K) \right\} = (\mathbf{a}_1 \bullet \mathbf{b}_1)(\mathbf{a}_2 \bullet \mathbf{b}_2)(\mathbf{a}_3 \bullet \mathbf{b}_3) \ ... \ (\mathbf{a}_N \bullet \mathbf{b}_N) \ . \tag{3.7}$$

Note from (3.6a,c) that if the orders of the tensors are not equal (M≠N) then the order of the dot product operator is <u>important</u>. However, when the orders of the tensors are equal (M=N) then the dot product operation yields a real number (3.6b) and the order of the operation is unimportant (i.e. the operation is commutative). For example,

$$(\mathbf{a}_1 \otimes \mathbf{a}_2) \bullet (\mathbf{b}_1 \otimes \mathbf{b}_2) = (\mathbf{a}_1 \bullet \mathbf{b}_1)\,(\mathbf{a}_2 \bullet \mathbf{b}_2) \ , \tag{3.8a}$$

$$(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3) \bullet (\mathbf{b}_1 \otimes \mathbf{b}_2) = \mathbf{a}_1\,(\mathbf{a}_2 \bullet \mathbf{b}_1)\,(\mathbf{a}_3 \bullet \mathbf{b}_2) \ , \tag{3.8b}$$

$$(\mathbf{a}_1 \otimes \mathbf{a}_2) \bullet (\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3) = (\mathbf{a}_1 \bullet \mathbf{b}_1)\,(\mathbf{a}_2 \bullet \mathbf{b}_2)\,\mathbf{b}_3 \ . \tag{3.8c}$$

<u>Cross Product (Special Case)</u>: The cross product of the tensor product of a string of M vectors $(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,...,\mathbf{a}_M)$ with the tensor product of another string of N vectors $(\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3,...,\mathbf{b}_N)$ is a tensor of order M if M≥N and of order N if N≥M, which is defined by

$(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3 \otimes ... \otimes \mathbf{a}_M) \times (\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3 \otimes ... \otimes \mathbf{b}_N)$

$$= (\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes ... \otimes \mathbf{a}_{M-N}) \left\{ \prod_{K=1}^{N} \otimes (\mathbf{a}_{M-N+K} \times \mathbf{b}_K) \right\} \quad \text{(for M>N)} \ , \tag{3.9a}$$

$(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3 \otimes ... \otimes \mathbf{a}_M) \times (\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3 \otimes ... \otimes \mathbf{b}_N)$

$$= (\mathbf{a}_1 \times \mathbf{b}_1) \otimes \left\{ \prod_{K=2}^{M} (\mathbf{a}_K \times \mathbf{b}_K) \right\} \quad \text{(for M=N)} \ , \tag{3.9b}$$

$(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3 \otimes ... \otimes \mathbf{a}_M) \times (\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3 \otimes ... \otimes \mathbf{b}_N)$

$$= \left\{ \prod_{K=1}^{M} (\mathbf{a}_K \times \mathbf{b}_K) \otimes \right\} (\mathbf{b}_{M+1} \otimes \mathbf{b}_{M+2} \otimes ... \otimes \mathbf{b}_N) \quad \text{(for M<N)} \ , \tag{3.9c}$$

Note from (3.9) that the order of the cross product operation is <u>important</u>. For examples we have

$$(\mathbf{a}_1 \otimes \mathbf{a}_2) \times (\mathbf{b}_1 \otimes \mathbf{b}_2) = (\mathbf{a}_1 \times \mathbf{b}_1) \otimes (\mathbf{a}_2 \times \mathbf{b}_2) \ , \tag{3.10a}$$

$$(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3) \times (\mathbf{b}_1 \otimes \mathbf{b}_2) = \mathbf{a}_1 \otimes (\mathbf{a}_2 \times \mathbf{b}_1) \otimes (\mathbf{a}_3 \times \mathbf{b}_2) \ , \tag{3.10b}$$

$$(\mathbf{a}_1 \otimes \mathbf{a}_2) \times (\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3) = (\mathbf{a}_1 \times \mathbf{b}_1) \otimes (\mathbf{a}_2 \times \mathbf{b}_2) \otimes \mathbf{b}_3 \ . \tag{3.10c}$$

<u>Juxtaposition (Special Case)</u>: The operation of juxtaposition of the tensor product of a string of M (M≥1) vectors $(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,...,\mathbf{a}_M)$ with another string of N (N≥1) vectors $(\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3,...,\mathbf{b}_N)$ is a tensor of order M+N–2 which is defined by

$(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3 \otimes ... \otimes \mathbf{a}_M)(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3 \otimes ... \otimes \mathbf{b}_N)$

$$= (\mathbf{a}_M \bullet \mathbf{b}_1) \ (\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3 \otimes ... \otimes \mathbf{a}_{M-1} \otimes \mathbf{b}_2 \otimes \mathbf{b}_3 \otimes ... \otimes \mathbf{b}_N) \ . \tag{3.11}$$

It is obvious from (3.7) that the order of the operation juxtaposition is important. For example,

$$\mathbf{a}_1 \mathbf{b}_1 = \mathbf{a}_1 \bullet \mathbf{b}_1 \ , \tag{3.12a}$$

$$(\mathbf{a}_1 \otimes \mathbf{a}_2)(\mathbf{b}_1 \otimes \mathbf{b}_2) = (\mathbf{a}_2 \bullet \mathbf{b}_1) \ (\mathbf{a}_1 \otimes \mathbf{b}_2) \ . \tag{3.12b}$$

Note from (3.11a) that the juxtaposition of a vector with another vector is the same as the dot product of the two vectors. In spite of this fact we will usually express the dot product between two vectors explicitly.

<u>Transpose (Special Case)</u>: The left transpose of order N of the tensor product of a string of M (M≥2N) vectors is denoted by a superscript LT(N) on the left-hand side of the string of vectors and is defined by

$${}^{LT(N)}[\ (\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes ... \otimes \mathbf{a}_N) \otimes (\mathbf{a}_{N+1} \otimes \mathbf{a}_{N+2} \otimes ... \otimes \mathbf{a}_{2N})\ ]$$

$$\otimes(\mathbf{a}_{2N+1}\otimes\mathbf{a}_{2N+2}\ldots\otimes\mathbf{a}_M)$$

$$= [\,(\mathbf{a}_{N+1}\otimes\mathbf{a}_{N+2}\otimes\ldots\otimes\mathbf{a}_{2N})\otimes(\mathbf{a}_1\otimes\mathbf{a}_2\otimes\ldots\otimes\mathbf{a}_N)\,]$$

$$\otimes(\mathbf{a}_{2N+1}\otimes\mathbf{a}_{2N+2}\ldots\otimes\mathbf{a}_M) \quad \text{for } M \geq 2N \tag{3.13}$$

Similarly, the right transpose of order N of the tensor product of a string of M ($M \geq 2N$) vectors is denoted by a superscript T(N) on the right-hand side of the string of vectors and is defined by

$$(\mathbf{a}_1\otimes\mathbf{a}_2\otimes\ldots\otimes\mathbf{a}_{M-2N})$$

$$\otimes[\,(\mathbf{a}_{M-2N+1}\otimes\mathbf{a}_{M-2N+2}\otimes\ldots\otimes\mathbf{a}_{M-N})\otimes(\mathbf{a}_{M-N+1}\otimes\mathbf{a}_{M-N+2}\ldots\otimes\mathbf{a}_M)\,]^{T(N)}$$

$$= (\mathbf{a}_1\otimes\mathbf{a}_2\otimes\ldots\otimes\mathbf{a}_{M-2N})$$

$$\otimes[\,(\mathbf{a}_{M-N+1}\otimes\mathbf{a}_{M-N+2}\ldots\otimes\mathbf{a}_M)\otimes(\mathbf{a}_{M-2N+1}\otimes\mathbf{a}_{M-2N+2}\otimes\ldots\otimes\mathbf{a}_{M-N})]$$

$$\text{for } M \geq 2N \tag{3.14}$$

The notation T(N) is used for the right transpose instead of the more cumbersome notation RT(N) because the right transpose is used most frequently in tensor manipulations. Similarly, for simplicity the left transpose of order 1 will merely be denoted by a superscript LT and the right transpose of order 1 will be denoted by a superscript T so that

$$^{LT}(\mathbf{a}_1\otimes\mathbf{a}_2)\otimes(\mathbf{a}_3\otimes\mathbf{a}_4\otimes\ldots\otimes\mathbf{a}_M) = (\mathbf{a}_2\otimes\mathbf{a}_1)\otimes(\mathbf{a}_3\otimes\mathbf{a}_4\otimes\ldots\otimes\mathbf{a}_M) \;, \tag{3.15a}$$

$$(\mathbf{a}_1\otimes\mathbf{a}_2\otimes\ldots\mathbf{a}_{M-2})\otimes(\mathbf{a}_{M-1}\otimes\mathbf{a}_M)^T = (\mathbf{a}_1\otimes\mathbf{a}_2\otimes\ldots\mathbf{a}_{M-2})\otimes(\mathbf{a}_M\otimes\mathbf{a}_{M-1}) \,. \tag{3.15b}$$

For example,

$$^{LT}(\mathbf{a}_1\otimes\mathbf{a}_2)\otimes\mathbf{a}_3 = (\mathbf{a}_2\otimes\mathbf{a}_1)\otimes\mathbf{a}_3 \;,\quad \mathbf{a}_1\otimes(\mathbf{a}_2\otimes\mathbf{a}_3)^T = \mathbf{a}_1\otimes(\mathbf{a}_3\otimes\mathbf{a}_2) \;, \tag{3.16a,b}$$

$$^{LT(2)}[\,(\mathbf{a}_1\otimes\mathbf{a}_2)\otimes(\mathbf{a}_3\otimes\mathbf{a}_4)\,] = (\mathbf{a}_3\otimes\mathbf{a}_4)\otimes(\mathbf{a}_1\otimes\mathbf{a}_2) \;, \tag{3.16c}$$

$$[\,(\mathbf{a}_1\otimes\mathbf{a}_2)\otimes(\mathbf{a}_3\otimes\mathbf{a}_4)\,]^{T(2)} = (\mathbf{a}_3\otimes\mathbf{a}_4)\otimes(\mathbf{a}_1\otimes\mathbf{a}_2) \;. \tag{3.16d}$$

From (3.16c,d) it can be seen that the right and left transposes of order 2 of the tensor product of a string of vectors of order 4 (2×2) are equal. In general the right and left transposes of order N of the tensor product of a string of vectors of order 2N are equal so that

$$^{LT(N)}[\,(\mathbf{a}_1\otimes\mathbf{a}_2\otimes\ldots\otimes\mathbf{a}_N)\otimes(\mathbf{a}_{N+1}\otimes\mathbf{a}_{N+2}\otimes\ldots\otimes\mathbf{a}_{2N})\,]$$

$$= [\ (\mathbf{a}_{N+1}\otimes\mathbf{a}_{N+2}\otimes\ldots\otimes\mathbf{a}_{2N})\otimes(\mathbf{a}_1\otimes\mathbf{a}_2\otimes\ldots\otimes\mathbf{a}_N)\ ]$$

$$= [\ (\mathbf{a}_1\otimes\mathbf{a}_2\otimes\ldots\otimes\mathbf{a}_N)\otimes(\mathbf{a}_{N+1}\otimes\mathbf{a}_{N+2}\otimes\ldots\otimes\mathbf{a}_{2N})\ ]^{T(N)}\ . \qquad (3.17)$$

Using the above definitions we are in a position to define the base tensors and components of tensors of any order on a Euclidean 3-space. To this end we recall that $\mathbf{e}_i$ are the orthonormal base vectors of a right-handed rectangular Cartesian coordinate system. It follows that $\mathbf{e}_i$ span the space of vectors.

Base Tensors: It also follows inductively that the tensor product of the string of M vectors

$$(\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\otimes\ldots\otimes\mathbf{e}_r\otimes\mathbf{e}_s\otimes\mathbf{e}_t)\ , \qquad (3.18)$$

with M free indices (i,j,k,...,r,s,t) are base tensors for all tensors of order M. This is because when (3.18) is in juxtaposition with an arbitrary vector **v** it yields scalar multiples of the base tensors of all tensors of order M–1, such that

$$(\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\otimes\ldots\otimes\mathbf{e}_r\otimes\mathbf{e}_s\otimes\mathbf{e}_t)\ \mathbf{v} = (\mathbf{e}_t \bullet \mathbf{v})\ (\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\otimes\ldots\otimes\mathbf{e}_r\otimes\mathbf{e}_s)\ , \qquad (3.19a)$$

$$\mathbf{v}\ (\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\otimes\ldots\otimes\mathbf{e}_r\otimes\mathbf{e}_s\otimes\mathbf{e}_t) = (\mathbf{e}_i \bullet \mathbf{v})\ (\mathbf{e}_j\otimes\mathbf{e}_k\otimes\ldots\otimes\mathbf{e}_r\otimes\mathbf{e}_s\otimes\mathbf{e}_t)\ . \qquad (3.19b)$$

Components of an Arbitrary Tensor: By definition the base tensors (3.18) span the space of tensors of order M so an arbitrary tensor **T** of order M may be expressed as a linear combination of the base tensors such that

$$\mathbf{T} = T_{ijk\ldots rst}\ (\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\otimes\ldots\otimes\mathbf{e}_r\otimes\mathbf{e}_s\otimes\mathbf{e}_t)\ , \qquad (3.20)$$

where the coefficients $T_{ijk\ldots rst}$ in (3.20) are the components of **T** relative to the coordinate system defined by the base vectors $\mathbf{e}_i$ and the summation convention is used over repeated indices in (3.20). Using the above operations these components may be calculated by

$$T_{ijk\ldots rst} = \mathbf{T} \bullet (\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\otimes\ldots\otimes\mathbf{e}_r\otimes\mathbf{e}_s\otimes\mathbf{e}_t)\ . \qquad (3.21)$$

Notice that the components of the tensor **T** are obtained by taking the dot product of the tensor with the base tensors of the space defining the order of the tensor, just as is the case for vectors (tensors of order one).

Tensor Product (General Case): Let **A** be a tensor of order M with components $A_{ij...mn}$ and let **B** be a tensor of order N with components $B_{rs...vw}$ then the tensor product of **A** and **B**

$$\mathbf{A}\otimes\mathbf{B} = A_{ij...mn}\, B_{rs...vw}\, (\, \mathbf{e}_i\otimes\mathbf{e}_j\otimes...\otimes\mathbf{e}_m\otimes\mathbf{e}_n\otimes\mathbf{e}_r\otimes\mathbf{e}_s\otimes...\mathbf{e}_v\otimes\mathbf{e}_w\,) \;, \tag{3.22}$$

is a tensor of order (M+N).

Dot Product (General Case): The dot product **A • B** of a tensor **A** of order M with a tensor **B** of order N is a tensor of order |M–N|. As examples let **A** and **B** be second order tensors with components A_{ij} and B_{ij} and let **C** be a fourth order tensor with components C_{ijkl}, then we have

$$\mathbf{A} \bullet \mathbf{B} = \mathbf{B} \bullet \mathbf{A} = A_{ij}\, B_{ij} \;, \tag{3.23a}$$

$$\mathbf{A} \bullet \mathbf{C} = A_{ij}\, C_{ijkl}\, \mathbf{e}_k \otimes \mathbf{e}_l \;, \quad \mathbf{C} \bullet \mathbf{A} = C_{ijkl}\, A_{kl}\, \mathbf{e}_i \otimes \mathbf{e}_j \;, \tag{3.23b,c}$$

$$\mathbf{A} \bullet \mathbf{C} \neq \mathbf{C} \bullet \mathbf{A} \;. \tag{3.23d}$$

Cross Product (General Case): The cross product **A** × **B** of a tensor **A** of order M with a tensor **B** of order N is a tensor of order M if M≥N and of order N if N≥M. As examples let **v** be a vector with components v_i and **A** and **B** be second order tensors with components A_{ir} and B_{js}. Then we have

$$\mathbf{A} \times \mathbf{v} = A_{ir}\, v_s\, \mathbf{e}_i\otimes(\mathbf{e}_r \times \mathbf{e}_s) = \varepsilon_{rst}\, A_{ir}\, v_s\, (\mathbf{e}_i\otimes\mathbf{e}_t) \;, \tag{3.24a}$$

$$\mathbf{v} \times \mathbf{A} = v_s\, A_{ir}\, (\mathbf{e}_s \times \mathbf{e}_i)\otimes\mathbf{e}_r = \varepsilon_{sit}\, v_s\, A_{ir}\, (\mathbf{e}_t\otimes\mathbf{e}_r) \;, \tag{3.24b}$$

$$\mathbf{A} \times \mathbf{B} = A_{ir}\, B_{js}\, (\mathbf{e}_i \times \mathbf{e}_j)\otimes(\mathbf{e}_r \times \mathbf{e}_s) = \varepsilon_{ijk}\varepsilon_{rst}\, A_{ir}\, B_{js}\, \mathbf{e}_k\otimes\mathbf{e}_t \;, \tag{3.24c}$$

$$\mathbf{B} \times \mathbf{A} = B_{js}\, A_{ir}\, (\mathbf{e}_j \times \mathbf{e}_i)\otimes(\mathbf{e}_s \times \mathbf{e}_r) = \varepsilon_{ijk}\varepsilon_{rst}\, A_{ir}\, B_{js}\, \mathbf{e}_k\otimes\mathbf{e}_t \;, \tag{3.24d}$$

$$\mathbf{A} \times \mathbf{v} \neq \mathbf{v} \times \mathbf{A} \;, \quad \mathbf{A} \times \mathbf{B} = \mathbf{B} \times \mathbf{A} \;. \tag{3.24e,f}$$

Note that in general the cross product operation is not commutative. However, from (3.24f) we observe that the cross product of two second order tensors is commutative.

Juxtaposition (General Case): Let **A** be a tensor of order M with components $A_{ij...mn}$ and **B** be a tensor of order N with components $B_{rs...vw}$. Then juxtaposition of **A** with **B** is denoted by

$$\mathbf{A}\,\mathbf{B} = A_{ij...mn}\, B_{rs...vw}\, (\mathbf{e}_i \otimes \mathbf{e}_j \otimes ... \otimes \mathbf{e}_m \otimes \mathbf{e}_n)\, (\mathbf{e}_r \otimes \mathbf{e}_s \otimes ... \mathbf{e}_v \otimes \mathbf{e}_w) \quad ,$$

$$= A_{ij...mn}\, B_{rs...vw}\, (\mathbf{e}_n \bullet \mathbf{e}_r)\, (\mathbf{e}_i \otimes \mathbf{e}_j \otimes ... \otimes \mathbf{e}_m \otimes \mathbf{e}_s \otimes ... \mathbf{e}_v \otimes \mathbf{e}_w) \quad ,$$

$$= A_{ij...mn}\, B_{rs...vw}\, \delta_{nr}\, (\mathbf{e}_i \otimes \mathbf{e}_j \otimes ... \otimes \mathbf{e}_m \otimes \mathbf{e}_s \otimes ... \mathbf{e}_v \otimes \mathbf{e}_w) \quad ,$$

$$= A_{ij...mn}\, B_{ns...vw}\, (\mathbf{e}_i \otimes \mathbf{e}_j \otimes ... \otimes \mathbf{e}_m \otimes \mathbf{e}_s \otimes ... \mathbf{e}_v \otimes \mathbf{e}_w)\,, \tag{3.25}$$

and is a tensor of order (M+N–2). Note that the juxtaposition of a tensor with a vector is the same as the dot product of the tensor with the vector.

Transpose of a Tensor: Let $\mathbf{T}$ be a tensor of order M with components $T_{ijkl...rstu}$ relative to the base vectors $\mathbf{e}_i$. Then with the help of (3.11)-(3.14) we define the Nth order (2N≤M) left transpose ${}^{LT(N)}\mathbf{T}$ and right transpose $\mathbf{T}^{T(N)}$ of $\mathbf{T}$. For example

$$\mathbf{T} = T_{ijkl...rstu}\, (\mathbf{e}_i \otimes \mathbf{e}_j) \otimes (\mathbf{e}_k \otimes \mathbf{e}_l) \otimes ... \otimes (\mathbf{e}_r \otimes \mathbf{e}_s) \otimes (\mathbf{e}_t \otimes \mathbf{e}_u) \quad , \tag{3.26a}$$

$${}^{LT}\mathbf{T} = T_{ijkl...rstu}\, (\mathbf{e}_j \otimes \mathbf{e}_i) \otimes (\mathbf{e}_k \otimes \mathbf{e}_l) \otimes ... \otimes (\mathbf{e}_r \otimes \mathbf{e}_s) \otimes (\mathbf{e}_t \otimes \mathbf{e}_u) \quad , \tag{3.26b}$$

$$\mathbf{T}^T = T_{ijkl...rstu}\, (\mathbf{e}_i \otimes \mathbf{e}_j) \otimes (\mathbf{e}_k \otimes \mathbf{e}_l) \otimes ... \otimes (\mathbf{e}_r \otimes \mathbf{e}_s) \otimes (\mathbf{e}_u \otimes \mathbf{e}_t) \quad , \tag{3.26c}$$

$${}^{LT(2)}\mathbf{T} = T_{ijkl...rstu}\, (\mathbf{e}_k \otimes \mathbf{e}_l) \otimes (\mathbf{e}_i \otimes \mathbf{e}_j) \otimes ... \otimes (\mathbf{e}_r \otimes \mathbf{e}_s) \otimes (\mathbf{e}_t \otimes \mathbf{e}_u) \quad , \tag{3.26d}$$

$$\mathbf{T}^{T(2)} = T_{ijkl...rstu}\, (\mathbf{e}_i \otimes \mathbf{e}_j) \otimes (\mathbf{e}_k \otimes \mathbf{e}_l) \otimes ... \otimes (\mathbf{e}_t \otimes \mathbf{e}_u) \otimes (\mathbf{e}_r \otimes \mathbf{e}_s) \quad , \tag{3.26e}$$

where we recall that the superscripts LT and T in (3.26b,c) stand for the left and right transpose of order 1. In particular note that the transpose operation does not change the order of the indices of the components of the tensor but merely changes the order of the base vectors. To see this more clearly let $\mathbf{T}$ be a second order tensor with components T_{ij} so that

$$\mathbf{T} = T_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j \;\; , \;\; \mathbf{T}^T = T_{ij}\, \mathbf{e}_j \otimes \mathbf{e}_i = {}^{LT}\mathbf{T}\,, \tag{3.27a,b}$$

It follows that for an arbitrary vector $\mathbf{v}$ we may deduce that

$$\mathbf{T}\,\mathbf{v} = \mathbf{v}\,\mathbf{T}^T \;\; , \;\; \mathbf{T}^T\,\mathbf{v} = \mathbf{v}\,\mathbf{T} \quad . \tag{3.28a,b}$$

Also, we note that the separate notation for the left transpose has been introduced to avoid confusion in interpreting an expression of the type $\mathbf{A}^T\mathbf{B}$ which is not equal to $\mathbf{A}^{LT}\mathbf{B}$.

Identity Tensor of Order 2M: The identity tensor of order 2M (M≥1) is denoted by $\mathbf{I}$(2M) and is a tensor that has the property that the dot product of $\mathbf{I}$(2M) with an arbitrary tensor $\mathbf{A}$ of order M yields the result $\mathbf{A}$, such that

$$\mathbf{I}(2M) \bullet \mathbf{A} = \mathbf{A} \bullet \mathbf{I}(2M) = \mathbf{A} \ . \tag{3.29}$$

Letting $\mathbf{e}_i \otimes \mathbf{e}_j \otimes ... \otimes \mathbf{e}_s \otimes \mathbf{e}_t$ be the base tensors of order M we may represent $\mathbf{I}$ in the form

$$\mathbf{I}(2M) = (\mathbf{e}_i \otimes \mathbf{e}_j \otimes ... \otimes \mathbf{e}_s \otimes \mathbf{e}_t) \otimes (\mathbf{e}_i \otimes \mathbf{e}_j \otimes ... \otimes \mathbf{e}_s \otimes \mathbf{e}_t) \ , \tag{3.30}$$

where we emphasize that summation over repeated indices is implied in (3.30). Since the second order identity tensor appears often in continuum mechanics it is convenient to denote it by $\mathbf{I}$. In view of (3.30) it follows that the second order identity $\mathbf{I}$ may be represented by

$$\mathbf{I} = \mathbf{e}_i \otimes \mathbf{e}_i \ . \tag{3.31}$$

Using (2.7) and (3.31) it may be shown that the components of the second order identity tensor are represented by the Kronecker delta symbol so that

$$\mathbf{I} \bullet (\mathbf{e}_i \otimes \mathbf{e}_j) = \delta_{ij} \ . \tag{3.32}$$

Zero Tensor of Order M: Since all components of the zero tensor of order M are 0 and since the order of the tensors in a given equation will usually be obvious from the context we will use the symbol 0 to denote the zero tensor of any order.

Lack of Commutativity: Note that in general, the operations of tensor product, dot product, cross product and juxtaposition are not commutative so the order of these operations must be preserved. Specifically, it follows that

$$\mathbf{A} \otimes \mathbf{B} \neq \mathbf{B} \otimes \mathbf{A} \ , \ \mathbf{A} \bullet \mathbf{B} \neq \mathbf{B} \bullet \mathbf{A} \ , \ \mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A} \ , \ \mathbf{A}\,\mathbf{B} \neq \mathbf{B}\,\mathbf{A} \ . \tag{3.33a,b,c,d}$$

Permutation Tensor: The permutation tensor $\boldsymbol{\varepsilon}$ is a third order tensor that may be defined such that for any two vectors $\mathbf{a}$ and $\mathbf{b}$ we have

$$(\mathbf{a} \otimes \mathbf{b}) \bullet \boldsymbol{\varepsilon} = \mathbf{a} \times \mathbf{b} \ . \tag{3.34}$$

Using (2.12) and (3.34) it may be shown that the components of the permutation tensor $\boldsymbol{\varepsilon}$ may be represented by the permutation symbol such that

$$\boldsymbol{\varepsilon} \bullet (\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k) = \varepsilon_{ijk} \ . \tag{3.35}$$

It also follows that

$$\boldsymbol{\varepsilon} \bullet (\mathbf{a} \otimes \mathbf{b}) = \mathbf{a} \times \mathbf{b} \ . \tag{3.36}$$

Hierarchy of Tensor Operations: To simplify the notation and reduce the need for using parentheses to clarify mathematical equations it is convenient to define the hierarchy of the tensor operations according to Table 3.1 with level 1 operations being performed before level 2 operations and so forth. Also, as is usual, the order in which operations in the same level are performed is determined by which operation appears in the most left-hand position in the equation.

Level	Tensor Operation
1	Left Transpose (LT) and Right Transpose (T)
2	Juxtaposition and Tensor product (⊗)
3	Cross product (×)
4	Dot product (•)
5	Addition and Subtraction

Table 3.1 Hierarchy of tensor operations

Gradient: Let x_i be the components of the position vector $\mathbf{x}$ associated with the rectangular Cartesian base vectors $\mathbf{e}_i$. The gradient of a scalar function f with respect to the position $\mathbf{x}$ is a vector denoted by grad f and represented by

$$\text{grad } f = \nabla f = \partial f/\partial \mathbf{x} = \partial f/\partial x_m \, \mathbf{e}_m = f_{,m} \, \mathbf{e}_m \ , \qquad (3.37)$$

where for convenience a comma is used to denote partial differentiation. Also, the gradient of a tensor function $\mathbf{T}$ of order M (M≥1) is a tensor of order M+1 denoted by grad $\mathbf{T}$ and represented by

$$\text{grad } \mathbf{T} = \partial \mathbf{T}/\partial \mathbf{x} = \partial \mathbf{T}/\partial x_m \otimes \mathbf{e}_m = \mathbf{T}_{,m} \otimes \mathbf{e}_m \ . \qquad (3.38)$$

Note that we write the derivative $\partial \mathbf{T}/\partial \mathbf{x}$ on the same line to indicate the order of the quantities. To see the importance of this, let $\mathbf{T}$ be a second order tensor with components T_{ij} so that

$$\text{grad } \mathbf{T} = \partial \mathbf{T}/\partial \mathbf{x} = \partial[T_{ij} \, \mathbf{e}_i \otimes \mathbf{e}_j]/\partial x_m \otimes \mathbf{e}_m = T_{ij,m} \, \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_m \ . \qquad (3.39)$$

Divergence: The divergence of a tensor **T** of order M (M≥1) is a tensor of order M–1 denoted by div **T** and represented by

$$\text{div}\,\mathbf{T} = \frac{\partial \mathbf{T}}{\partial x_k} \bullet \mathbf{e}_k. \tag{3.40}$$

For example if **T** is a second order tensor then from (3.31),(3.39) and (3.40) we have

$$\text{div}\,\mathbf{T} = T_{ij,j}\,\mathbf{e}_i \;. \tag{3.41}$$

Curl: The curl of a vector **v** with components v_i is a vector denoted by curl **v** and represented by

$$\text{curl}\,\mathbf{v} = -\frac{\partial \mathbf{v}}{\partial x_j} \times \mathbf{e}_j = -\,v_{i,j}\,\varepsilon_{ijk}\,\mathbf{e}_k = v_{i,j}\,\varepsilon_{jik}\,\mathbf{e}_k \;. \tag{3.42}$$

Also, the curl of a tensor **T** of order M (M≥1) is a tensor of order M denoted by curl **T** and represented by

$$\text{curl}\,\mathbf{T} = -\frac{\partial \mathbf{T}}{\partial x_k} \times \mathbf{e}_k \;. \tag{3.43}$$

For example, if **T** is a second order tensor with components T_{ij} then

$$\text{curl}\,\mathbf{T} = -\,T_{ij,k}\,\varepsilon_{jkm}\,\mathbf{e}_i \otimes \mathbf{e}_m \,. \tag{3.44}$$

Laplacian: The Laplacian of a tensor **T** of order M is a tensor of order M denoted by $\nabla^2\mathbf{T}$ and represented by

$$\nabla^2\mathbf{T} = \text{div}\,(\,\text{grad}\,\mathbf{T}\,) = [\,\mathbf{T}_{,i} \otimes \mathbf{e}_i\,]_{,j} \bullet \mathbf{e}_j = \mathbf{T}_{,mm} \;. \tag{3.45}$$

Divergence Theorem: Let **n** be the unit outward normal to a surface ∂P of a region P, da be the element of area of ∂P, dv be the element of volume of P, and **T** be an arbitrary tensor of any order. Then the divergence theorem states that

$$\int_{\partial P} \mathbf{T}\,\mathbf{n}\,da = \int_P \text{div}\,\mathbf{T}\,dv \;. \tag{3.46}$$

4. Additional Definitions and Results

In order to better understand this definition of juxtaposition and in order to connect this definition with the usual rules for matrix multiplication let **A**, **B**, **C** be second order tensors with components A_{ij}, B_{ij}, C_{ij}, respectively, and define **C** by

$$\mathbf{C} = \mathbf{A}\mathbf{B} \ . \tag{4.1}$$

Using the representation (3.18) for each of these tensors it follows that

$$\mathbf{C} = A_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j\, B_{mn}\, \mathbf{e}_m \otimes \mathbf{e}_n = A_{ij}\, B_{mn}\, (\mathbf{e}_j \bullet \mathbf{e}_m)\, \mathbf{e}_i \otimes \mathbf{e}_n = A_{im}\, B_{mn}\, \mathbf{e}_i \otimes \mathbf{e}_n \ , \tag{4.2a}$$

$$C_{ij} = \mathbf{C} \bullet \mathbf{e}_i \otimes \mathbf{e}_j = A_{rm}\, B_{mn}\, (\mathbf{e}_r \otimes \mathbf{e}_n) \bullet (\mathbf{e}_i \otimes \mathbf{e}_j) = A_{im}\, B_{mj} \ . \tag{4.2b}$$

Examination of the result (4.2b) indicates that the second index of **A** is summed with the first index of **B** which is consistent with the usual operation of row times column inherent in the definition of matrix multiplication.

<u>Symmetric</u>: The second order tensor **A** with the $9=3^2$ components A_{ij} referred to the base vectors $\mathbf{e}_i$ is said to be symmetric if

$$\mathbf{A} = \mathbf{A}^T \ , \ A_{ij} = A_{ji} \ . \tag{4.3a,b}$$

It follows from (3.25) that if **A** is symmetric and **v** is an arbitrary vector with components v_i then

$$\mathbf{A}\,\mathbf{v} = \mathbf{v}\,\mathbf{A} \ , \ A_{ij}\, v_j = v_j\, A_{ji} \ . \tag{4.4a,b}$$

<u>Skew-Symmetric</u>: The second order tensor **A** with the $9=3^2$ components A_{ij} referred to the base vectors $\mathbf{e}_i$ is said to be skew-symmetric if

$$\mathbf{A} = -\,\mathbf{A}^T \ , \ A_{ij} = -\,A_{ji} \ . \tag{4.5a,b}$$

It also follows from (3.17) that if **A** is skew-symmetric and **v** is an arbitrary vector with components v_i then

$$\mathbf{A}\,\mathbf{v} = -\,\mathbf{v}\,\mathbf{A} \ , \ A_{ij}\, v_j = -\,v_j\, A_{ji} \ . \tag{4.6a,b}$$

Using these definitions we may observe that an arbitrary second order tensor **B** , with components B_{ij}, may be separated uniquely into its symmetric part denoted by $\mathbf{B}_{sym}$, with components $B_{(ij)}$, and its skew-symmetric part denoted by $\mathbf{B}_{skew}$, with components $B_{[ij]}$, such that

$$\mathbf{B} = \mathbf{B}_{sym} + \mathbf{B}_{skew} \ , \ B_{ij} = B_{(ij)} + B_{[ij]} \ , \tag{4.7a,b}$$

$$\mathbf{B}_{sym} = \frac{1}{2}(\mathbf{B} + \mathbf{B}^T) = \mathbf{B}_{sym}^T \quad , \quad B_{(ij)} = \frac{1}{2}(B_{ij} + B_{ji}) = B_{(ji)} \quad , \tag{4.7c,d}$$

$$\mathbf{B}_{skew} = \frac{1}{2}(\mathbf{B} - \mathbf{B}^T) = -\mathbf{B}_{skew}^T \quad , \quad B_{[ij]} = \frac{1}{2}(B_{ij} - B_{ji}) = -B_{[ji]} \quad . \tag{4.7e,f}$$

Trace: The trace operation is defined as the dot product of an arbitrary second order tensor $\mathbf{T}$ with the second order identity tensor $\mathbf{I}$. Letting T_{ij} be the components of $\mathbf{T}$ we have

$$\mathbf{T} \bullet \mathbf{I} = T_{ij}\,(\mathbf{e}_i \otimes \mathbf{e}_j) \bullet (\mathbf{e}_m \otimes \mathbf{e}_m) = T_{ij}\,(\mathbf{e}_i \bullet \mathbf{e}_m)(\mathbf{e}_j \bullet \mathbf{e}_m) = T_{ij}\,\delta_{im}\,\delta_{jm}$$

$$= T_{ij}\,\delta_{ij} = T_{jj} \quad . \tag{4.8}$$

Deviatoric Tensor: The second order tensor $\mathbf{A}$ with the $9=3^2$ components A_{ij} referred to the base vectors $\mathbf{e}_i$ is said to be deviatoric if

$$\mathbf{A} \bullet \mathbf{I} = 0 \quad , \quad A_{mm} = 0 \quad . \tag{4.9a,b}$$

Spherical and Deviatoric Parts: Using these definitions we may observe that an arbitrary second order tensor $\mathbf{T}$, with components T_{ij}, may be separated uniquely into its spherical part denoted by $T\,\mathbf{I}$, with components $T\,\delta_{ij}$, and its deviatoric part denoted by $\mathbf{T}'$, with components T'_{ij}, such that

$$\mathbf{T} = T\,\mathbf{I} + \mathbf{T}' \quad , \quad T_{ij} = T\,\delta_{ij} + T'_{ij} \quad , \tag{4.10a,b}$$

$$\mathbf{T}' \bullet \mathbf{I} = 0 \quad , \quad T'_{mm} = 0 \quad . \tag{4.10c,d}$$

Taking the dot product of (4.10a) with the second order identity $\mathbf{I}$ it may be shown that T is the mean value of the diagonal terms of $\mathbf{T}$

$$T = \frac{1}{3}\,\mathbf{T} \bullet \mathbf{I} = \frac{1}{3}\,T_{mm} \quad . \tag{4.11}$$

For later convenience it is useful to consider properties of the dot product between strings of second order tensors. To this end, let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ be second order tensors, with components A_{ij}, B_{ij}, C_{ij}, D_{ij}, respectively. Then it can be shown that

$$\mathbf{A} \bullet (\mathbf{BCD}) = A_{ij}B_{im}C_{mn}D_{nj} \quad , \quad \mathbf{A} \bullet (\mathbf{BCD}) = (\mathbf{B}^T\mathbf{A}) \bullet (\mathbf{CD}) \quad , \tag{4.12a,b}$$

$$\mathbf{A} \bullet (\mathbf{BCD}) = (\mathbf{AD}^T) \bullet (\mathbf{BC}) \quad , \quad \mathbf{A} \bullet (\mathbf{BCD}) = (\mathbf{B}^T\mathbf{AD}^T) \bullet \mathbf{C} \quad . \tag{4.12c,d}$$

5. Transformation Relations

Consider two right handed orthonormal rectangular Cartesian coordinate systems with base vectors $\mathbf{e}_i$ and $\mathbf{e}'_i$, and define the transformation tensor $\mathbf{A}$ by

$$\mathbf{A} = \mathbf{e}_m \otimes \mathbf{e}'_m \ . \tag{5.1}$$

It follows from the definition (5.1) that $\mathbf{A}$ is an orthogonal tensor

$$\mathbf{A}\,\mathbf{A}^T = (\mathbf{e}_m \otimes \mathbf{e}'_m)\,(\mathbf{e}'_n \otimes \mathbf{e}_n) = (\mathbf{e}'_m \bullet \mathbf{e}'_n)\,(\mathbf{e}_m \otimes \mathbf{e}_n) \ , \tag{5.2a}$$

$$= \delta'_{mn}\,(\mathbf{e}_m \otimes \mathbf{e}_n) = (\mathbf{e}_m \otimes \mathbf{e}_m) = \mathbf{I} \ , \tag{5.2b}$$

$$\mathbf{A}^T\mathbf{A} = (\mathbf{e}'_m \otimes \mathbf{e}_m)\,(\mathbf{e}_n \otimes \mathbf{e}'_n) = (\mathbf{e}_m \bullet \mathbf{e}_n)\,(\mathbf{e}'_m \otimes \mathbf{e}'_n) \ , \tag{5.2c}$$

$$= \delta_{mn}\,(\mathbf{e}'_m \otimes \mathbf{e}'_n) = (\mathbf{e}'_m \otimes \mathbf{e}'_m) = \mathbf{I} \ . \tag{5.2d}$$

It also follows that

$$\mathbf{e}_i = \mathbf{A}\,\mathbf{e}'_i = (\mathbf{e}_m \otimes \mathbf{e}'_m)\,\mathbf{e}'_i = \mathbf{e}_m\,(\mathbf{e}'_m \bullet \mathbf{e}'_i) = \mathbf{e}_m\,\delta_{mi} \ , \tag{5.3a}$$

$$\mathbf{e}'_i = \mathbf{A}^T\,\mathbf{e}_i \ , \tag{5.3b}$$

where in obtaining (5.3b) we have multiplied (5.2a) by $\mathbf{A}^T$ and have used the orthogonality condition (5.2c).

These equations can be written in equivalent component form by noting that the components A_{ij} of $\mathbf{A}$ referred to the base vectors $\mathbf{e}_i$ and the components A'_{ij} of $\mathbf{A}$ referred to the base vectors $\mathbf{e}'_i$ are defined by

$$A_{ij} = \mathbf{A} \bullet (\mathbf{e}_i \otimes \mathbf{e}_j) = (\mathbf{e}_m \otimes \mathbf{e}'_m) \bullet (\mathbf{e}_i \otimes \mathbf{e}_j) = (\mathbf{e}_m \bullet \mathbf{e}_i)\,(\mathbf{e}'_m \bullet \mathbf{e}_j)$$

$$= \delta_{mi}\,(\mathbf{e}'_m \bullet \mathbf{e}_j) = \mathbf{e}'_i \bullet \mathbf{e}_j \ , \tag{5.4a}$$

$$A'_{ij} = \mathbf{A} \bullet (\mathbf{e}'_i \otimes \mathbf{e}'_j) = (\mathbf{e}_m \otimes \mathbf{e}'_m) \bullet (\mathbf{e}'_i \otimes \mathbf{e}'_j) = (\mathbf{e}_m \bullet \mathbf{e}'_i)\,(\mathbf{e}'_m \bullet \mathbf{e}'_j)$$

$$= (\mathbf{e}_m \bullet \mathbf{e}'_i)\,\delta_{mj} = \mathbf{e}_j \bullet \mathbf{e}'_i = \mathbf{e}'_i \bullet \mathbf{e}_j \ . \tag{5.4b}$$

It is important to emphasize that these results indicate that the first index of A_{ij} (or A'_{ij}) is identified with the primed coordinate system $\mathbf{e}'_i$ and the second index is identified with the unprimed coordinate system $\mathbf{e}_i$. This identification is a consequence of the definition (5.1) and is arbitrary in the sense that one could introduce an alternative definition where the order of the vectors in (5.1) is reversed. However, once the definition (5.1) is introduced it is essential to maintain consistency throughout the text. Also, note from

(5.4a,b) that the components of **A** referred to either the unprimed or the primed coordinate systems are equal

$$A_{ij} = A'_{ij} \quad . \tag{5.5}$$

Using the expressions (5.4) and the results (5.5) we may rewrite (5.3) in the forms

$$\mathbf{e}_i = (A_{mn}\, \mathbf{e}'_m \otimes \mathbf{e}'_n)\, \mathbf{e}'_i = A_{mi}\, \mathbf{e}'_m \quad , \tag{5.6a}$$

$$\mathbf{e}'_i = (A_{mn}\, \mathbf{e}_n \otimes \mathbf{e}_m)\, \mathbf{e}_i = A_{in}\, \mathbf{e}_n \quad . \tag{5.6b}$$

Again, we note that in (5.6) the first index of A_{ij} refers to the primed coordinate system and the second index refers to the unprimed coordinate system.

One of the most fundamental property of a tensor **T** is that the tensor is independent of the particular coordinate system with respect to which we desire to express it. Specifically, we note that all the tensor properties (3.1)-(3.15) have been defined without regard to any particular coordinate system. Furthermore, we emphasize that since physical laws cannot depend on our arbitrary choice of a coordinate system it is essential to express the mathematical representation of these physical laws using tensors. For this reason tensors are essential in continuum mechanics.

Although an arbitrary tensor **T** of order M is independent of the choice of a coordinate system, the components $T_{ijk...rst}$ of **T** with respect to the base vectors $\mathbf{e}_i$ are defined by (3.21) and explicitly depend on the choice of the coordinate system that defines $\mathbf{e}_i$. It follows by analogy to (3.21) that the components $T'_{ijk...rst}$ of **T** relative to the base vectors $\mathbf{e}_i'$ are defined by

$$T'_{ijk...rst} = \mathbf{T} \bullet (\mathbf{e}'_i \otimes \mathbf{e}'_j \otimes \mathbf{e}'_k \otimes ... \otimes \mathbf{e}'_r \otimes \mathbf{e}'_s \otimes \mathbf{e}'_t) \quad , \tag{5.7}$$

so that **T** admits the alternative representation

$$\mathbf{T} = T'_{ijk...rst}\; \mathbf{e}'_i \otimes \mathbf{e}'_j \otimes \mathbf{e}'_k \otimes ... \otimes \mathbf{e}'_r \otimes \mathbf{e}'_s \otimes \mathbf{e}'_t \quad . \tag{5.8}$$

Now, since **T** admits both of the representations (3.20) and (5.8) it follows that the components $T_{ijk..rst}$ and $T'_{ijk...rst}$ must be related to each other. To determine this relation we merely substitute (5.6) into (3.21) and (5.7) and use (3.20) and (5.8) to obtain

$$T_{ijk...rst} = \mathbf{T} \bullet (A_{li}\mathbf{e}_l'\otimes A_{mj}\mathbf{e}_m'\otimes A_{nk}\mathbf{e}_n'\otimes ...\otimes A_{ur}\mathbf{e}_u'\otimes A_{vs}\mathbf{e}_v'\otimes A_{wt}\mathbf{e}_w')$$

$$= A_{li}A_{mj}A_{nk}...A_{ur}A_{vs}A_{wt}\ \mathbf{T} \bullet (\mathbf{e}_l'\otimes\mathbf{e}_m'\otimes\mathbf{e}_n'\otimes ...\otimes\mathbf{e}_u'\otimes\mathbf{e}_v'\otimes\mathbf{e}_w')$$

$$= A_{li}A_{mj}A_{nk}...A_{ur}A_{vs}A_{wt}\ T'_{lmn...uvw}\ , \tag{5.9a}$$

$$T'_{ijk...rst} = \mathbf{T} \bullet (A_{il}\mathbf{e}_l\otimes A_{jm}\mathbf{e}_m\otimes A_{kn}\mathbf{e}_n\otimes ...\otimes A_{ru}\mathbf{e}_u\otimes A_{sv}\mathbf{e}_v\otimes A_{tw}\mathbf{e}_w)$$

$$= A_{il}A_{jm}A_{kn}...A_{ru}A_{sv}A_{tw}\ \mathbf{T} \bullet (\mathbf{e}_l\otimes\mathbf{e}_m\otimes\mathbf{e}_n\otimes ...\otimes\mathbf{e}_u\otimes\mathbf{e}_v\otimes\mathbf{e}_w)$$

$$= A_{il}A_{jm}A_{kn}...A_{ru}A_{sv}A_{tw}\ T_{lmn...uvw}\ . \tag{5.9b}$$

For example, if **v** is a vector with components v_i and v_i' then

$$\mathbf{v} = v_i\ \mathbf{e}_i = v_i'\ \ \mathbf{e}_i'\ , \tag{5.10a}$$

$$v_i = A_{mi}\ v_m'\ ,\ v_i' = A_{im}\ v_m\ , \tag{5.10b,c}$$

and if **T** is a second order tensor with components T_{ij} and T'_{ij} then

$$\mathbf{T} = T_{ij}\ \mathbf{e}_i\otimes\mathbf{e}_j\ = T'_{ij}\ \mathbf{e}_i'\otimes\mathbf{e}_j'\ , \tag{5.11a}$$

$$T_{ij} = A_{mi}\ A_{nj}\ T'_{mn}\ ,\ T_{ij} = A^T_{im}\ T_{mn}'A_{nj}\ , \tag{5.11b,c}$$

$$T'_{ij} = A_{im}\ A_{jn}\ T_{mn}\ ,\ T'_{ij} = A_{im}T_{mn}\ A^T_{nj}\ . \tag{5.11d,e}$$

6. Bodies, Configurations, Motions, Mass, Mass Density

In an abstract sense a body B is a set of material particles which are denoted by Y (see Fig. 6.1). In mechanics a body is assumed to be smooth and can be put into correspondence with a domain of Euclidean 3-Space. Bodies are seen only in their configurations, i.e., the regions of Euclidean 3-Space they occupy at each instant of time t ($-\infty < t < \infty$). In the following we will refer all position vectors to the origin of a fixed rectangular Cartesian coordinate system.

Present Configuration: The present configuration of the body is the region of Euclidean 3-Space occupied by the body at the present time t. Let $\mathbf{x}$ be the position vector which identifies the place occupied by the particle Y at the time t. Since we have assumed that the body can be mapped smoothly into a domain of Euclidean 3-Space we may write

$$\mathbf{x} = \bar{\mathbf{x}}\,(Y,t) \quad . \tag{6.1}$$

In (6.1), Y refers to the particle, t refers to the present time, $\mathbf{x}$ refers to the value of the function $\bar{\mathbf{x}}$ which characterizes the mapping. It is assumed that $\bar{\mathbf{x}}$ is differentiable as many times as desired both with respect to Y and t. Also, for each t it is assumed that (6.1) is invertible so that

$$Y = \bar{\mathbf{x}}^{-1}(\mathbf{x},t) = \tilde{Y}(\mathbf{x},t) \quad . \tag{6.2}$$

Motion: The mapping (6.1) is called a motion of the body because it specifies how each particle Y moves through space as time progresses.

Reference Configuration: Often it is convenient to select one particular configuration, called a reference configuration, and refer everything concerning the body and its motion to this configuration. The reference configuration need not necessarily be an actual configuration occupied by the body and in particular, the reference configuration need not be the initial configuration.

Let $\mathbf{X}$ be the position vector of the particle Y in the reference configuration κ. Then the mapping from Y to the place $\mathbf{X}$ in the reference configuration may be written as

$$\mathbf{X} = \bar{\mathbf{X}}(Y) \quad . \tag{6.3}$$

In (6.3), $\mathbf{X}$ refers to the value of the function $\bar{\mathbf{X}}$ which characterizes the mapping. It is important to note that the mapping (6.3) does not depend on time because the reference configuration is a single constant configuration. The mapping (6.3) is assumed to be invertible and differentiable as many times as desired. Specifically, the inverse mapping is given by

$$Y = \bar{\mathbf{X}}^{-1}(\mathbf{X}) = \hat{Y}(\mathbf{X}) \quad . \tag{6.4}$$

It follows that the mapping from the reference configuration to the present configuration can be obtained by substituting (6.4) into (6.1) to deduce that

$$\mathbf{x} = \bar{\mathbf{x}}(\hat{Y}(\mathbf{X}),t) = \hat{\mathbf{x}}(\mathbf{X},t) \quad . \tag{6.5}$$

From (6.5) it is obvious that the functional form of the mapping $\hat{\mathbf{x}}$ depends on the specific choice of the reference configuration. Further in this regard we emphasize that the choice of the reference configuration is similar to the choice of coordinates in that it is arbitrary to the extent that a one-to-one correspondence exists between the material particles Y and their locations $\mathbf{X}$ in the reference configuration. Also, the inverse of the mapping (6.5) may be written in the form

$$\mathbf{X} = \tilde{\mathbf{X}}(\mathbf{x},t) \quad . \tag{6.6}$$

<u>Representations</u>: There are several methods of describing properties of a body. In the following we specifically consider three possible representations. To this end, let f be an arbitrary function characterizing a property of the body, and admit the following three representations

$$f = \bar{f}(Y,t) \quad , \quad f = \hat{f}(\mathbf{X},t) \quad , \quad f = \tilde{f}(\mathbf{x},t) \quad . \tag{6.7a,b,c}$$

For definiteness, in (6.7) we have distinguished between the value of the function and its functional form. Whenever, this is necessary we will consistently denote functions that depend on Y with and overbar $(\bar{\ })$, functions that depend on $\mathbf{X}$ with a hat $(\hat{\ })$, and functions that depend on $\mathbf{x}$ with a tilde $(\tilde{\ })$. Furthermore, in view of the mappings (6.4) and (6.6) the functional forms $\bar{f}$, $\hat{f}$, $\tilde{f}$ are related by

$$\hat{f}(\mathbf{X},t) = \bar{f}(\hat{Y}(\mathbf{X}),t) \ , \ \ \tilde{f}(\mathbf{x},t) = \hat{f}\,(\tilde{\mathbf{X}}(\mathbf{x},t),t) \quad . \tag{6.8a,b}$$

The representation (6.7a) is called <u>material</u> because the material point Y is used as an independent variable. The representation (6.7b) is called <u>referential</u> or <u>Lagrangian</u> because the position **X** of a material point in the reference configuration is the independent variable, and the representation (6.7c) is called <u>spatial</u> or <u>Eulerian</u> because the current position **x** in space is used as an independent variable. However, we emphasize that in view of our smoothness assumption, any two of these representations may be placed in a one-to-one correspondence with each other.

Here we will use both the coordinate free forms of equations as well as their indicial counterparts. To this end, let $\mathbf{e}_A$ be a constant orthonormal basis associated with the reference configuration and let $\mathbf{e}_i$ be a constant orthonormal basis associated with the present configuration. For our purposes it is sufficient to take these basis to coincide so that

$$\mathbf{e}_i \bullet \mathbf{e}_A = \delta_{iA} \ , \tag{6.9}$$

where δ_{iA} is the usual Kronecker delta symbol. In the following we will refer all tensor quantities to either of these bases and for clarity we will use upper case letters as indices of quantities associated with the reference configuration and lower case letters as indices of quantities associated with the present configuration. For example

$$\mathbf{X} = X_A \, \mathbf{e}_A \ , \ \mathbf{x} = x_i \, \mathbf{e}_i \ , \tag{6.10}$$

where X_A are the rectangular Cartesian components of the position vector **X** and x_i are the rectangular Cartesian components of the position vector **x** and the usual summation convention over repeated indices is used. It follows that the mapping (6.5) may be written in the form

$$x_i = \hat{x}_i \, (X_A,t) \ . \tag{6.11}$$

<u>Velocity and Acceleration</u>: The velocity **v** of a material point Y is defined as the rate of change of position of the material point. Since the function $\bar{\mathbf{x}}(Y,t)$ characterizes the position of the material point Y at any time t it follows that the velocity is given by

$$\mathbf{v} = \dot{\mathbf{x}} = \frac{\partial \bar{\mathbf{x}}(Y,t)}{\partial t} \ , \ v_i = \dot{x}_i = \frac{\partial \bar{x}_i(Y,t)}{\partial t} \ , \tag{6.12a,b}$$

where a superposed dot is used to denote partial differentiation with respect to time t holding the material particle Y fixed. Similarly, the acceleration **a** of a material point Y is defined by

$$\mathbf{a} = \dot{\mathbf{v}} = \frac{\partial \bar{\mathbf{v}}(Y,t)}{\partial t} \quad , \quad a_i = \dot{v}_i = \frac{\partial \bar{v}_i(Y,t)}{\partial t} \quad . \tag{6.13a,b}$$

Notice that in view of the mappings (6.4) and (6.6) the velocity and acceleration can be expressed as functions of either (**X**,t) or (**x**,t).

Material Derivative: The material derivative of an arbitrary function f is defined by

$$\dot{f} = \left. \frac{\partial \bar{f}(Y,t)}{\partial t} \right|_Y \quad . \tag{6.14}$$

It is important to emphasize that the material derivative which is denoted by a superposed dot is defined to be the rate of change of the function holding the material particle Y fixed. In this sense the velocity **v** is the material derivative of the position **x** and the acceleration **a** is the material derivative of the velocity **v**. Recalling from (6.7) that the function f can be represented using either the material, Lagrangian, or Eulerian representations, it follows from the chain rule of differentiation that $\dot{f}$ admits the additional representations

$$\dot{f} = \frac{\partial \hat{f}(\mathbf{X},t)}{\partial t} \dot{t} + [\partial \hat{f}(\mathbf{X},t)/\partial \mathbf{X}] \dot{\mathbf{X}} = \frac{\partial \hat{f}(\mathbf{X},t)}{\partial t} \quad , \tag{6.15a}$$

$$\dot{f} = \frac{\partial \hat{f}(\mathbf{X},t)}{\partial t} \dot{t} + [\partial \hat{f}(\mathbf{X},t)/\partial X_A] \dot{X}_A = \frac{\partial \hat{f}(\mathbf{X},t)}{\partial t} \quad , \tag{6.15b}$$

$$\dot{f} = \frac{\partial \tilde{f}(\mathbf{x},t)}{\partial t} \dot{t} + [\partial \tilde{f}(\mathbf{x},t)/\partial \mathbf{x}] \dot{\mathbf{x}} = \frac{\partial \tilde{f}(\mathbf{x},t)}{\partial t} + [\partial \tilde{f}(\mathbf{x},t)/\partial \mathbf{x}] \bullet \mathbf{v} \quad , \tag{6.15c}$$

$$\dot{f} = \frac{\partial \tilde{f}(\mathbf{x},t)}{\partial t} \dot{t} + [\partial \tilde{f}(\mathbf{x},t)/\partial x_m] \dot{x}_m = \frac{\partial \tilde{f}(\mathbf{x},t)}{\partial t} + [\partial \tilde{f}(\mathbf{x},t)/\partial x_m] \, v_m \quad , \tag{6.15d}$$

where in (6.15a) we have used the fact that the mapping (6.3) from the material point Y to its location **X** in the reference configuration is independent of time so that $\dot{\mathbf{X}}$ vanishes. It is important to emphasize that the physics of the material derivative defined by (6.14) remains unchanged even though its specific functional form (6.15) for different representations may change.

Mass and Mass Density: Each part P of the body is assumed to be endowed with a positive measure M(P) (i.e. a real number > 0) called the mass of the part P. Letting v be the volume of the part P in the present configuration at time t, and assuming that the mass M(P) is an absolutely continuous function there exists a positive measure $\rho(\mathbf{x},t)$ defined by

$$\rho(\mathbf{x},t) = \lim_{v \to 0} \frac{M(P)}{v} \quad . \tag{6.16}$$

In (6.16) $\mathbf{x}$ is the point occupied by the part P of the body at time t in the limit as v approaches zero. The function ρ is called the mass density of the body at the point $\mathbf{x}$ in the present configuration at time t. It follows that the mass M(P) of the part P may be determined by integration of the mass density such that

$$M(P) = \int_P \rho \, dv \ , \tag{6.17}$$

where dv is the element of volume in the present configuration.

Similarly, we can define the mass density $\rho_0(\mathbf{X},t)$ of the part P_0 of the body in the reference configuration such that the mass $M(P_0)$ of the part P_0 is given by

$$M(P_0) = \int_{P_0} \rho_0 \, dV \ , \tag{6.18}$$

where dV is the element of volume in the reference configuration. It should be emphasized that at this stage in the development the mass of a material part of the body denoted by P or P_0 can depend on time.

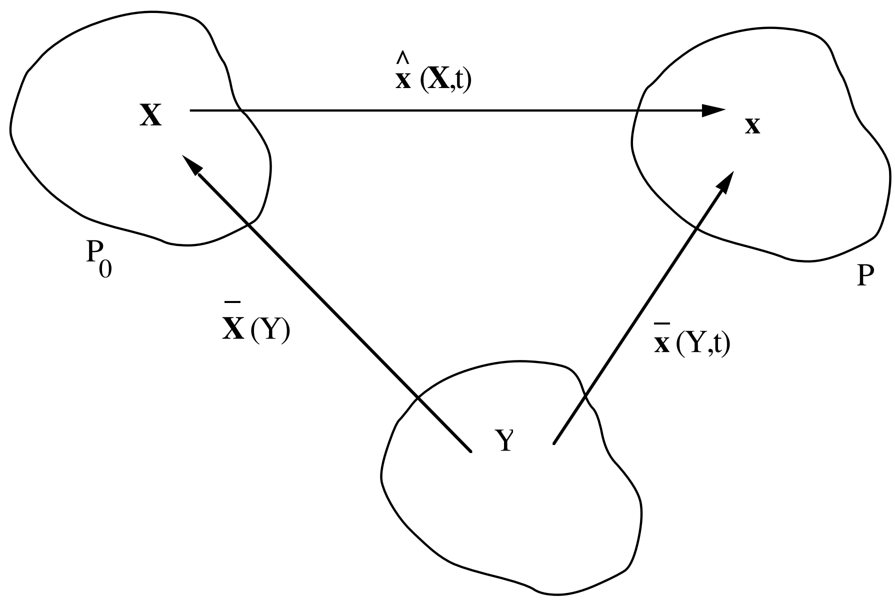

Fig. 6.1 Configurations

7. Deformation Gradient and Deformation Measures

In order to describe the deformation of the body from the reference configuration to the present configuration let us model the body in its reference configuration as a finite collection of neighboring tetrahedrons. As the number of tetrahedrons increases we can approximate a body having an arbitrary shape. If we can determine the deformation of each of these tetrahedrons from the reference configuration to the present configuration then we can determine the shape (and volume) of the body in the present configuration by simply connecting the neighboring tetrahedrons. Since a tetrahedron is characterized by a triad of three vectors we realize that the deformation of an arbitrary elemental tetrahedron (infinitesimally small) can be determined if we can determine the deformation of an arbitrary material line element. This is because the material line element can be identified with each of the base vectors of the tetrahedron.

Deformation Gradient: For this reason it is sufficient to determine the deformation of a material line element $d\mathbf{X}$ in the reference configuration to the material line element $d\mathbf{x}$ in the present configuration. Recalling that the mapping $\mathbf{x} = \hat{\mathbf{x}}(\mathbf{X},t)$ defines the position $\mathbf{x}$ in the present configuration of any material point $\mathbf{X}$ in the reference configuration, it follows that

$$d\mathbf{x} = (\partial\hat{\mathbf{x}}/\partial\mathbf{X})\, d\mathbf{X} = \mathbf{F}\, d\mathbf{X} \ , \qquad (7.1a)$$

$$dx_i = (\partial\hat{x}_i/\partial X_A)\, dX_A = x_{i,A}\, dX_A = F_{iA}\, dX_A \ , \qquad (7.1b)$$

$$\mathbf{F} = (\partial\hat{\mathbf{x}}/\partial\mathbf{X}) \ , \ F_{iA} = x_{i,A} \ , \qquad (7.1c,d)$$

where $\mathbf{F}$ is the deformation gradient with components F_{iA}. Throughout the text a comma denotes partial differentiation with respect to X_A if the index is a capital letter and with respect to x_i if the index is a lower case letter. Since the mapping $\hat{\mathbf{x}}(\mathbf{X},t)$ is invertible we require

$$\det \mathbf{F} \neq 0 \ , \ \det (x_{i,A}) \neq 0 \ . \qquad (7.2a,b)$$

However, for our purposes we wish to retain the possibility that the reference configuration could coincide with the present configuration at one time ($\mathbf{x}=\mathbf{X};\mathbf{F}=\mathbf{I}$) so we require

$$\det \mathbf{F} > 0 \ , \ \det(x_{i,A}) > 0 \ . \tag{7.3a,b}$$

Right and Left Green Deformation Tensors: The magnitude ds of the material line element d**x** in the present configuration may be calculated using (7.1) such that

$$(ds)^2 = d\mathbf{x} \bullet d\mathbf{x} = \mathbf{F}\, d\mathbf{X} \bullet \mathbf{F}\, d\mathbf{X} = d\mathbf{X} \bullet \mathbf{F}^T\mathbf{F}\, d\mathbf{X} = d\mathbf{X} \bullet \mathbf{C}\, d\mathbf{X} \ , \tag{7.4a}$$

$$(ds)^2 = dx_i\, dx_i = F_{iA}\, dX_A\, F_{iB}\, dX_B = dX_A\, x_{i,A} x_{i,B}\, dX_B = dX_A\, C_{AB}\, dX_B \ , \tag{7.4b}$$

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} \ , \ \ C_{AB} = F_{iA}\, F_{iB} = x_{i,A}\, x_{i,B} \ , \tag{7.4c,d}$$

where **C** is called the right Green deformation tensor. Similarly, the magnitude dS of the material line element d**X** in the reference configuration may be calculated by inverting (7.1a) to obtain

$$d\mathbf{X} = \mathbf{F}^{-1}\, d\mathbf{x} \ , \ \ dX_A = X_{A,i}\, dx_i \ , \tag{7.5a,b}$$

which yields

$$(dS)^2 = d\mathbf{X} \bullet d\mathbf{X} = \mathbf{F}^{-1}d\mathbf{x} \bullet \mathbf{F}^{-1}d\mathbf{x} = d\mathbf{x} \bullet \mathbf{F}^{-T}\mathbf{F}^{-1}d\mathbf{x} = d\mathbf{x} \bullet \mathbf{c}\, d\mathbf{x} \ , \tag{7.6a}$$

$$(dS)^2 = dX_A\, dX_A = X_{A,i}dx_i\, X_{A,j}dx_j = dx_i\, X_{A,i}\, X_{A,j}\, dx_j = dx_i\, c_{ij}\, dx_j \ , \tag{7.6b}$$

$$\mathbf{c} = \mathbf{F}^{-T}\mathbf{F}^{-1} \ , \ \ c_{ij} = X_{A,i}\, X_{A,j} \ . \tag{7.6c,d}$$

where **c** is the Cauchy deformation tensor. It is also convenient to define the left Green deformation tensor **B** by

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T \ , \ \ B_{ij} = F_{iA}\, F_{jA} = x_{i,A}\, x_{j,A} \ , \tag{7.7a,b}$$

and note that

$$\mathbf{c} = \mathbf{B}^{-1} \ . \tag{7.8}$$

Stretch and Extension: The stretch λ of a material line element is defined in terms of the ratio of the lengths ds and dS of the line element in the present and reference configurations, respectively, such that

$$\lambda = \frac{ds}{dS} \ . \tag{7.9}$$

Also, the extension E of the same material line element is defined by

$$E = \lambda - 1 \ . \tag{7.10}$$

It follows from these definitions that the stretch is always positive. Also, the stretch is greater than one and the extension is greater than zero when the material line element is extended relative to its reference length.

For convenience let **S** be the unit vector defining the direction of the line element d**X** and let **s** be the unit vector defining the direction of the associated line element d**x**. It follows from (7.4a) and (7.6a) that

$$d\mathbf{X} = \mathbf{S}\, dS \ , \ dX_A = S_A\, dS \ , \ \mathbf{S} \bullet \mathbf{S} = S_A S_A = 1 \ , \qquad (7.11a,b,c)$$

$$d\mathbf{x} = \mathbf{s}\, ds \ , \ dx_i = s_i\, ds \ , \ \mathbf{s} \bullet \mathbf{s} = s_i\, s_i = 1 \ . \qquad (7.11d,e,f)$$

Thus using (7.1),(7.6),(7.9) and (7.11) it follows that

$$\lambda\, \mathbf{s} = \mathbf{F}\, \mathbf{S} \ , \ \lambda\, s_i = x_{i,A}\, S_A \ , \qquad (7.12a,b)$$

$$\lambda^2 = \mathbf{S} \bullet \mathbf{C}\, \mathbf{S} \ , \ \lambda^2 = S_A\, C_{AB}\, S_B \ , \qquad (7.12c,d)$$

$$\frac{1}{\lambda^2} = \mathbf{s} \bullet \mathbf{c}\, \mathbf{s} \ , \ \frac{1}{\lambda^2} = s_i\, c_{ij}\, s_j \ . \qquad (7.12e,f)$$

Since the stretch is positive it also follows from (7.12c,d) that the **C** is a positive definite tensor. Similarly, it can be shown that **B** in (7.7a) is also a positive definite tensor. Notice from (7.12c) that the stretch of a line element depends not only on the value of **C** at the material point **X** and the time t, but it depends on the orientation **S** of the line element in the reference configuration.

<u>A Pure Measure of Dilatation (Volume Change):</u> In order to discuss the relative volume change of a material element it is convenient to first prove that for any nonsingular second order tensor **F** and any two vectors **a** and **b** that

$$\mathbf{Fa} \times \mathbf{Fb} = \det(\mathbf{F})\, \mathbf{F}^{-T}(\mathbf{a} \times \mathbf{b}) \ . \qquad (7.13)$$

To prove this it is first noted that the quantity $\mathbf{F}^{-T}(\mathbf{a} \times \mathbf{b})$ is a vector that is orthogonal to plane formed by the vectors **Fa** and **Fb** since

$$\mathbf{F}^{-T}(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{Fa} = (\mathbf{a} \times \mathbf{b}) \bullet (\mathbf{F}^{-T})^T\mathbf{Fa} = (\mathbf{a} \times \mathbf{b}) \bullet \mathbf{F}^{-1}\mathbf{Fa} = (\mathbf{a} \times \mathbf{b}) \bullet \mathbf{a} = 0 \ ,$$

$$\mathbf{F}^{-T}(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{Fb} = (\mathbf{a} \times \mathbf{b}) \bullet \mathbf{F}^{-1}\mathbf{Fb} = (\mathbf{a} \times \mathbf{b}) \bullet \mathbf{b} = 0 \ . \qquad (7.14)$$

This means that the quantity $(\mathbf{Fa} \times \mathbf{Fb})$ must be a vector that is parallel to $\mathbf{F}^{-T}(\mathbf{a} \times \mathbf{b})$ so that

$$\mathbf{Fa} \times \mathbf{Fb} = \alpha\, \mathbf{F}^{-T}(\mathbf{a} \times \mathbf{b}) \ . \qquad (7.15)$$

Next, the value of the scalar α is determined by noting that both sides of equation (7.15) must be linear functions of **a** and **b**. This means that α is independent of the vectors **a** and **b**. Moreover, letting **c** be an arbitrary vector it follows that

$$\mathbf{Fa} \times \mathbf{Fb} \bullet \mathbf{Fc} = \alpha\, \mathbf{F}^{-T}(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{Fc} = \alpha\, (\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c} \ . \tag{7.16}$$

The proof is finished by considering the rectangular Cartesian base vectors $\mathbf{e}_i$ and taking $\mathbf{a}$=$\mathbf{e}_1$, $\mathbf{b}$=$\mathbf{e}_2$, $\mathbf{c}$=$\mathbf{e}_3$ to deduce that

$$\alpha = \mathbf{Fe}_1 \times \mathbf{Fe}_2 \bullet \mathbf{Fe}_3 = \det(\mathbf{F}) \ . \tag{7.17}$$

This expression can be recognized as the determinant of the tensor **F** since it represents the scalar triple product of the columns of **F** when it is expressed in rectangular Cartesian components.

Now, it will be shown that the determinant J of the deformation gradient **F**

$$J = \det \mathbf{F} \ , \tag{7.18}$$

is a pure measure of dilatation. To this end, consider an elemental material volume defined by the line elements $d\mathbf{X}^1$, $d\mathbf{X}^2$, $d\mathbf{X}^3$ in the reference configuration and defined by the associated line elements $d\mathbf{x}^1$, $d\mathbf{x}^2$, $d\mathbf{x}^3$ in the present configuration. Thus, the elemental volumes dV in the reference configuration and dv in the present configuration are given by

$$dV = d\mathbf{X}^1 \times d\mathbf{X}^2 \bullet d\mathbf{X}^3 \ , \ dv = d\mathbf{x}^1 \times d\mathbf{x}^2 \bullet d\mathbf{x}^3 \ . \tag{7.19a,b}$$

Since (7.1a) defines the mapping of each line element from the reference configuration to the present configuration it follows that

$$dv = \mathbf{F}d\mathbf{X}^1 \times \mathbf{F}d\mathbf{X}^2 \bullet \mathbf{F}d\mathbf{X}^3 = J\, \mathbf{F}^{-T}(d\mathbf{X}^1 \times d\mathbf{X}^2) \bullet \mathbf{F}d\mathbf{X}^3 \ ,$$

$$= J\, (d\mathbf{X}^1 \times d\mathbf{X}^2) \bullet \mathbf{F}^{-1}\mathbf{F}d\mathbf{X}^3 = J\, d\mathbf{X}^1 \times d\mathbf{X}^2 \bullet d\mathbf{X}^3 \ , \tag{7.20a}$$

$$dv = J\, dV \ . \tag{7.20b}$$

This means that J is a pure measure of dilatation. It also follows from (7.4c) and (7.19) that the scalar I_3 defined by

$$I_3 = \det \mathbf{C} = J^2 \ , \tag{7.21}$$

is another pure measure of dilatation.

Pure Measures of Distortion (Shape Change): In general, the deformation gradient **F** characterizes the dilatation (volume change) and distortion (shape change) of a material element. Therefore, whenever **F** is a unimodular tensor (its determinant J equals unity) **F** is a pure measure of distortion. Using this idea which originated with Flory (1961) we separate **F** into its dilatational part $J^{1/3}\mathbf{I}$ and its distortional part $\mathbf{F}'$ such that

$$\mathbf{F} = (J^{1/3}\mathbf{I})\,\mathbf{F}' = J^{1/3}\,\mathbf{F}' \ , \quad \mathbf{F}' = J^{-1/3}\,\mathbf{F} \ , \ \det \mathbf{F}' = 1 \ . \qquad (7.22a,b,c)$$

Note that since $\mathbf{F}'$ is unimodular (7.22c) it is a pure measure of distortion. Also note that the use of a prime here should not be confused with the earlier use of a prime to denote the deviatoric part of a tensor (4.10). In this regard we emphasize that in general $\mathbf{F}'$ is not a deviatoric tensor. Similarly, we may separate $\mathbf{C}$ into its dilatational part $I_3^{1/3}\mathbf{I}$ and its distortional part $\mathbf{C}'$ such that

$$\mathbf{C} = (I_3^{1/3}\mathbf{I})\,\mathbf{C}' = I_3^{1/3}\,\mathbf{C}' \ , \ \mathbf{C}' = I_3^{-1/3}\,\mathbf{C} \ , \ \det \mathbf{C}' = 1 \ . \qquad (7.23a,b,c)$$

Strain Measures: Using (7.4) and (7.6) it follows that the change in length of a line element can be expressed in the following forms

$$(ds)^2 - (dS)^2 = d\mathbf{X} \bullet (\mathbf{C} - \mathbf{I})\, d\mathbf{X} = d\mathbf{X} \bullet (2\mathbf{E})\, d\mathbf{X} \ , \qquad (7.24a)$$

$$(ds)^2 - (dS)^2 = dX_A\,(C_{AB} - \delta_{AB})\, dX_B = dX_A(2E_{AB})\, dX_B \ , \qquad (7.24b)$$

$$(ds)^2 - (dS)^2 = d\mathbf{x} \bullet (\mathbf{I} - \mathbf{c})\, d\mathbf{x} = d\mathbf{x} \bullet (2\mathbf{e})\, d\mathbf{x} \ , \qquad (7.24c)$$

$$(ds)^2 - (dS)^2 = dx_i\,(\delta_{ij} - c_{ij})\, dx_j = dx_i(2e_{ij})\, dx_j \ , \qquad (7.24d)$$

where $\mathbf{E}$ is the Lagrangian strain and $\mathbf{e}$ is the Almansi strain defined by

$$2\,\mathbf{E} = \mathbf{C} - \mathbf{I} \ , \ 2\,\mathbf{e} = \mathbf{I} - \mathbf{c} \ . \qquad (7.25a,b)$$

Furthermore, in view of the separation (7.23) it is sometimes convenient to define a scalar measure of dilatational strain E and a tensorial measure of distortional strain $\mathbf{E}'$ by

$$2\,E = I_3 - 1 \ , \ 2\,\mathbf{E}' = \mathbf{C}' - \mathbf{I} \ . \qquad (7.26a,b)$$

Eigenvalues of **C** and **B**: In appendix A we briefly review the notions of eigenvalues, eigenvectors and the principal invariants of a tensor. Using the definitions (7.4c),(7.7a), and (A3) we first show that the principal invariants of $\mathbf{C}$ and $\mathbf{B}$ are equal. To this end, we use the properties of the dot product given by (4.12) to deduce that

$$\mathbf{C} \bullet \mathbf{I} = \mathbf{F}^T\mathbf{F} \bullet \mathbf{I} = \mathbf{F} \bullet \mathbf{F} = \mathbf{F}\mathbf{F}^T \bullet \mathbf{I} = \mathbf{B} \bullet \mathbf{I} \ , \qquad (7.27a)$$

$$\mathbf{C} \bullet \mathbf{C} = \mathbf{F}^T\mathbf{F} \bullet \mathbf{F}^T\mathbf{F} = \mathbf{F} \bullet \mathbf{F}\mathbf{F}^T\mathbf{F} = \mathbf{F}\mathbf{F}^T \bullet \mathbf{F}\mathbf{F}^T = \mathbf{B} \bullet \mathbf{B} \ , \qquad (7.27b)$$

$$\det \mathbf{C} = \det \mathbf{F}^T\mathbf{F} = \det \mathbf{F}^T \det \mathbf{F} = (\det \mathbf{F})^2 = \det \mathbf{F}\mathbf{F}^T = \det \mathbf{B} \ . \qquad (7.27c)$$

If follows from (A3) that the principal invariants of $\mathbf{C}$ and $\mathbf{B}$ are equal

$$I_1(\mathbf{C}) = I_1(\mathbf{B}) \ , \ I_2(\mathbf{C}) = I_2(\mathbf{B}) \ , \ I_3(\mathbf{C}) = I_3(\mathbf{B}) \ . \qquad (7.28a,b,c)$$

Furthermore, using (7.12c) we may deduce that the eigenvalues of $\mathbf{C}$ are also the squares of the principal values of stretch λ, which are determined by the characteristic equation

$$\det(\mathbf{C} - \lambda^2\mathbf{I}) = -\lambda^6 + \lambda^4 I_1(\mathbf{C}) - \lambda^2 I_2(\mathbf{C}) + I_3(\mathbf{C}) = \det(\mathbf{B} - \lambda^2\mathbf{I}) = 0 \quad . \tag{7.29}$$

Displacement Vector: The displacement vector **u** is the vector that connects the position **X** of a material point in the reference configuration to its position **x** in the present configuration so that

$$\mathbf{u} = \mathbf{x} - \mathbf{X} \;,\; \mathbf{x} = \mathbf{X} + \mathbf{u} \;,\; \mathbf{X} = \mathbf{x} - \mathbf{u} \;, \tag{7.30a,b,c}$$

$$\mathbf{u} = u_A\, \mathbf{e}_A = u_i\, \mathbf{e}_i \;. \tag{7.30d}$$

It follows from the definition (7.1c) of the deformation gradient **F** that

$$\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X} = \partial(\mathbf{X} + \mathbf{u})/\partial\mathbf{X} = \mathbf{I} + \partial\hat{\mathbf{u}}/\partial\mathbf{X} \;, \tag{7.31a}$$

$$\mathbf{F}^{-1} = \partial\mathbf{X}/\partial\mathbf{x} = \partial(\mathbf{x} - \mathbf{u})/\partial\mathbf{x} = \mathbf{I} - \partial\tilde{\mathbf{u}}/\partial\mathbf{x} \;. \tag{7.31b}$$

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = (\mathbf{I} + \partial\hat{\mathbf{u}}/\partial\mathbf{X})^T (\mathbf{I} + \partial\hat{\mathbf{u}}/\partial\mathbf{X})$$

$$= \mathbf{I} + \partial\hat{\mathbf{u}}/\partial\mathbf{X} + (\partial\hat{\mathbf{u}}/\partial\mathbf{X})^T + (\partial\hat{\mathbf{u}}/\partial\mathbf{X})^T(\partial\hat{\mathbf{u}}/\partial\mathbf{X}) \;, \tag{7.31c}$$

$$C_{AB} = \delta_{AB} + \hat{u}_{A,B} + \hat{u}_{B,A} + \hat{u}_{M,A}\,\hat{u}_{M,B} \;, \tag{7.31d}$$

$$\mathbf{c} = \mathbf{B}^{-1} = \mathbf{F}^{-T}\mathbf{F}^{-1} = (\mathbf{I} - \partial\tilde{\mathbf{u}}/\partial\mathbf{x})^T (\mathbf{I} - \partial\tilde{\mathbf{u}}/\partial\mathbf{x})$$

$$= \mathbf{I} - \partial\tilde{\mathbf{u}}/\partial\mathbf{x} - (\partial\tilde{\mathbf{u}}/\partial\mathbf{x})^T + (\partial\tilde{\mathbf{u}}/\partial\mathbf{x})^T(\partial\tilde{\mathbf{u}}/\partial\mathbf{x}) \;, \tag{7.31e}$$

$$c_{ij} = \delta_{ij} - \tilde{u}_{i,j} - \tilde{u}_{j,i} + \tilde{u}_{m,i}\,\tilde{u}_{m,j} \;. \tag{7.31f}$$

Then, with the help of the definitions (7.25) the strains **E** and **e** may be expressed in terms of the displacement **u** by

$$\mathbf{E} = \frac{1}{2}\left[\partial\hat{\mathbf{u}}/\partial\mathbf{X} + (\partial\hat{\mathbf{u}}/\partial\mathbf{X})^T + (\partial\hat{\mathbf{u}}/\partial\mathbf{X})^T(\partial\hat{\mathbf{u}}/\partial\mathbf{X})\right] \;, \tag{7.32a}$$

$$E_{AB} = \frac{1}{2}(\hat{u}_{A,B} + \hat{u}_{B,A} + \hat{u}_{M,A}\,\hat{u}_{M,B}) \;, \tag{7.32b}$$

$$\mathbf{e} = \frac{1}{2}\left[\partial\tilde{\mathbf{u}}/\partial\mathbf{x} + (\partial\tilde{\mathbf{u}}/\partial\mathbf{x})^T - (\partial\tilde{\mathbf{u}}/\partial\mathbf{x})^T(\partial\tilde{\mathbf{u}}/\partial\mathbf{x})\right] \;, \tag{7.32c}$$

$$e_{ij} = \frac{1}{2}(\tilde{u}_{i,j} + \tilde{u}_{j,i} - \tilde{u}_{m,i}\,\tilde{u}_{m,j}) \;. \tag{7.32d}$$

Since these expressions have been obtained without any approximation they are exact and are sometimes referred to as finite strain measures. Notice the different signs in front of the quadratic terms in displacement appearing in the expressions (7.32a) and (7.32c).

Area Element: The element of area dA formed by the elemental parallelogram associated with the material line elements $d\mathbf{X}^1$ and $d\mathbf{X}^2$ in the reference configuration, and the element of area da formed by the corresponding line elements $d\mathbf{x}^1$ and $d\mathbf{x}^2$ in the present configuration are given by

$$\mathbf{N}\, dA = d\mathbf{X}^1 \times d\mathbf{X}^2 \ , \ \ \mathbf{n}\, da = d\mathbf{x}^1 \times d\mathbf{x}^2 \ , \tag{7.33a,b}$$

where $\mathbf{N}$ and $\mathbf{n}$ are the unit vectors normal to the material surfaces defined by $d\mathbf{X}^1$, $d\mathbf{X}^2$ and $d\mathbf{x}^1$, $d\mathbf{x}^2$, respectively. It follows from (7.1a) and (7.13) that

$$\mathbf{n}\, da = \mathbf{F}d\mathbf{X}^1 \times \mathbf{F}d\mathbf{X}^2 = J\, \mathbf{F}^{-T}(d\mathbf{X}^1 \times d\mathbf{X}^2) = J\, \mathbf{F}^{-T}\mathbf{N}\, dA \ . \tag{7.34}$$

It is important to emphasize that the line element that was normal to the material surface in the reference configuration does not necessarily remain normal to the material surface.

8. Polar Decomposition Theorem

The polar decomposition theorem states that any invertible second order tensor **F** can be uniquely decomposed into the polar forms

$$\mathbf{F} = \mathbf{RM} = \mathbf{NR} \ , \ F_{iA} = R_{iM}M_{MA} = N_{im}R_{mA} \ , \qquad (8.1a,b)$$

where **R** is an orthogonal tensor

$$\mathbf{R}^T\mathbf{R} = \mathbf{I} \ , \ R_{mA}R_{mB} = \delta_{AB} \ , \qquad (8.2a,b)$$

$$\mathbf{R}\mathbf{R}^T = \mathbf{I} \ , \ R_{iM}R_{jM} = \delta_{ij} \ , \qquad (8.2c,d)$$

and **M** and **N** are symmetric positive definite tensors so that for an arbitrary vector **v** we have

$$\mathbf{M}^T = \mathbf{M} \ , \ M_{BA} = M_{AB} \ , \qquad (8.3a,b)$$

$$\mathbf{v} \bullet \mathbf{Mv} > 0 \ , \quad v_A M_{AB} v_B > 0 \quad \text{for } \mathbf{v} \neq 0 \ , \qquad (8.3c,d)$$

$$\mathbf{N}^T = \mathbf{N} \ , \ N_{ji} = N_{ij} \ , \qquad (8.3e,f)$$

$$\mathbf{v} \bullet \mathbf{Nv} > 0 \ , \quad v_i N_{ij} v_j > 0 \quad \text{for } \mathbf{v} \neq 0 \ . \qquad (8.3g,h)$$

To prove this theorem we first consider the following Lemma.

Lemma: If **S** is an invertible second order tensor then $\mathbf{S}^T\mathbf{S}$ and $\mathbf{S}\mathbf{S}^T$ are positive definite tensors.

Proof: (i) Let

$$\mathbf{w} = \mathbf{Sv} \ , \quad w_i = S_{ij}v_j \ . \qquad (8.4a,b)$$

Since **S** is invertible it follows that

$$\mathbf{w} = 0 \ \text{ if and only if } \ \mathbf{v} = 0 \ , \ \mathbf{w} \neq 0 \text{ if and only if } \mathbf{v} \neq 0 \ . \qquad (8.5a,b)$$

Consider

$$\mathbf{w} \bullet \mathbf{w} = \mathbf{Sv} \bullet \mathbf{Sv} = \mathbf{v} \bullet \mathbf{S}^T\mathbf{Sv} \ , \ w_m w_m = S_{mi}v_i S_{mj}v_j = v_i S_{im}{}^T S_{mj} v_j \ . \qquad (8.6a,b)$$

Since $\mathbf{w} \bullet \mathbf{w} > 0$ whenever $\mathbf{v} \neq 0$ it follows that $\mathbf{S}^T\mathbf{S}$ is positive definite.

(ii) Alternatively, let

$$\mathbf{w} = \mathbf{S}^T\mathbf{v} \ , \quad w_i = S_{ij}{}^T v_j = S_{ji} v_j \ . \qquad (8.7a,b)$$

Similarly, consider

$$\mathbf{w} \bullet \mathbf{w} = \mathbf{S}^T\mathbf{v} \bullet \mathbf{S}^T\mathbf{v} = \mathbf{v} \bullet \mathbf{S}\mathbf{S}^T\mathbf{v} \ , \ w_m w_m = S_{im}v_i S_{jm}v_j = v_i S_{im} S_{mj}{}^T v_j \ . \qquad (8.8a,b)$$

Since $\mathbf{w} \bullet \mathbf{w} > 0$ whenever $\mathbf{v} \neq 0$ it follows that $\mathbf{S}\mathbf{S}^T$ is positive definite.

To prove the polar decomposition theorem we first prove existence of the forms **F=RM** and **F=NR** and then prove uniqueness of the quantities **R**,**M**,**N**.

Existence: (i) Since **F** is invertible the tensor $\mathbf{F}^T\mathbf{F}$ is symmetric and positive definite so there exists a symmetric positive definite square root **M**

$$\mathbf{M} = (\mathbf{F}^T\mathbf{F})^{1/2} \;,\; \mathbf{M}^2 = \mathbf{F}^T\mathbf{F} \;,\; M_{AM}M_{MB} = F_{mA}F_{mB} \;. \tag{8.9a,b,c}$$

Then let $\mathbf{R}_1$ be defined by

$$\mathbf{R}_1 = \mathbf{F}\mathbf{M}^{-1} \;,\; \mathbf{F} = \mathbf{R}_1\mathbf{M} \;. \tag{8.10a,b}$$

To prove that $\mathbf{R}_1$ is an orthogonal tensor consider

$$\mathbf{R}_1\mathbf{R}_1{}^T = \mathbf{F}\mathbf{M}^{-1}(\mathbf{F}\mathbf{M}^{-1})^T = \mathbf{F}\mathbf{M}^{-1}\mathbf{M}^{-T}\mathbf{F}^T = \mathbf{F}(\mathbf{M}^2)^{-1}\mathbf{F}^T$$

$$= \mathbf{F}(\mathbf{F}^T\mathbf{F})^{-1}\mathbf{F}^T = \mathbf{F}(\mathbf{F}^{-1}\mathbf{F}^{-T})\mathbf{F}^T = \mathbf{I} \;, \tag{8.11a}$$

$$\mathbf{R}_1{}^T\mathbf{R}_1 = \mathbf{M}^{-T}\mathbf{F}^T\mathbf{F}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M}^2\mathbf{M}^{-1} = \mathbf{I} \;. \tag{8.11b}$$

(ii) Similarly, since **F** is invertible the tensor $\mathbf{F}\mathbf{F}^T$ is symmetric and positive definite so there exists a symmetric positive definite square root **N**

$$\mathbf{N} = (\mathbf{F}\mathbf{F}^T)^{1/2} \;,\; \mathbf{N}^2 = \mathbf{F}\mathbf{F}^T \;,\; N_{im}N_{mj} = F_{iM}F_{jM} \;. \tag{8.12a,b,c}$$

Then let $\mathbf{R}_2$ be defined by

$$\mathbf{R}_2 = \mathbf{N}^{-1}\mathbf{F} \;,\; \mathbf{F} = \mathbf{N}\mathbf{R}_2 \;. \tag{8.13a,b}$$

To prove that $\mathbf{R}_2$ is an orthogonal tensor consider

$$\mathbf{R}_2\mathbf{R}_2{}^T = \mathbf{N}^{-1}\mathbf{F}(\mathbf{N}^{-1}\mathbf{F})^T = \mathbf{N}^{-1}\mathbf{F}\mathbf{F}^T\mathbf{N}^{-T} = \mathbf{N}^{-1}\mathbf{N}^2\mathbf{N}^{-1} = \mathbf{I} \;, \tag{8.14a}$$

$$\mathbf{R}_2{}^T\mathbf{R}_2 = \mathbf{F}^T\mathbf{N}^{-T}\mathbf{N}^{-1}\mathbf{F} = \mathbf{F}^T\mathbf{N}^{-2}\mathbf{F} = \mathbf{F}^T(\mathbf{F}\mathbf{F}^T)^{-1}\mathbf{F}$$

$$= \mathbf{F}^T\mathbf{F}^{-T}\mathbf{F}^{-1}\mathbf{F} = \mathbf{I} \;. \tag{8.14b}$$

Uniqueness: (i) Assume that $\mathbf{R}_1$ and **M** are not unique so that

$$\mathbf{F} = \mathbf{R}_1\mathbf{M} = \mathbf{R}_1^*\mathbf{M}^* \;. \tag{8.15}$$

Then consider

$$\mathbf{F}^T\mathbf{F} = \mathbf{M}^2 = (\mathbf{R}_1^*\mathbf{M}^*)^T\mathbf{R}_1^*\mathbf{M}^* = \mathbf{M}^{*T}\mathbf{R}_1^{*T}\mathbf{R}_1^*\mathbf{M}^* = \mathbf{M}^{*2} \;. \tag{8.16}$$

However, since **M** and $\mathbf{M}^*$ are both symmetric and positive definite we deduce that **M** is unique

$$\mathbf{M} = \mathbf{M}^* \;. \tag{8.17}$$

Using (8.17) in (8.15) we have

$$\mathbf{R}_1\mathbf{M} = \mathbf{R}_1^*\mathbf{M} \ , \tag{8.18}$$

so that by multiplication of (8.18) on the left by $\mathbf{M}^{-1}$ we may deduce that $\mathbf{R}_1$ is unique

$$\mathbf{R}_1 = \mathbf{R}_1^* \ . \tag{8.19}$$

(ii) Similarly, assume that $\mathbf{R}_2$ and $\mathbf{N}$ are not unique so that

$$\mathbf{F} = \mathbf{N}\mathbf{R}_2 = \mathbf{N}^*\mathbf{R}_2^* \ . \tag{8.20}$$

Then consider

$$\mathbf{F}\mathbf{F}^T = \mathbf{N}^2 = \mathbf{N}^*\mathbf{R}_2^* \, (\mathbf{N}^*\mathbf{R}_2^*)^T = \mathbf{N}^*\mathbf{R}_2^*\mathbf{R}_2^{*T}\mathbf{N}^{*T} = \mathbf{N}^{*2} \ . \tag{8.21}$$

However, since $\mathbf{N}$ and $\mathbf{N}^*$ are both symmetric and positive definite we deduce that $\mathbf{N}$ is unique

$$\mathbf{N} = \mathbf{N}^* \ . \tag{8.22}$$

Using (8.22) in (8.20) we have

$$\mathbf{N}\mathbf{R}_2 = \mathbf{N}\mathbf{R}_2^* \ , \tag{8.23}$$

so that by multiplication of (8.23) on the right by $\mathbf{N}^{-1}$ we may deduce that $\mathbf{R}_2$ is unique

$$\mathbf{R}_2 = \mathbf{R}_2^* \ . \tag{8.24}$$

Finally, we must prove that $\mathbf{R}_1$=$\mathbf{R}_2$=$\mathbf{R}$. To this end let

$$\mathbf{A} = \mathbf{R}_1\mathbf{M}\mathbf{R}_1^T = \mathbf{F}\mathbf{R}_1^T \ . \tag{8.25}$$

Clearly, $\mathbf{A}$ is symmetric so that

$$\mathbf{A}^2 = \mathbf{A}\mathbf{A}^T = \mathbf{F}\mathbf{R}_1^T \, (\mathbf{F}\mathbf{R}_1^T)^T = \mathbf{F}\mathbf{R}_1^T\mathbf{R}_1\mathbf{F}^T = \mathbf{F}\mathbf{F}^T = \mathbf{N}^2 \ . \tag{8.26}$$

Since $\mathbf{A}$ and $\mathbf{N}$ are symmetric it follows with the help of (8.25) and (8.10b) that

$$\mathbf{N} = \mathbf{A} = \mathbf{F}\mathbf{R}_1^T = \mathbf{N}\mathbf{R}_2\mathbf{R}_1^T \ . \tag{8.27}$$

Now, multiplying (8.27) on the left by $\mathbf{N}^{-1}$ and on the right by $\mathbf{R}_1$ we deduce that

$$\mathbf{R}_1 = \mathbf{R}_2 = \mathbf{R} \ , \tag{8.28}$$

which completes the proof.

To explain the physical interpretation of the polar decomposition theorem recall from (7.1a) that a line element $d\mathbf{X}$ in the reference configuration is transformed by $\mathbf{F}$ into the line element $d\mathbf{x}$ in the present configuration and define the elemental vectors $d\mathbf{X}'$ and $d\mathbf{x}'$ such that

$$d\mathbf{x} = \mathbf{RM}\, d\mathbf{X} \;\Rightarrow\; d\mathbf{X}' = \mathbf{M}\, d\mathbf{X} \;,\; d\mathbf{x} = \mathbf{R}\, d\mathbf{X}' \;, \tag{8.29a,b,c}$$

$$dx_i = R_{iA}M_{AB}\, dX_B \Rightarrow dX_A' = M_{AB}\, dX_B \;,\; dx_i = R_{iA}\, dX_A' \;, \tag{8.29d,e,f}$$

and

$$d\mathbf{x} = \mathbf{NR}\, d\mathbf{X} \;\Rightarrow\; d\mathbf{x}' = \mathbf{R}\, d\mathbf{X} \;,\; d\mathbf{x} = \mathbf{N}\, d\mathbf{x}' \;, \tag{8.30a,b,c}$$

$$dx_i = N_{ij}R_{jB}\, dX_B \;\Rightarrow\; dx_j' = R_{jB}\, dX_B \;,\; dx_i = N_{ij}\, dx_j' \;. \tag{8.30d,e,f}$$

In general a line element experiences both stretching and rotation as it deforms from $d\mathbf{X}$ to $d\mathbf{x}$. However, the polar decomposition theorem separates the deformation into stretching and pure rotation. To see this use (7.4a) together with (8.29) and consider

$$ds^2 = d\mathbf{x} \bullet d\mathbf{x} = \mathbf{R}\, d\mathbf{X}' \bullet \mathbf{R}\, d\mathbf{X}' = d\mathbf{X}' \bullet \mathbf{R}^T\mathbf{R}\, d\mathbf{X}' = d\mathbf{X}' \bullet d\mathbf{X}' \;. \tag{8.31}$$

It follows that the magnitude of $d\mathbf{X}'$ is the same as that of $d\mathbf{x}$ so that all the stretching occurs during the transformation from $d\mathbf{X}$ to $d\mathbf{X}'$ and that the transformation from $d\mathbf{X}'$ to $d\mathbf{x}$ is a pure rotation. Similarly, with the help of (7.6a) and (8.30) we have

$$d\mathbf{x}' \bullet d\mathbf{x}' = \mathbf{R}\, d\mathbf{X} \bullet \mathbf{R}\, d\mathbf{X} = d\mathbf{X} \bullet \mathbf{R}^T\mathbf{R}\, d\mathbf{X} = d\mathbf{X} \bullet d\mathbf{X} = dS^2 \;. \tag{8.32}$$

It follows that the magnitude of $d\mathbf{x}'$ is the same as that of $d\mathbf{X}$ so that all the stretching occurs during the transformation from $d\mathbf{x}'$ to $d\mathbf{x}$ and that the transformation from $d\mathbf{X}$ to $d\mathbf{x}'$ is a pure rotation.

Although the transformations from $d\mathbf{X}$ to $d\mathbf{X}'$ and from $d\mathbf{x}'$ to $d\mathbf{x}$ contain all the stretching they also tend to rotate a general line element. However, if we consider the special line element $d\mathbf{X}$ which is parallel to any of the three principal directions of $\mathbf{M}$ then the transformation from $d\mathbf{X}$ to $d\mathbf{X}'$ is a pure stretch without rotation (see Fig. 8.1a) because

$$d\mathbf{X}' = \mathbf{M}\, d\mathbf{X} = \lambda\, d\mathbf{X} \;, \tag{8.33}$$

where λ is the stretch defined by (7.9). It then follows that for this line element

$$d\mathbf{x} = \mathbf{F}\, d\mathbf{X} = \mathbf{RM}\, d\mathbf{X} = \mathbf{R}\, \lambda d\mathbf{X} = \lambda\, d\mathbf{x}' \;, \tag{8.34a}$$

$$d\mathbf{x} = \mathbf{F}\, d\mathbf{X} = \mathbf{NR}\, d\mathbf{X} = \mathbf{N}\, d\mathbf{x}' = \lambda\, d\mathbf{x}' \;, \tag{8.34b}$$

so that $d\mathbf{x}'$ is also parallel to a principal direction of $\mathbf{N}$, which means that the transformation from $d\mathbf{x}'$ to $d\mathbf{x}$ is a pure stretch without rotation (see Fig. 8.1b). This also means that the rotation tensor **R** describes the complete rotation of line elements which are either parallel to principal directions of **M** in the reference configuration or parallel to principal directions of **N** in the present configuration.

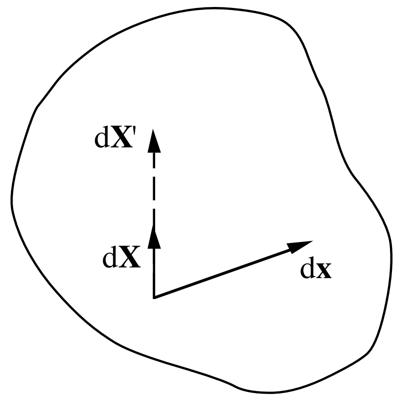

Fig. 8.1a: Pure stretching followed by pure rotation; **F=RM**; d**X**'=**M** d**X**; d**x**=**R** d**X**'.

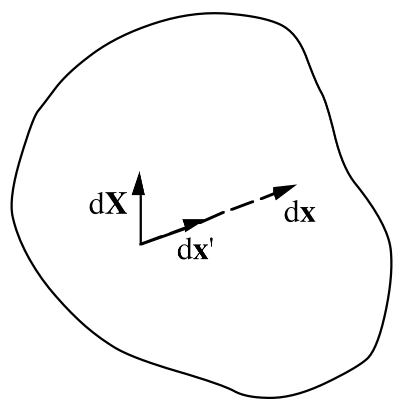

Fig. 8.1b: Pure rotation followed by pure stretching; **F=NR**; d**x**'=**R**d**X**; d**x**=**N**d**x**'.

9. Velocity Gradient and Rate of Deformation Tensors

The gradient of the velocity **v** with respect to the present position **x** is denoted by **L** and is defined by

$$\mathbf{L} = \partial \mathbf{v}/\partial \mathbf{x} \ , \ L_{ij} = \frac{\partial v_i}{\partial x_j} = v_{i,j} \ . \tag{9.1a,b}$$

The symmetric part of **L** is called the rate of deformation tensor and is denoted by **D**, while the skew symmetric part of **L** is called the spin tensor and is denoted by **W**. Thus

$$\mathbf{L} = \mathbf{D} + \mathbf{W} \ , \ v_{i,j} = D_{ij} + W_{ij} \ , \tag{9.2a,b}$$

$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) = \mathbf{D}^T \ , \ D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) = D_{ji} \ , \tag{9.2c,d}$$

$$\mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T) = -\mathbf{W}^T \ , \ W_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i}) = -W_{ji} \ . \tag{9.2e,f}$$

Using the chain rule of differentiation, the continuity of the derivatives, and the definition of the material derivative it follows that

$$\dot{\mathbf{F}} = \frac{\partial}{\partial t}\left(\partial \hat{\mathbf{x}}/\partial \mathbf{X}\right) = \partial^2 \hat{\mathbf{x}}/\partial t \partial \mathbf{X} = \partial(\partial \hat{\mathbf{x}}/\partial t)/\partial \mathbf{X} = \partial \hat{\mathbf{v}}/\partial \mathbf{X}$$

$$= (\partial \tilde{\mathbf{v}}/\partial \mathbf{x})\,(\partial \hat{\mathbf{x}}/\partial \mathbf{X}) = \mathbf{LF} \ , \tag{9.3a}$$

$$\dot{\overline{(x_{i,A})}} = \frac{\partial}{\partial t}(\hat{x}_{i,A}) = \frac{\partial^2 \hat{x}_i}{\partial t \partial X_A} = \frac{\partial}{\partial X_A}\left(\frac{\partial \hat{x}_i}{\partial t}\right) = \hat{v}_{i,A} = \tilde{v}_{i,m}\,\hat{x}_{m,A} \ . \tag{9.3b}$$

Now let us consider the material derivative of **C**

$$\dot{\mathbf{C}} = \dot{\overline{\mathbf{F}^T\mathbf{F}}} = \dot{\mathbf{F}}^T\mathbf{F} + \mathbf{F}^T\dot{\mathbf{F}} = (\mathbf{LF})^T\mathbf{F} + \mathbf{F}^T(\mathbf{LF}) = \mathbf{F}^T(\mathbf{L}^T + \mathbf{L})\mathbf{F} = 2\,\mathbf{F}^T\mathbf{DF} \ , \tag{9.4a}$$

$$\dot{C}_{AB} = \dot{\overline{(x_{i,A})}}\ x_{i,B} + x_{i,A}\ \dot{\overline{(x_{i,B})}} = v_{i,m}x_{m,A}x_{i,B} + x_{i,A}v_{i,m}x_{m,B}$$

$$= x_{m,A}\,(v_{i,m} + v_{m,i})\,x_{i,B} = 2\,x_{m,A}\,D_{im}\,x_{i,B} = 2\,x_{m,A}D_{mi}x_{i,B} \ . \tag{9.4b}$$

Furthermore, since the spin tensor **W** is skew symmetric there exits a unique vector $\boldsymbol{\omega}$ called the axial vector of **W** such that for any vector **a**

$$\mathbf{W}\,\mathbf{a} = \boldsymbol{\omega} \times \mathbf{a} \ , \ W_{ij}\,a_j = \varepsilon_{ikj}\,\omega_k\,a_j \ . \tag{9.5a,b}$$

Since (9.5b) must be true for any vector **a** and **W** and $\boldsymbol{\omega}$ are independent of **a** it follows that

$$W_{ij} = \varepsilon_{ikj}\,\omega_k = \varepsilon_{jik}\,\omega_k = -\,\varepsilon_{ijk}\,\omega_k \ . \tag{9.6}$$

Multiplying (9.6) by ε_{ijm} and using the identity

$$\varepsilon_{ijk}\varepsilon_{ijm} = 2\,\delta_{km} \ , \tag{9.7}$$

we may solve for ω_m in terms of W_{ij} to obtain

$$\omega_m = -\frac{1}{2}\ \varepsilon_{ijm}\, W_{ij} \ . \tag{9.8}$$

Substituting (9.2f) into (9.8) we have

$$\omega_m = -\frac{1}{2}\ \varepsilon_{ijm}\, v_{i,j} = \frac{1}{2}\ \varepsilon_{jim}\, v_{i,j} = \frac{1}{2}\ \varepsilon_{mji}\, v_{i,j} \ , \tag{9.9a}$$

$$\boldsymbol{\omega} = \frac{1}{2}\ \text{curl}\ \mathbf{v} = \frac{1}{2}\ \nabla \times \mathbf{v} \ , \tag{9.9b}$$

where the symbol ∇ denotes the gradient operator

$$\nabla\phi = \phi_{,i}\, \mathbf{e}_i \ . \tag{9.10}$$

10. Deformation: Interpretations and Examples

In order to interpret the various deformation measures we recall from (7.11) and (7.12) that

$$\lambda \mathbf{s} = \mathbf{F}\mathbf{S} \ , \ \lambda s_i = x_{i,A} S_A \ , \ \lambda = \frac{ds}{dS} \ , \tag{10.1a,b,c}$$

$$\mathbf{s} = \frac{d\mathbf{x}}{ds} \ , \ \mathbf{s} \bullet \mathbf{s} = 1 \ , \ \mathbf{S} = \frac{d\mathbf{X}}{dS} \ , \ \mathbf{S} \bullet \mathbf{S} = 1 \ , \tag{10.1d,e,f}$$

where $\mathbf{S}$ is the unit vector in the direction of the material line element $d\mathbf{X}$ of length dS, $\mathbf{s}$ is the unit vector in the direction of the material line element $d\mathbf{x}$ of length ds, and λ is the stretch. Now from (7.12c) and the definition (7.25a) of Lagrangian strain $\mathbf{E}$ we may write

$$\lambda^2 = \mathbf{S} \bullet \mathbf{C}\mathbf{S} = 1 + 2\, \mathbf{S} \bullet \mathbf{E}\mathbf{S} = 1 + 2\, S_A E_{AB} S_B \ . \tag{10.2}$$

Also, the extension E defined by (7.10) becomes

$$E = \frac{ds - dS}{dS} = \lambda - 1 = \sqrt{1 + 2\, S_A E_{AB} S_B} - 1 \ . \tag{10.3}$$

For the purpose of interpreting the diagonal components of the strain tensor let us calculate the extensions E_1, E_2, E_3 of the line elements which were parallel to the coordinate axes with base vectors $\mathbf{e}_A$ in the reference configuration. Thus, from (10.3) we have

$$E = E_1 = \sqrt{1 + 2E_{11}} - 1 \quad \text{for } \mathbf{S} = \mathbf{e}_1 \ , \tag{10.4a}$$

$$E = E_2 = \sqrt{1 + 2E_{22}} - 1 \quad \text{for } \mathbf{S} = \mathbf{e}_2 \ , \tag{10.4b}$$

$$E = E_3 = \sqrt{1 + 2E_{33}} - 1 \quad \text{for } \mathbf{S} = \mathbf{e}_3 \ . \tag{10.4c}$$

This clearly shows that the diagonal components of the strain tensor are measures of the extensions of line elements which were parallel to the coordinate directions in the reference configuration.

To interpret the off-diagonal components of the strain tensor E_{AB} as measures of shear we consider two material line elements $d\mathbf{X}$ and $d\bar{\mathbf{X}}$ which are deformed into $d\mathbf{x}$ and

$d\bar{\mathbf{x}}$, respectively. Letting $\bar{\mathbf{S}}$, $d\bar{S}$ and $\bar{\mathbf{s}}$, $d\bar{s}$ be the directions and lengths of the line elements $d\bar{\mathbf{X}}$ and $d\bar{\mathbf{x}}$, respectively, we have from (10.1a)

$$\bar{\lambda}\ \bar{\mathbf{s}} = \mathbf{F}\,\bar{\mathbf{S}}\ ,\ \bar{\lambda}\ = \frac{d\bar{s}}{d\bar{S}}\ . \tag{10.5a,b}$$

Notice that there is no over bar on $\mathbf{F}$ in (10.5) because (10.1a) is valid for any line element, including the particular line element $d\bar{\mathbf{X}}$. It follows that the angle Θ between the undeformed line elements $d\mathbf{X}$, $d\bar{\mathbf{X}}$ and the angle θ between the deformed line elements $d\mathbf{x}$, $d\bar{\mathbf{x}}$ may be calculated by (see Fig. 10.1)

$$\cos\Theta = \frac{d\mathbf{X}}{dS} \bullet \frac{d\bar{\mathbf{X}}}{d\bar{S}} = \mathbf{S} \bullet \bar{\mathbf{S}},\ \cos\theta = \frac{d\mathbf{x}}{ds} \bullet \frac{d\bar{\mathbf{x}}}{d\bar{s}} = \mathbf{s} \bullet \bar{\mathbf{s}}\ . \tag{10.6a,b}$$

Then with the help of (10.1a), (10.5a) and (7.25a) we deduce that

$$\cos\theta = \frac{\mathbf{S} \bullet \mathbf{C}\bar{\mathbf{S}}}{\lambda\bar{\lambda}} = \frac{2\mathbf{S} \bullet \mathbf{E}\bar{\mathbf{S}} + \mathbf{S} \bullet \bar{\mathbf{S}}}{\lambda\bar{\lambda}}\ . \tag{10.7}$$

Furthermore, using (10.2) and (10.6a) we have

$$\cos\theta = \frac{2\,S_A E_{AB} \bar{S}_B + \cos\Theta}{\sqrt{1 + 2S_M E_{MN} S_N}\ \sqrt{1 + 2\bar{S}_R E_{RS} \bar{S}_S}}\ . \tag{10.8}$$

Defining the change in the angle between the two line elements by ψ (10.8) becomes

$$\theta = \Theta - \psi\ , \tag{10.9a}$$

$$\cos\Theta\,\cos\psi + \sin\Theta\,\sin\psi = \frac{2\,S_A E_{AB} \bar{S}_B + \cos\Theta}{\sqrt{1 + 2S_M E_{MN} S_N}\ \sqrt{1 + 2\bar{S}_R E_{RS} \bar{S}_S}}\ . \tag{10.9b}$$

Notice that in general the change in angle ψ depends on the original angle Θ and on all of the components of strain.

As a specific example consider two line elements which in the reference configuration are orthogonal and aligned along the coordinate axes so that

$$\mathbf{S} = \mathbf{e}_1 \ , \ \bar{\mathbf{S}} = \mathbf{e}_2 \ , \ \Theta = \frac{\pi}{2} \ . \qquad (10.10a,b,c)$$

Then, (10.9b) reduces to

$$\sin \psi = \frac{2E_{12}}{\sqrt{1 + 2E_{11}} \ \sqrt{1 + 2E_{22}}} \ . \qquad (10.11)$$

Thus, the shear depends on the normal components of strain as well as on the off-diagonal components of strain . However, if the strain is small (i.e. $E_{AB} << 1$) then (10.11) may be approximated by

$$\psi \approx 2E_{12} \ , \qquad (10.12)$$

which shows that the off-diagonal terms are related to shear deformations.

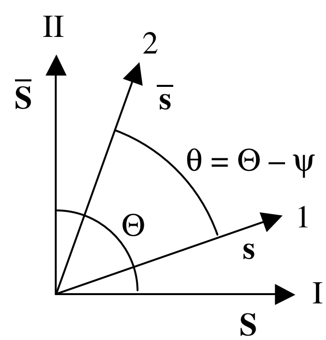

Fig. 10.1 Shear Angle: Points I, II in the reference configuration move to points 1, 2 in the present configuration.

To provide a physical interpretation of the rate of deformation it is convenient to use (9.2a) and (9.3a) and take the material derivative of (10.1a) to deduce that

$$\dot{\lambda} \, \mathbf{s} + \lambda \, \dot{\mathbf{s}} = \dot{\mathbf{F}} \, \mathbf{S} = \mathbf{LFS} = \lambda \, \mathbf{Ls} = \lambda \, (\mathbf{D} + \mathbf{W}) \, \mathbf{s} \ . \qquad (10.13)$$

In (10.13) we have used the fact that **S** is a material direction so its material derivative vanishes. Since **s** is a unit vector it can only rotate so that its rate of change is perpendicular to itself

$$\mathbf{s} \bullet \mathbf{s} = 1 \Rightarrow \dot{\mathbf{s}} \bullet \mathbf{s} + \mathbf{s} \bullet \dot{\mathbf{s}} = 2 \, \dot{\mathbf{s}} \bullet \mathbf{s} = 0 \ \Rightarrow \dot{\mathbf{s}} \bullet \mathbf{s} = 0 \ . \qquad (10.14a,b,c)$$

Thus, taking the dot product of (10.13) with **s** we have

$$\dot{\lambda} = \lambda\,(\mathbf{D} + \mathbf{W}) \bullet (\mathbf{s}\otimes\mathbf{s}) = \lambda\,\mathbf{D} \bullet (\mathbf{s}\otimes\mathbf{s}) = \lambda\,\mathbf{s} \bullet \mathbf{D}\mathbf{s} \;, \qquad (10.15)$$

where we have used the fact that the inner product of the skew-symmetric tensor **W** with the symmetric tensor **n⊗n** vanishes. It follows that the rate of deformation tensor **D** is directly related to the rate of change of stretch.

Substituting (10.15) into (10.13) we obtain

$$\dot{\mathbf{s}} = \mathbf{W}\mathbf{s} + [\mathbf{D} - \{\mathbf{s} \bullet \mathbf{D}\mathbf{s}\}\mathbf{I}]\,\mathbf{s} \;, \qquad (10.16)$$

which shows that in general the rate of rotation of **s** is dependent on both the tensors **D** and **W**. However, if **s** is parallel to a principal direction of **D** then

$$\mathbf{D}\mathbf{s} = \{\mathbf{s} \bullet \mathbf{D}\mathbf{s}\}\,\mathbf{s} \;,\; \dot{\mathbf{s}} = \mathbf{W}\mathbf{s} \;. \qquad (10.17a,b)$$

This shows that the spin tensor **W** controls the rate of rotation of the line element d**x** which in the present configuration is parallel to a principal direction of **D**. Furthermore, using (9.5a) we see that the axial vector **ω** determines the rate of rotation of **s** for this case

$$\dot{\mathbf{s}} = \boldsymbol{\omega} \times \mathbf{s} \;. \qquad (10.18)$$

<u>Example: Extension and Contraction (Fig. 10.2)</u>

By way of example let X_A be the Cartesian components of **X** and x_i be the Cartesian components of **x** and let the Cartesian base vectors $\mathbf{e}_A$ and $\mathbf{e}_i$ coincide ($\mathbf{e}_i = \delta_{iA}\mathbf{e}_A$) and consider the motion defined by

$$x_1 = e^{at}\,X_1 \;,\; x_2 = e^{-bt}\,X_2 \;,\; x_3 = X_3 \;, \qquad (10.19a,b,c)$$

where a,b are positive numbers. The inverse mapping is given by

$$X_1 = e^{-at}\,x_1 \;,\; X_2 = e^{bt}\,x_2 \;,\; X_3 = x_3 \;. \qquad (10.20a,b,c)$$

It follows that

$$\mathbf{F} = \begin{pmatrix} e^{at} & 0 & 0 \\ 0 & e^{-bt} & 0 \\ 0 & 0 & 1 \end{pmatrix},\; \mathbf{C} = \begin{pmatrix} e^{2at} & 0 & 0 \\ 0 & e^{-2bt} & 0 \\ 0 & 0 & 1 \end{pmatrix},\; 2\mathbf{E} = \begin{pmatrix} e^{2at}-1 & 0 & 0 \\ 0 & e^{-2bt}-1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (10.21a,b,c)$$

In order to better understand the deformation we calculate the stretch λ and the extension E of line elements which were parallel to the coordinate directions in the reference configuration

$$\text{For} \quad \mathbf{S} = \mathbf{e}_1 \;,\; \lambda = e^{at} \geq 1 \;, E = e^{at} - 1 \geq 0 \;, \quad \text{(extension)} \;, \tag{10.22a}$$

$$\text{For} \quad \mathbf{S} = \mathbf{e}_2 \;,\; \lambda = e^{-bt} \leq 1 \;, E = e^{-bt} - 1 \leq 0 \;, \quad \text{(contraction)}, \tag{10.22b}$$

$$\text{For} \quad \mathbf{S} = \mathbf{e}_3 \;,\; \lambda = 1 \;, \quad E = 0 \;, \qquad \text{(no deformation)}. \tag{10.22c}$$

Next we consider the rate of deformation and deduce that

$$v_1 = ax_1 \;,\; v_2 = -bx_2 \;,\; v_3 = 0 \;, \tag{10.23a,b,c}$$

$$\mathbf{L} = \mathbf{D} = \begin{pmatrix} a & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & 0 \end{pmatrix} \;,\; \mathbf{W} = 0 \;,\; \boldsymbol{\omega} = 0 \,. \tag{10.23d,e,f}$$

The principal directions of $\mathbf{D}$ are $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ so since $\mathbf{W}=0$ we realize that the line elements that are parallel to these principal directions in the present configuration experience pure stretching without rotation

$$\text{For} \quad \mathbf{s} = \mathbf{e}_1 \;,\; \frac{\dot{\lambda}}{\lambda} = a > 0 \;,\; \dot{\mathbf{s}} = 0 \;, \quad \text{(rate of extension)} \;, \tag{10.24a}$$

$$\text{For} \quad \mathbf{s} = \mathbf{e}_2 \;,\; \frac{\dot{\lambda}}{\lambda} = -b < 0 \;,\; \dot{\mathbf{s}} = 0 \;, \quad \text{(rate of contraction)} \;, \tag{10.24b}$$

$$\text{For} \quad \mathbf{s} = \mathbf{e}_3 \;,\; \frac{\dot{\lambda}}{\lambda} = 0 \;, \quad \dot{\mathbf{s}} = 0 \;, \quad \text{(no deformation)} \;. \tag{10.24c}$$

We emphasize that although $\mathbf{W}$ vanishes this does not mean that no line elements rotate during this motion.

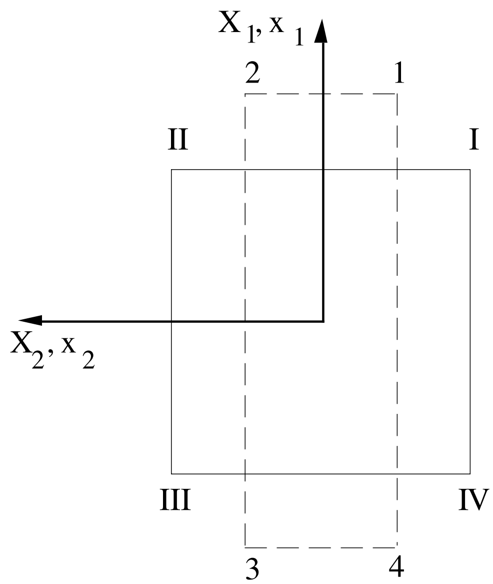

Fig. 10.2 Extension and Contraction: Points I,II,III,IV in the reference configuration move to points 1,2,3,4 in the present configuration.

<u>Example: Simple Shear (Fig. 10.3)</u>

In order to clarify the meaning of the spin tensor **W** consider the simple shearing deformation which is defined by

$$x_1 = X_1 + \kappa(t)\, X_2 \ , \ x_2 = X_2 \ , \ x_3 = X_3 \ , \tag{10.25a,b,c}$$

where κ is a monotonically increasing nonnegative function of time

$$\kappa \ge 0 \ , \ \dot{\kappa} > 0 \ . \tag{10.26a,b}$$

The inverse mapping is given by

$$X_1 = x_1 - \kappa\, x_2 \ , \ X_2 = x_2 \ , \ X_3 = x_3 \ , \tag{10.27a,b,c}$$

and it follows that

$$\mathbf{F} = \begin{pmatrix} 1 & \kappa & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \ \mathbf{C} = \begin{pmatrix} 1 & \kappa & 0 \\ \kappa & 1+\kappa^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \ 2\mathbf{E} = \begin{pmatrix} 0 & \kappa & 0 \\ \kappa & \kappa^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{10.28a,b,c}$$

In order to better understand the deformation we calculate the stretch λ and the extension E of line elements which were parallel to the coordinate directions in the reference configuration

For $\mathbf{S} = \mathbf{e}_1$, $\lambda = 1$, $E = 0$, (no deformation) , (10.29a)

For $\mathbf{S} = \mathbf{e}_2$, $\lambda = \sqrt{1+\kappa^2}$, $E = \sqrt{1+\kappa^2} - 1 \geq 0$, (extension), (10.29b)

For $\mathbf{S} = \mathbf{e}_3$, $\lambda = 1$, $E = 0$, (no deformation). (10.29c)

Notice that the result (10.29b) could be obtained by direct calculation using elementary geometry. Next we consider the rate of deformation and deduce that

$$v_1 = \dot{\kappa}\, x_2 \ , \ v_2 = 0 \ , \ v_3 = 0 \ , \tag{10.30a,b,c}$$

$$\mathbf{L} = \begin{pmatrix} 0 & \dot{\kappa} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \ \mathbf{D} = \frac{1}{2}\begin{pmatrix} 0 & \dot{\kappa} & 0 \\ \dot{\kappa} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{10.30d,e}$$

$$\mathbf{W} = \frac{1}{2}\begin{pmatrix} 0 & \dot{\kappa} & 0 \\ -\dot{\kappa} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \boldsymbol{\omega} = -\frac{1}{2}\dot{\kappa}\ \mathbf{e}_3 \ , \tag{10.30f,g}$$

Thus, the principal directions of $\mathbf{D}$ are $\frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2)$, $\frac{1}{\sqrt{2}}(-\mathbf{e}_1 + \mathbf{e}_2)$, $\mathbf{e}_3$ so with the help of (10.15) and (10.18) we may deduce that

For $\mathbf{s} = \frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2)$, $\frac{\dot{\lambda}}{\lambda} = \frac{1}{2}\dot{\kappa} > 0$, (rate of extension) , (10.31a)

For $\mathbf{s} = \frac{1}{\sqrt{2}}(-\mathbf{e}_1 + \mathbf{e}_2)$, $\frac{\dot{\lambda}}{\lambda} = -\frac{1}{2}\dot{\kappa} < 0$, (rate of contraction) , (10.31b)

For $\mathbf{s} = \mathbf{e}_3$, $\frac{\dot{\lambda}}{\lambda} = 0$, (no deformation) . (10.31c)

It follows from (10.30g) that the material line elements in (10.31) are rotating in the clockwise direction about the $\mathbf{e}_3$ axis with angular speed $\frac{1}{2}\dot{\kappa}$. Finally we note that the motion is isochoric (no change in volume) since

$$J = \det \mathbf{F} = 1 \ , \ \mathbf{D} \bullet \mathbf{I} = 0 \ . \tag{10.32a,b}$$

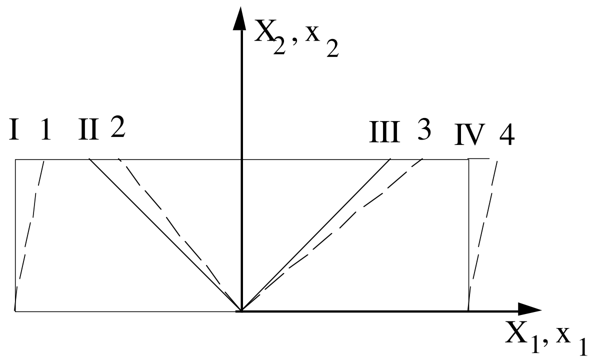

Fig. 10.3 Simple Shear: Points I,II,III,IV in the reference configuration move to points 1,2,3,4 in the present configuration.

11. Superposed Rigid Body Motions

In this section we consider a group of motions associated with configurations P^+ which differ from an arbitrary prescribed motion such as (6.5)

$$\mathbf{x} = \hat{\mathbf{x}}(\mathbf{X},t) \quad , \tag{11.1}$$

by only superposed rigid body motions of the entire body, i.e., motions which in addition to the prescribed motion include purely rigid motions of the body.

To this end, consider a material point $\mathbf{X}$ of the body, which in the present configuration P at time t occupies the location $\mathbf{x}$ as specified by (11.1). Suppose that under a superposed rigid body motion the material point which is at $\mathbf{x}$ at time t in the configuration P moves to the location $\mathbf{x}^+$ at time t^+

$$t^+ = t + a \quad , \tag{11.2}$$

in the configuration P^+, where a is a constant. Throughout the text we denote quantities associated with the configuration P^+ using the same symbol as associated with the configuration P but with a superposed (+). Thus, we introduce the vector function $\mathbf{x}^+$ and write

$$\mathbf{x}^+ = \hat{\bar{\mathbf{x}}}^+(\mathbf{X},t^+) = \hat{\mathbf{x}}^+(\mathbf{X},t) \quad , \tag{11.3}$$

where we have used (11.2) and have distinguished between the two functions $\hat{\bar{\mathbf{x}}}^+$ and $\hat{\mathbf{x}}^+$ in (11.3) to indicate the absence of the constant a in the argument of $\hat{\mathbf{x}}^+$.

Similarly, consider another material point $\mathbf{Y}$ of the body, which in the present configuration P at time t occupies the location $\mathbf{y}$ as specified by

$$\mathbf{y} = \hat{\mathbf{x}}(\mathbf{Y},t) \quad . \tag{11.4}$$

It is important to emphasize that the function $\hat{\mathbf{x}}$ in (11.4) is the same function as that in (11.1). Furthermore, suppose that under the same superposed rigid body motion the material point which is at $\mathbf{y}$ at time t in the configuration P moves to the location $\mathbf{y}^+$ at time t^+. Then, with the help of (11.3) we may write

$$\mathbf{y}^+ = \hat{\bar{\mathbf{x}}}^+(\mathbf{Y},t^+) = \hat{\mathbf{x}}^+(\mathbf{Y},t) \quad . \tag{11.5}$$

Recalling the inverse relationships

$$\mathbf{X} = \tilde{\mathbf{X}}^{-1}(\mathbf{x},t) \ , \ \mathbf{Y} = \tilde{\mathbf{X}}^{-1}(\mathbf{y},t) \ , \tag{11.6a,b}$$

the function $\hat{\mathbf{x}}^+$ on the right hand sides of (11.3) and (11.5) may be expressed as different functions of $\mathbf{x}$,t and $\mathbf{y}$,t, respectively, such that

$$\mathbf{x}^+ = \hat{\mathbf{x}}^+(\tilde{\mathbf{X}}^{-1}(\mathbf{x},t),t) = \tilde{\mathbf{x}}^+(\mathbf{x},t) \ , \tag{11.7a}$$

$$\mathbf{y}^+ = \hat{\mathbf{x}}^+(\tilde{\mathbf{X}}^{-1}(\mathbf{y},t),t) = \tilde{\mathbf{x}}^+(\mathbf{y},t) \ . \tag{11.7b}$$

Since the superposed motion of the body is restricted to be rigid, the magnitude of the relative displacement $\mathbf{y}^+ - \mathbf{x}^+$ must remain equal to the magnitude of the relative displacement $\mathbf{y} - \mathbf{x}$ for all pairs of material points $\mathbf{X},\mathbf{Y}$, and for all time. Hence,

$$[\, \tilde{\mathbf{x}}^+(\mathbf{y},t) - \tilde{\mathbf{x}}^+(\mathbf{x},t)] \bullet [\, \tilde{\mathbf{x}}^+(\mathbf{y},t) - \tilde{\mathbf{x}}^+(\mathbf{x},t)] = (\mathbf{y} - \mathbf{x}) \bullet (\mathbf{y} - \mathbf{x}) \ , \tag{11.8a}$$

$$[\, \tilde{x}^+_m(\mathbf{y},t) - \tilde{x}^+_m(\mathbf{x},t)] \ [\tilde{x}^+_m(\mathbf{y},t) - \tilde{x}^+_m(\mathbf{x},t)] = (y_m - x_m)\,(y_m - x_m) \ , \tag{11.8b}$$

for all $\mathbf{x},\mathbf{y}$ in the region occupied by the body at time t.

Since $\mathbf{x},\mathbf{y}$ are independent, we may differentiate (11.8) consecutively with respect to $\mathbf{x}$ and $\mathbf{y}$ to obtain

$$-2\,[\, \partial\tilde{\mathbf{x}}^+(\mathbf{x},t)/\partial\mathbf{x}\,]^T\,[\, \tilde{\mathbf{x}}^+(\mathbf{y},t) - \tilde{\mathbf{x}}^+(\mathbf{x},t)] = -2\,(\mathbf{y} - \mathbf{x}) \ , \tag{11.9a}$$

$$[\, \partial\tilde{\mathbf{x}}^+(\mathbf{x},t)/\partial\mathbf{x}\,]^T\,[\, \partial\tilde{\mathbf{x}}^+(\mathbf{y},t)/\partial\mathbf{y}\,] = \mathbf{I} \ , \tag{11.9b}$$

$$-2\,[\, \partial\tilde{x}^+_m(\mathbf{x},t)/\partial x_i\,]\,[\tilde{x}^+_m(\mathbf{y},t) - \tilde{x}^+_m(\mathbf{x},t)] = -2\,(y_i - x_i) \ , \tag{11.9c}$$

$$[\, \partial\tilde{x}^+_m(\mathbf{x},t)/\partial x_i\,]\,[\, \partial\tilde{x}^+_m(\mathbf{y},t)/\partial y_j\,] = \delta_{ij} \ . \tag{11.9d}$$

It follows from (11.9b) that the determinant of the tensor $\partial\tilde{\mathbf{x}}^+(\mathbf{x},t)/\partial\mathbf{x}$ does not vanish so that this tensor is invertible and (11.9b) may be rewritten in the alternative form

$$[\, \partial\tilde{\mathbf{x}}^+(\mathbf{x},t)/\partial\mathbf{x}\,]^T = [\, \partial\tilde{\mathbf{x}}^+(\mathbf{y},t)/\partial\mathbf{y}\,]^{-1} \ , \tag{11.10}$$

for all $\mathbf{x},\mathbf{y}$ in the region and all t. Thus, each side of the equation must be a tensor function of time only, say $\mathbf{Q}^T(t)$, so that

$$\partial\tilde{\mathbf{x}}^+(\mathbf{x},t)/\partial\mathbf{x} = \mathbf{Q}(t) \ , \tag{11.11}$$

for all $\mathbf{x}$ in the region and all time t. Since (11.11) is independent of $\mathbf{x}$ we also have

$$\partial\tilde{\mathbf{x}}^+(\mathbf{y},t)/\partial\mathbf{y} = \mathbf{Q}(t) \ , \tag{11.12}$$

so that (11.9b) yields

$$\mathbf{Q}^T(t)\,\mathbf{Q}(t) = \mathbf{I} \;,\; \det \mathbf{Q} = \pm 1 \;, \qquad (11.13a,b)$$

which shows that $\mathbf{Q}$ is an orthogonal tensor.

Since (11.7a) represents a superposed rigid body motion it must include the trivial motion

$$\tilde{\mathbf{x}}^+(\mathbf{x},t) = \mathbf{x} \;,\; \mathbf{Q} = \mathbf{I} \;,\; \det \mathbf{Q} = +1 \;. \qquad (11.14a,b,c)$$

Furthermore, since the motions are assumed to be continuous and det $\mathbf{Q}$ cannot vanish, we must <u>always</u> have

$$\det \mathbf{Q} = +1 \;, \qquad (11.15)$$

so that $\mathbf{Q}(t)$ is a proper orthogonal function of time only

$$\mathbf{Q}^T(t)\,\mathbf{Q}(t) = \mathbf{Q}(t)\,\mathbf{Q}^T(t) = \mathbf{I} \;,\; \det \mathbf{Q} = +1 \;. \qquad (11.15a,b)$$

Integrating (11.11) we obtain the general solution in the form

$$\mathbf{x}^+ = \tilde{\mathbf{x}}^+(\mathbf{x},t) = \mathbf{c}(t) + \mathbf{Q}(t)\,\mathbf{x} \;, \qquad (11.16)$$

where $\mathbf{c}(t)$ is an arbitrary function of time only. In (11.16) the function $\mathbf{c}(t)$ represents an arbitrary translation of the body and the function $\mathbf{Q}(t)$ represents an arbitrary rotation of the body.

By definition the superposed part of the motion defined by (11.16) is a rigid body motion. This means that the lengths of line elements are preserved and the angles between two line elements are also preserved so that

$$|\mathbf{x}^+ - \mathbf{y}^+|^2 = (\mathbf{x}^+ - \mathbf{y}^+) \bullet (\mathbf{x}^+ - \mathbf{y}^+) = \mathbf{Q}(\mathbf{x}-\mathbf{y}) \bullet \mathbf{Q}(\mathbf{x}-\mathbf{y}) = (\mathbf{x}-\mathbf{y}) \bullet \mathbf{Q}^T\mathbf{Q}(\mathbf{x}-\mathbf{y})$$
$$= (\mathbf{x}-\mathbf{y}) \bullet \mathbf{I}\,(\mathbf{x}-\mathbf{y}) = (\mathbf{x}-\mathbf{y}) \bullet (\mathbf{x}-\mathbf{y}) = |\mathbf{x}-\mathbf{y}|^2 \;, \qquad (11.17a)$$

$$\cos\theta^+ = \frac{(\mathbf{x}^+ - \mathbf{y}^+)}{|\mathbf{x}^+ - \mathbf{y}^+|} \bullet \frac{(\mathbf{x}^+ - \mathbf{z}^+)}{|\mathbf{x}^+ - \mathbf{z}^+|} = \frac{\mathbf{Q}(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \bullet \frac{\mathbf{Q}(\mathbf{x}-\mathbf{z})}{|\mathbf{x}-\mathbf{z}|}$$

$$= \frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \bullet \frac{\mathbf{Q}^T\mathbf{Q}(\mathbf{x}-\mathbf{z})}{|\mathbf{x}-\mathbf{z}|} = \frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \bullet \frac{(\mathbf{x}-\mathbf{z})}{|\mathbf{x}-\mathbf{z}|} = \cos\theta \qquad (11.17b)$$

where $\mathbf{x},\mathbf{y},\mathbf{z}$ are arbitrary points in the body which move to $\mathbf{x}^+,\mathbf{y}^+,\mathbf{z}^+$ under superposed rigid body motion (SRBM). Furthermore, this means that areas, and volumes are preserved under SRBM. To show this we use (11.16) with $\mathbf{x}=\hat{\mathbf{x}}(\mathbf{X},t)$ to calculate the deformation gradient $\mathbf{F}^+$ from the reference configuration to the superposed configuration

$$\mathbf{F}^+ = \partial\mathbf{x}^+/\partial\mathbf{X} = \mathbf{Q}(\partial\mathbf{x}/\partial\mathbf{X}) = \mathbf{QF} \quad , \tag{11.18}$$

so that from (7.20b), (7.34) and (11.18) we have

$$J^+ = \frac{dv^+}{dV} = \det \mathbf{F}^+ = \det(\mathbf{QF}) = (\det \mathbf{Q})(\det \mathbf{F}) = J \quad , \tag{11.19a}$$

$$\mathbf{n}^+ \, da^+ = d\mathbf{x}^{1+} \times d\mathbf{x}^{2+} = J^+(\mathbf{F}^+)^{-T}\mathbf{N} \, dA = J \, \mathbf{QF}^{-T}\mathbf{N} \, dA = \mathbf{Qn} \, da \; , \tag{11.19b}$$

$$(da^+)^2 = \mathbf{n}^+ da^+ \bullet \mathbf{n}^+ da^+ = \mathbf{Qn} \, da \bullet \mathbf{Qn} \, da = \mathbf{n} \bullet \mathbf{Q}^T\mathbf{Qn} \, (da)^2 = (da)^2 \quad , \tag{11.19c}$$

$$\mathbf{n}^+ = \mathbf{Qn} \quad . \tag{11.19d}$$

For later convenience it is desirable to calculate expressions for the velocity and rate of deformation tensors associated with the superposed configuration. To this end, we take the material derivative of (11.13a) to deduce that

$$\dot{\mathbf{Q}}^T\mathbf{Q} + \mathbf{Q}^T\dot{\mathbf{Q}} = 0 \;\; \Rightarrow \;\; \dot{\mathbf{Q}} = \mathbf{\Omega}\,\mathbf{Q} \;\; , \;\; \mathbf{\Omega}^T = -\,\mathbf{\Omega} \quad , \tag{11.20a,b,c}$$

where $\mathbf{\Omega}$ is a skew-symmetric tensor function of time only. Letting $\boldsymbol{\omega}$ be the axial vector of $\mathbf{\Omega}$ we recall that for an arbitrary vector $\mathbf{a}$

$$\mathbf{\Omega}\,\mathbf{a} = \boldsymbol{\omega} \times \mathbf{a} \quad . \tag{11.21}$$

Thus, by taking the material derivative of (11.16) we may calculate the velocity $\mathbf{v}^+$ of the material point in the superposed configuration

$$\mathbf{v}^+ = \dot{\mathbf{x}}^+ = \dot{\mathbf{c}} + \dot{\mathbf{Q}}\,\mathbf{x} + \mathbf{Q}\,\dot{\mathbf{x}} = \dot{\mathbf{c}} + \mathbf{\Omega}\,\mathbf{Q}\,\mathbf{x} + \mathbf{Q}\,\mathbf{v} \quad , \tag{11.22a}$$

$$\mathbf{v}^+ = \dot{\mathbf{c}} + \mathbf{\Omega}\,(\mathbf{x}^+ - \mathbf{c}) + \mathbf{Q}\,\mathbf{v} = \dot{\mathbf{c}} + \boldsymbol{\omega} \times (\mathbf{x}^+ - \mathbf{c}) + \mathbf{Q}\,\mathbf{v} \quad . \tag{11.22b}$$

It follows that the velocity gradient $\mathbf{L}^+$ and rate of deformation tensors $\mathbf{D}^+$ and $\mathbf{W}^+$ associated with the superposed configuration are given by

$$\mathbf{L}^+ = \partial\mathbf{v}^+/\partial\mathbf{x}^+ = \mathbf{Q}(\partial\mathbf{v}/\partial\mathbf{x})(\partial\mathbf{x}/\partial\mathbf{x}^+) + \mathbf{\Omega} = \mathbf{QLQ}^T + \mathbf{\Omega} \quad , \tag{11.23a}$$

$$\mathbf{D}^+ = \mathbf{QDQ}^T \;\; , \;\; \mathbf{W}^+ = \mathbf{QWQ}^T + \mathbf{\Omega} \quad , \tag{11.23b,c}$$

where we have used the condition (11.20c) and have differentiated (11.16) to obtain

$$\partial\mathbf{x}^+/\partial\mathbf{x} = \mathbf{Q} \;\; , \;\; \partial\mathbf{x}/\partial\mathbf{x}^+ = \mathbf{Q}^T \quad . \tag{11.24a,b}$$

Up to this point we have been discussing superposed rigid body motions that are in addition to the general motion $\hat{\mathbf{x}}(\mathbf{X},t)$ of a deformable body. However, the kinematics of rigid body motions may be obtained as a special case by identifying $\mathbf{x}$ with its value $\mathbf{X}$ in the fixed reference configuration so that distortion and dilatation of the body are eliminated and (11.22b) yields

$$\mathbf{x} = \mathbf{X} \;\Rightarrow\; \dot{\mathbf{x}}^+ = \dot{\mathbf{c}} + \boldsymbol{\omega} \times (\mathbf{x}^+ - \mathbf{c}) \,. \tag{11.25}$$

In this form it is easy to recognize that $\mathbf{c}(t)$ represents the translation of a point moving with the rigid body and $\boldsymbol{\omega}$ is the angular velocity of the rigid body.

12. Material Line, Material Surface and Material Volume

Recall that a material point Y is mapped into its location $\mathbf{X}$ in the reference configuration. Since this mapping is independent of time, lines, surfaces, and volumes which remain constant in the reference configuration always contain the same material points and therefore are called material.

Material Line: A material line is a fixed curve in the reference configuration that may be parameterized by its archlength S which is independent of time. It follows that the Lagrangian representation of a material line becomes

$$\mathbf{X} = \mathbf{X}(S) \ . \tag{12.1}$$

Alternatively, using the mapping (6.5) we may determine the Eulerian representation of the same material line in the form

$$\mathbf{x} = \mathbf{x}(S,t) = \hat{\mathbf{x}}(\mathbf{X}(S),t) \ . \tag{12.2}$$

Material Surface: A material surface is a fixed surface in the reference configuration that may be parameterized by two coordinates S_1 and S_2 that are independent of time. It follows that the Lagrangian representation of a material surface becomes

$$\mathbf{X} = \mathbf{X}(S_1,S_2) \quad \text{or} \quad \hat{F}(\mathbf{X}) = 0 \ . \tag{12.3a,b}$$

Alternatively, using the mapping (6.5) and its inverse (6.6) we may determine the Eulerian representation of the same material surface in the form

$$\mathbf{x} = \mathbf{x}(S_1,S_2,t) = \hat{\mathbf{x}}(\mathbf{X}(S_1,S_2),t) \quad \text{or} \quad \tilde{F}(\mathbf{x},t) = \hat{F}(\tilde{\mathbf{X}}(\mathbf{x},t)) = 0 \ . \tag{12.4a,b}$$

Lagrange's criterion for a material surface: The surface defined by the constraint $\tilde{f}(\mathbf{x},t)=0$ is material if and only if

$$\dot{\tilde{f}} = \frac{\partial \tilde{f}}{\partial t} + \partial \tilde{f}/\partial \mathbf{x} \bullet \mathbf{v} = 0 \ . \tag{12.5}$$

Proof: In general we can use the mapping (6.5) to deduce that

$$\hat{f}(\mathbf{X},t) = \tilde{f}(\hat{\mathbf{x}}(\mathbf{X},t),t) \ . \tag{12.6}$$

It follows from (12.5) and (12.6) that

$$\dot{\hat{f}}(\mathbf{X},t) = \frac{\partial \hat{f}}{\partial t} = \dot{\tilde{f}} = 0 \ , \tag{12.7}$$

so that $\hat{f}$ is independent of time and the surface $\hat{f}=0$ is fixed in the reference configuration and thus $\hat{f}=\tilde{f}=0$ characterizes a material surface. Alternatively, if $\hat{f}$ is independent of time then $\dot{\hat{f}}=0$ and $\dot{\tilde{f}}=0$.

Material Region: A material region is a region of space bounded by a closed material surface. For example if ∂P_0 is a closed material surface in the reference configuration then the region of space P_0 enclosed by ∂P_0 is a material region that contains the same material points for all time. Alternatively, using the mapping (6.5) each point of the material surface ∂P_0 maps into a point on the closed material surface ∂P in the present configuration so the region P enclosed by ∂P is the associated material region in the present configuration.

13. The Transport Theorem

In this section we develop the transport theorem that allows us to calculate the time derivative of the integral over a material region P in the present configuration whose closed boundary ∂P is changing with time. By way of introduction let us consider the simpler one-dimensional case and recall that

$$\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} f(x,t)\,dx = \int_{\alpha(t)}^{\beta(t)} \frac{\partial f(x,t)}{\partial t}\,dx + f(\beta(t),t)\,\dot{\beta} - f(\alpha(t),t)\,\dot{\alpha} \quad , \tag{13.1}$$

where f(x,t) is an arbitrary function of position x and time t, and α(t),β(t) define the changing boundaries of integration. What is important to notice is that the rate of change of the boundaries enter in the calculation in (13.1).

To develop the generalization of (13.1) to three dimensions it is most convenient to consider an arbitrary scalar or tensor valued function φ which admits the representations

$$\phi = \tilde{\phi}(\mathbf{x},t) = \hat{\phi}(\mathbf{X},t) \quad . \tag{13.2}$$

By mapping the material region P from the present configuration back to the reference configuration P_0 we may easily calculate the derivative of the integral of φ over the changing region P as follows

$$\frac{d}{dt}\int_P \tilde{\phi}(\mathbf{x},t)\,dv = \frac{d}{dt}\int_{P_0} \hat{\phi}(\mathbf{X},t)\,J\,dV \quad , \tag{13.3a}$$

$$= \int_{P_0} \frac{\partial}{\partial t}\{\hat{\phi}(\mathbf{X},t)\,J\}\Big|_{\mathbf{X}}\,dV \quad , \tag{13.3b}$$

$$= \int_{P_0} \overline{\{\hat{\phi}(\mathbf{X},t)\,J\}}^{\bullet}\,dV \quad , \tag{13.3c}$$

$$= \int_{P_0} \{\dot{\phi}\,J + \hat{\phi}\,\dot{J}\}\,dV \quad , \tag{13.3d}$$

$$= \int_{P_0} \{\dot{\phi} + \hat{\phi}\ \mathrm{div}\,\mathbf{v}\}\,J\,dV \quad , \tag{13.3e}$$

so that in summary we have

$$\frac{d}{dt}\int_P \tilde{\phi}(\mathbf{x},t)\,dv = \int_P \{\dot{\phi} + \tilde{\phi}\ \text{div}\,\mathbf{v}\}\,dv \ , \tag{13.4}$$

where $\dot{\phi}$ is the usual material derivative of ϕ

$$\dot{\phi} = \frac{\partial\hat{\phi}(\mathbf{X},t)}{\partial t} = \frac{\partial\tilde{\phi}(\mathbf{x},t)}{\partial t} + \partial\tilde{\phi}(\mathbf{x},t)/\partial\mathbf{x} \bullet \mathbf{v} \ . \tag{13.5}$$

Now, substituting (13.5) into (13.4) and using the divergence theorem we have

$$\frac{d}{dt}\int_P \tilde{\phi}(\mathbf{x},t)\,dv = \int_P \Big\{\frac{\partial\tilde{\phi}(\mathbf{x},t)}{\partial t} + \partial\tilde{\phi}(\mathbf{x},t)/\partial\mathbf{x} \bullet \mathbf{v} + \tilde{\phi}\ \text{div}\,\mathbf{v}\Big\}\,dv \ , \tag{13.6a}$$

$$= \int_P \frac{\partial\tilde{\phi}(\mathbf{x},t)}{\partial t}\,dv + \int_P \text{div}\{\tilde{\phi}\,\mathbf{v}\}\,dv \ , \tag{13.6b}$$

$$= \int_P \frac{\partial\tilde{\phi}(\mathbf{x},t)}{\partial t}\,dv + \int_{\partial P} \tilde{\phi}\,\mathbf{v} \bullet \mathbf{n}\,da \ , \tag{13.6c}$$

where $\mathbf{n}$ is the unit outward normal to the material surface ∂P. It should be emphasized that the time differentiation and the integration operations commute in (13.3b) because the region P_0 is independent of time. In contrast, the time differentiation and the integration operations in (13.6c) do not commute because the region P depends on time. However, sometimes in fluid mechanics the region P in space at time t is considered to be a <u>control volume</u> and is identified as the fixed region $\bar{P}$ with boundary $\partial\bar{P}$ and the time differentiation is interchanged with the integration operations to obtain

$$\frac{d}{dt}\int_P \tilde{\phi}(\mathbf{x},t)\,dv = \frac{\partial}{\partial t}\int_{\bar{P}} \tilde{\phi}(\mathbf{x},t)\,dv + \int_{\partial\bar{P}} \tilde{\phi}\,\mathbf{v} \bullet \mathbf{n}\,da \ . \tag{13.7}$$

However, in (13.7) it is <u>essential</u> to interpret the partial differentiation operation as differentiation with respect to time holding $\mathbf{x}$ fixed. To avoid possible confusion it is preferable to use the form (13.6c) instead of (13.7).

14. Conservation of Mass

Recall from (6.17) and (6.18) that the mass M(P) of the part P in the present configuration and the mass $M(P_0)$ of the part P_0 in the reference configuration are determined by integrating the mass densities ρ and ρ_0, respectively. The conservation of mass states that mass of a material region remains constant. Since the material region P_0 in the reference configuration is mapped into the material region P in the present configuration it follows that the conservation of mass requires

$$\int_P \rho \, dv = \int_{P_0} \rho_0 \, dV \quad , \tag{14.1}$$

for every part P (or P_0) of the body. Furthermore, since P_0 and ρ_0 are independent of time we may also write

$$\frac{d}{dt}\int_P \rho \, dv = 0 \quad . \tag{14.2}$$

The equations (14.1) and (14.2) are called global equations because they are stated with reference to a finite region of space. In order to derive the local forms of these equations we first recall that by using (7.20b) the integral over P may be converted to an integral over P_0 such that

$$\int_P \rho \, dv = \int_{P_0} \rho \, J \, dV \quad . \tag{14.3}$$

It then follows that the statement (14.1) may be rewritten in the form

$$\int_{P_0} [\rho J - \rho_0] \, dV = 0 \quad . \tag{14.4}$$

Now, assuming that the integrand in (14.4) is a continuous function of space and assuming that (14.4) holds for all arbitrary parts P_0 of the body we may use the theorem proved in Appendix B to deduce that

$$\rho \, J = \rho_0 \quad , \tag{14.5}$$

at every point of the body. The form (14.5) is the Lagrangian representation of the local form of conservation of mass. It is considered a local form because it holds at every point in the body.

Alternatively, we may use the transport theorem (13.4) to rewrite (14.2) in the form

$$\int_P [\dot{\rho} + \rho \operatorname{div} \mathbf{v}]\, dv = 0 \ . \tag{14.6}$$

Now, assuming that the integrand in (14.6) is a continuous function of space and assuming that (14.6) holds for all arbitrary parts P of the body we may use the theorem proved in Appendix B to deduce that

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0 \ , \tag{14.7}$$

at every point of the body. The form (14.7) is the Eulerian representation of the local form of conservation of mass. Note that the result (14.7) may also be deduced directly from (14.5) by using equation (P4.3) and the condition that $\dot{\rho}_0$=0.

For later convenience we use the transport theorem (13.4) with $\phi = \rho f$ to deduce that

$$\frac{d}{dt}\int_P \rho\, f\, dv = \int_P [\, \overline{\rho\, f}^{\,\bullet} + \rho\, f \operatorname{div} \mathbf{v}\,]\, dv$$

$$= \int_P [\, \rho\, \dot{f} + f\,(\dot{\rho} + \rho \operatorname{div} \mathbf{v})]\, dv \ . \tag{14.8}$$

Now using the local form (14.7) of conservation of mass, equation (14.8) reduces to

$$\frac{d}{dt}\int_P \rho\, f\, dv = \int_P [\, \rho\, \dot{f}\,]\, dv \ . \tag{14.9}$$

15. Balances of Linear and Angular Momentum

In the previous section we discussed the conservation of mass equation, which can be thought of as an equation to determine the mass density ρ. For the purely mechanical theory it is necessary to add two additional balance laws called the balances of linear and angular momentum.

Balance of Linear Momentum: In words the balance of linear momentum states that the rate of change of linear momentum of an arbitrary part P of a body is equal to the total external force applied to that part of the body. These external forces are separated into two types: body forces which act at each point of the part P and surface tractions that act at each point of the surface ∂P of P. The body force per unit mass is denoted by the vector $\mathbf{b}$ and the surface traction per unit area is denoted by the stress vector $\mathbf{t}(\mathbf{n})$, which depends explicitly on the unit outward normal $\mathbf{n}$ to the surface ∂P. Then, the global form of the balance of linear momentum may be expressed as

$$\frac{d}{dt}\int_P \rho\,\mathbf{v}\,dv = \int_P \rho\,\mathbf{b}\,dv + \int_{\partial P} \mathbf{t}(\mathbf{n})\,da \;, \tag{15.1}$$

where ρ is the mass density and the velocity $\mathbf{v}$ is the linear momentum per unit mass.

Balance of Angular Momentum: In words the balance of angular momentum states that the rate of change of angular momentum of an arbitrary part P of a body is equal to the total external moment applied to that part of the body by the body force and the surface tractions. In this statement the angular momentum and the moment are referred to an arbitrary but fixed point. Letting $\mathbf{x}$ be the position vector relative to a fixed origin of an arbitrary point in P, the global form of the balance of angular momentum may be expressed as

$$\frac{d}{dt}\int_P \mathbf{x}\times\rho\,\mathbf{v}\,dv = \int_P \mathbf{x}\times\rho\,\mathbf{b}\,dv + \int_{\partial P} \mathbf{x}\times\mathbf{t}(\mathbf{n})\,da \;, \tag{15.2}$$

16. Existence of the Stress Tensor

Consider an arbitrary part P of the body with closed boundary ∂P and let P be divided by a material surface s into two parts P_1 and P_2 with closed boundaries ∂P_1 and ∂P_2, respectively. Furthermore, let the intersection of ∂P_1 and ∂P be denoted by $\partial P'$ and the intersection of ∂P_2 and ∂P be denoted by $\partial P''$ (see Fig. 16.1). Mathematically, we may summarize these definitions by

$$P = P_1 \cup P_2 \ , \ \ \partial P' = \partial P_1 \cap \partial P \ , \ \ \partial P'' = \partial P_2 \cap \partial P \ , \tag{16.1a,b,c}$$

$$\partial P = \partial P' \cup \partial P'' \ , \ \ \partial P_1 = \partial P' \cup s \ , \ \ \partial P_2 = \partial P'' \cup s \ . \tag{16.1d,e,f}$$

Also, let $\mathbf{n}$ be the unit normal to the surface s measured outward from the part P_1 (see Fig. 16.1).

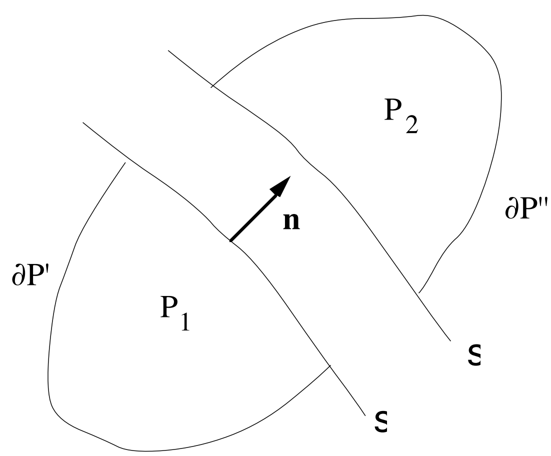

Fig. 16.1 Parts P_1 and P_2 of an arbitrary part P of a body

Now recall that the balance of linear momentum is assumed to apply to an arbitrary part of the body so its application to the parts P, P_1 and P_2 yields

$$\frac{d}{dt}\int_P \rho\, \mathbf{v}\, dv - \int_P \rho\, \mathbf{b}\, dv - \int_{\partial P} \mathbf{t}(\mathbf{n})\, da = 0 \ , \tag{16.2a}$$

$$\frac{d}{dt}\int_{P_1} \rho\, \mathbf{v}\, dv - \int_{P_1} \rho\, \mathbf{b}\, dv - \int_{\partial P_1} \mathbf{t}(\mathbf{n})\, da = 0 \ , \tag{16.2b}$$

$$\frac{d}{dt}\int_{P_2} \rho\, \mathbf{v}\, dv - \int_{P_2} \rho\, \mathbf{b}\, dv - \int_{\partial P_2} \mathbf{t}(\mathbf{n})\, da = 0 \ , \tag{16.2c}$$

where $\mathbf{n}$ in (16.2a,b,c) is considered to be the unit outward normal to the part and is not to be confused with the particular definition of $\mathbf{n}$ associated with the surface S. Since the regions P,P_1,P_2 are material and since the local form (14.7) of the conservation of mass is assumed to hold in each of these parts, the result (14.9) may be used to deduce that

$$\frac{d}{dt}\int_P \rho\,\mathbf{v}\,dv = \int_P \rho\,\dot{\mathbf{v}}\,dv = \int_{P_1} \rho\,\dot{\mathbf{v}}\,dv + \int_{P_2} \rho\,\dot{\mathbf{v}}\,dv \ , \tag{16.3a}$$

$$\frac{d}{dt}\int_{P_1} \rho\,\mathbf{v}\,dv = \int_{P_1} \rho\,\dot{\mathbf{v}}\,dv \ , \quad \frac{d}{dt}\int_{P_2} \rho\,\mathbf{v}\,dv = \int_{P_2} \rho\,\dot{\mathbf{v}}\,dv \ . \tag{16.3b,c}$$

Also, using (16.1) we obtain

$$\int_P \rho\,\mathbf{b}\,dv = \int_{P_1} \rho\,\mathbf{b}\,dv + \int_{P_2} \rho\,\mathbf{b}\,dv \ , \tag{16.4a}$$

$$\int_{\partial P} \mathbf{t}(\mathbf{n})\,da = \int_{\partial P'} \mathbf{t}(\mathbf{n})\,da + \int_{\partial P''} \mathbf{t}(\mathbf{n})\,da \ , \tag{16.4b}$$

$$\int_{\partial P_1} \mathbf{t}(\mathbf{n})\,da = \int_{\partial P'} \mathbf{t}(\mathbf{n})\,da + \int_S \mathbf{t}(\mathbf{n})\,da \ , \tag{16.4c}$$

$$\int_{\partial P_2} \mathbf{t}(\mathbf{n})\,da = \int_{\partial P''} \mathbf{t}(\mathbf{n})\,da + \int_S \mathbf{t}(-\mathbf{n})\,da \ , \tag{16.4d}$$

where in (16.4d) we note that the unit outward normal to S when it is considered a part of P_2 is $(-\mathbf{n})$. Thus, with the help of (16.3) and (16.4) the equations (16.2) may be rewritten in the forms

$$\begin{aligned}&\Big[\int_{P_1} \rho\,\dot{\mathbf{v}}\,dv - \int_{P_1} \rho\,\mathbf{b}\,dv - \int_{\partial P'} \mathbf{t}(\mathbf{n})\,da\Big] \\ &+ \Big[\int_{P_2} \rho\,\dot{\mathbf{v}}\,dv - \int_{P_2} \rho\,\mathbf{b}\,dv - \int_{\partial P'} \mathbf{t}(\mathbf{n})\,da\Big] = 0 \ ,\end{aligned} \tag{16.5a}$$

$$\int_{P_1} \rho\,\dot{\mathbf{v}}\,dv - \int_{P_1} \rho\,\mathbf{b}\,dv - \int_{\partial P'} \mathbf{t}(\mathbf{n})\,da - \int_S \mathbf{t}(\mathbf{n})\,da = 0 \ , \tag{16.5b}$$

$$\int_{P_2} \rho\,\dot{\mathbf{v}}\,dv - \int_{P_2} \rho\,\mathbf{b}\,dv - \int_{\partial P'} \mathbf{t}(\mathbf{n})\,da - \int_S \mathbf{t}(-\mathbf{n})\,da = 0 \ . \tag{16.5c}$$

Next we subtract (16.5b) and (16.5c) from (16.5a) to deduce that

$$\int_S \left[\mathbf{t}\,(\mathbf{n}) + \mathbf{t}\,(-\mathbf{n})\right] \mathrm{da} = 0 \ . \tag{16.6}$$

Since (16.6) must hold for arbitrary material surfaces s and since we assume that the integrand is a continuous function of points on s it follows by a result similar to that developed in Appendix B that

$$\mathbf{t}\,(-\,\mathbf{n}) = -\,\mathbf{t}\,(\mathbf{n}) \ , \tag{16.7}$$

must hold for all points on s. Note that this result, which is called Cauchy's Lemma, is the analogue of Newton's law of action and reaction because it states that the stress vector applied by part P_2 on part P_1 is equal in magnitude and opposite in direction to the stress vector applied by part P_1 on part P_2.

In general, the stress vector **t** is a function of position **x**, time t, and the unit outward normal **n** to the surface on which it is applied

$$\mathbf{t} = \mathbf{t}\,(\mathbf{x},t\,;\,\mathbf{n}) \ . \tag{16.8}$$

It follows that the state of stress at a point **x** and at time t must be determined by the infinite number of stress vectors obtained by considering all possible orientations (**n**) of planes passing through **x** at time t. However, it is not necessary to consider all possible orientations. To verify this statement we first note that the simplest polyhedron is a tetrahedron with four faces. Secondly, we note that any three-dimensional region of space can by approximated to any degree of accuracy using a finite collection of tetrahedrons. Therefore, if we can analyze the state of stress in a simple tetrahedron we can in principle analyze the stress at a point in an arbitrary body. To this end, consider the tetrahedron with three faces that are perpendicular to the Cartesian base vectors $\mathbf{e}_i$, and whose fourth face is defined by the unit outward normal vector **n** (see Fig. 16.2). Let: the vertex D (Fig. 16.2) be located at an arbitrary point **y** in the part P of the body; the surfaces perpendicular to $\mathbf{e}_i$ have surface areas S_i, respectively; and the slanted surface whose normal is **n** have surface area S.

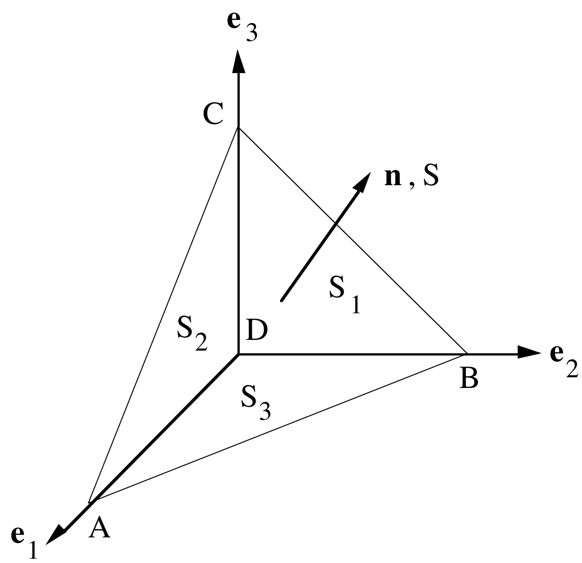

Fig. 16.2 An Elemental Tetrahedron

Denoting $\mathbf{x}_{AD}$, $\mathbf{x}_{BD}$, $\mathbf{x}_{CD}$ as the vectors from the vertex D to the vertices A,B,C, respectively, it follows by vector algebra that

$$2\,S\,\mathbf{n} = (\mathbf{x}_{BD} - \mathbf{x}_{AD}) \times (\mathbf{x}_{CD} - \mathbf{x}_{AD}) \ , \tag{16.9a}$$

$$2\,S\,\mathbf{n} = (\mathbf{x}_{BD} \times \mathbf{x}_{CD}) + (\mathbf{x}_{CD} \times \mathbf{x}_{AD}) + (\mathbf{x}_{AD} \times \mathbf{x}_{BD}) \ , \tag{16.9b}$$

$$2\,S\,\mathbf{n} = 2\,S_1\,\mathbf{e}_1 + 2\,S_2\,\mathbf{e}_2 + 2\,S_3\,\mathbf{e}_3 \ , \tag{16.9c}$$

so that the areas S_j may be related to S and **n** by the formula

$$S_j = \mathbf{e}_j \bullet S\,\mathbf{n} = S\,n_j \ , \tag{16.10}$$

where n_j are the Cartesian components of **n**. Also, the volume of the tetrahedron is given by

$$V_{tet} = \frac{1}{6}\ (\mathbf{x}_{BD}-\mathbf{x}_{AD}) \times (\mathbf{x}_{CD}-\mathbf{x}_{AD}) \bullet \mathbf{x}_{CD} = \frac{1}{6}\ (2\,S\,\mathbf{n}) \bullet \mathbf{x}_{CD} = \frac{1}{3}\ S\,h \ , \tag{16.11}$$

where we have used (16.9a). In (16.11) S is the area of the slanted side ABC of the tetrahedron and h is the height of the tetrahedron measured normal to the slanted side.

Now with the help of the result (14.9) the balance of linear momentum may be written in the form

$$\int_P \rho \{\dot{\mathbf{v}} - \mathbf{b}\} \, dv = \int_{\partial P} \mathbf{t}(\mathbf{n}) \, da \ . \tag{16.12}$$

Then taking P to be the region of the tetrahedron the balance of linear momentum (16.12) becomes

$$\int_P \rho \{\dot{\mathbf{v}} - \mathbf{b}\} \, dv = \int_S \mathbf{t}(\mathbf{n}) \, da + \sum_{j=1}^{3} \int_{S_j} \mathbf{t}(-\mathbf{e}_j) \, da \ . \tag{16.13}$$

However, by Cauchy's Lemma (16.7)

$$\mathbf{t}(-\mathbf{e}_j) = -\mathbf{t}(\mathbf{e}_j) \ . \tag{16.14}$$

Defining the three vectors $\mathbf{T}_j$ to be the stress vectors applied to the surfaces whose outward normals are $\mathbf{e}_j$

$$\mathbf{T}_j = \mathbf{t}(\mathbf{e}_j) \ , \tag{16.15}$$

we may rewrite (16.13) in the form

$$\int_P \rho \{\dot{\mathbf{v}} - \mathbf{b}\} \, dv = \int_S \mathbf{t}(\mathbf{n}) \, da - \sum_{j=1}^{3} \int_{S_j} \mathbf{T}_j \, da \ . \tag{16.16}$$

Assuming that the term $\rho(\dot{\mathbf{v}} - \mathbf{b})$ is bounded and recalling that

$$\left| \int_P f \, dv \right| \leq \int_P | f | \, dv \ , \tag{16.17}$$

it follows that there exists a positive constant K such that

$$\left| \int_P \rho \{\dot{\mathbf{v}} - \mathbf{b}\} \, dv \right| \leq \int_P \left| \rho \{\dot{\mathbf{v}} - \mathbf{b}\} \right| dv$$

$$\leq \int_P K \, dv = K \int_P dv = K \frac{1}{3} Sh \ . \tag{16.18}$$

Further, assuming that the stress vector is a continuous function of position **x** and the normal **n**, the mean value theorem for integrals states that there exist points on the surfaces S, S_i for which the values $\mathbf{t}^*(\mathbf{n}), \mathbf{T}_j^*$ of the quantities $\mathbf{t}(\mathbf{n})$, $\mathbf{T}_j$ evaluated at these points are related to the integrals such that

$$\int_S \mathbf{t}(\mathbf{n}) \, da = \mathbf{t}^*(\mathbf{n}) \, S \ , \quad \sum_{j=1}^{3} \int_{S_j} \mathbf{T}_j \, da = \mathbf{T}_j^* \, S_j = \mathbf{T}_j^* \, n_j \, S \ , \tag{16.19a,b}$$

where we have used the result (16.10) and summation is implied over the repeated index j. Then with the help of (16.16) and (16.19) equation (16.18) yields the result that

$$|\mathbf{t}^*(\mathbf{n}) - \mathbf{T}_j^* n_j| \leq \frac{1}{3}\,Kh \ , \tag{16.20}$$

where we have divided by the positive area S. Now, considering the set of similar tetrahedrons with the same vertex and with diminishing heights h it follows from (16.20) that as h approaches zero we may deduce that

$$\mathbf{t}^*(\mathbf{n}) = \mathbf{T}_j^{\,*} n_j \ . \tag{16.21}$$

However, in this limit all functions are evaluated at the same point **x** so we may suppress the star notation and write

$$\mathbf{t}(\mathbf{n}) = \mathbf{T}_j\, n_j \ . \tag{16.22}$$

Also, since **x** was an arbitrary point in the above argument it follows that (16.22) must hold for all points **x** and all normals **n**.

In words the result (16.22) states that the stress vector on an arbitrary surface may be expressed as a linear combination of the stress vectors applied to the surfaces whose unit normals are in the coordinate directions $\mathbf{e}_j$, and that the coefficients are the components of the normal **n**. Notice that by introducing the definition

$$\mathbf{T} = \mathbf{T}_j \otimes \mathbf{e}_j \ , \tag{16.23}$$

equations (16.15) and (16.22) may be written in the alternative forms

$$\mathbf{T}_j = \mathbf{T}\,\mathbf{e}_j \ , \ \ \mathbf{t}(\mathbf{n}) = \mathbf{T}\,\mathbf{n} \ , \tag{16.24a,b}$$

It follows from (16.24b) that since **T** transforms an arbitrary vector **n** into a vector **t**, **T** must be a second order tensor. This tensor **T** is called the Cauchy stress tensor and its Cartesian components T_{ij} are given by

$$T_{ij} = (\mathbf{e}_i \otimes \mathbf{e}_j) \bullet \mathbf{T} = \mathbf{e}_i \bullet \mathbf{T}_j \ , \tag{16.25}$$

so that the component form of (16.24) becomes

$$t_i = T_{ij}\, n_j \ , \tag{16.26}$$

where t_i are the Cartesian components of **t**. Furthermore, in view of (16.15) it follows that components T_{ij} of $\mathbf{T}_j$ are the components of the stress vectors on the surfaces whose outward normals are $\mathbf{e}_j$ (see Fig. 16.3) and that the first index i of T_{ij} refers to the direction of the component of the stress vector and the second index j of T_{ij} refers to the plane on which the stress vector acts.

It is important to emphasize that the stress tensor **T**(**x**,t) is a function of position **x** and time t and in particular is independent of the normal **n**. Therefore the state of stress at a point is characterized by the stress tensor **T**. On the other hand, the stress vector **t**(**x**,t ;**n**) includes an explicit dependence on the normal **n** and characterizes the force per unit area acting on the particular plane defined by **n** that passes through the point **x** at time t.

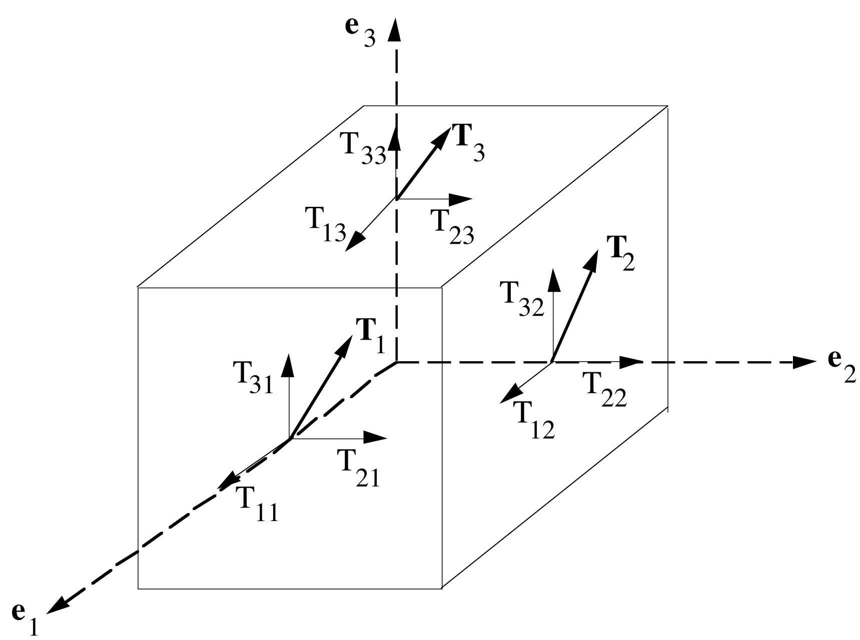

Fig. 16.3 Components of Stress Tensor

The stress vector **t** on any surface can be separated into a component normal to the surface and a component parallel to the surface such that

$$\mathbf{t} = \mathbf{t}_n + \mathbf{t}_s \ , \ \mathbf{t}_n = \sigma\,\mathbf{n} \ , \ \mathbf{t}_s = \tau\,\mathbf{s} \ , \tag{16.27a,b,c}$$

where the normal component σ, the magnitude of the shearing component τ, and the shearing direction **s** are defined by

$$\sigma = \mathbf{t} \bullet \mathbf{n} \ , \ \tau = |\,\mathbf{t}_s\,| = \left[\mathbf{t} \bullet \mathbf{t} - \sigma^2\right]^{1/2} \ , \tag{16.28a,b}$$

$$\mathbf{s} = \frac{\mathbf{t}_s}{\tau} = \frac{\mathbf{t} - \sigma\,\mathbf{n}}{\tau} \ , \ \mathbf{s} \bullet \mathbf{s} = 1 \ . \tag{16.28c,d}$$

It is important to note that σ and τ are functions of the state of the material through the value of the stress tensor **T** at the point of interest and are functions of the normal **n** to the plane of interest.

Sometimes a failure criterion for a brittle material is formulated in terms of a critical value of tensile stress whereas a failure criterion (like the Tresca condition) for a metal is formulated in terms of a critical value of the shear stress. Consequently, it is natural to determine the maximum values of the normal stress σ and the shear stress τ. To this end the equations (16.28a,b) are rewritten in the forms

$$\sigma = \mathbf{T} \bullet (\mathbf{n} \otimes \mathbf{n}) \ , \ \tau^2 = \mathbf{T}^2 \bullet (\mathbf{n} \otimes \mathbf{n}) - \sigma^2 \ . \qquad (16.29a,b)$$

Then, it is necessary to search for critical values of σ and τ as functions **n**. However, it is important to remember that the components of **n** are not independent because **n** must be a unit vector

$$\mathbf{n} \bullet \mathbf{n} = 1 \ . \qquad (16.30)$$

Appendix C reviews the method of Lagrange Multipliers which is used to determine critical values of functions subject to constrains, and Appendix D determines the critical values of σ and τ. In particular, it is recalled that the critical values of σ occur on the planes whose normals are in the principal directions of the stress tensor **T**. Also, letting $\{\sigma_1, \sigma_2, \sigma_3\}$ be the ordered principal values of **T** and $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ be the associated principal directions of **T**

$$\mathbf{T}\,\mathbf{p}_1 = \sigma_1\,\mathbf{p}_1 \ , \ \mathbf{T}\,\mathbf{p}_2 = \sigma_2\,\mathbf{p}_2 \ , \mathbf{T}\,\mathbf{p}_3 = \sigma_3\,\mathbf{p}_3 \ , \qquad (16.31a,b,c)$$

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \ , \qquad (16.31d)$$

it can be shown that σ is bounded by the values σ_1 and σ_3

$$\sigma_1 \geq \sigma \geq \sigma_3 \ . \qquad (16.32)$$

Therefore, the maximum value of tensile stress σ equals σ_1 and it occurs on the plane whose normal is in the direction $\mathbf{p}_1$. Moreover, it can be shown that the stress vector acting on this critical plane has no shearing component

$$\mathbf{t} = \sigma_1\,\mathbf{n} \ , \ \sigma = \sigma_1 \ , \ \tau = 0 \quad \text{for} \quad \mathbf{n} = \pm\,\mathbf{p}_1 \ . \qquad (16.33a,b,c)$$

In Appendix D it is also shown that the maximum shear stress τ_{max} occurs on a plane which bisects the planes defined by the maximum tensile stress $\mathbf{p}_1$ and the minimum tensile stress $\mathbf{p}_3$ such that

$$\sigma = \frac{\sigma_1 + \sigma_3}{2} \quad , \quad \tau_{max} = \tau = \frac{\sigma_1 - \sigma_3}{2} \quad \text{for } \mathbf{n} = \pm \frac{1}{\sqrt{2}} (\mathbf{p}_1 \pm \mathbf{p}_3). \qquad (16.34a,b)$$

Notice that on this plane the normal stress σ does not necessarily vanish so that the stress vector $\mathbf{t}$ does not apply a pure shear stress on the plane where τ is maximum.

17. Local Forms of Balance Laws

Assuming sufficient continuity and using the local form of conservation of mass together with the result (14.9) we may deduce that

$$\frac{d}{dt}\int_P \rho\, \mathbf{v}\, dv = \int_P \rho\, \dot{\mathbf{v}}\, dv \ , \tag{17.1a}$$

$$\frac{d}{dt}\int_P \mathbf{x} \times \rho\, \mathbf{v}\, dv = \int_P \rho\, \overline{\mathbf{x} \times \mathbf{v}}^{\bullet}\, dv = \int_P \mathbf{x} \times \rho\, \dot{\mathbf{v}}\, dv \ . \tag{17.1b}$$

Also, we may use the relationship (16.24) between the stress vector **t**, the stress tensor **T**, and the unit normal **n** together with the divergence theorem (3.46) to obtain

$$\int_{\partial P} \mathbf{t}\, da = \int_{\partial P} \mathbf{T}\, \mathbf{n}\, da = \int_P \mathrm{div}\, \mathbf{T}\, dv \ , \tag{17.2a}$$

$$\int_{\partial P} \mathbf{x} \times \mathbf{t}\, da = \int_{\partial P} \mathbf{x} \times \mathbf{T}\mathbf{n}\, da = \int_P \mathrm{div}\, (\mathbf{x} \times \mathbf{T})\, dv = \int_P \left[\mathbf{e}_j \times \mathbf{T}_j + \mathbf{x} \times \mathrm{div}\, \mathbf{T}\right] dv \ , \tag{17.2b}$$

where in (17.2b) we have used (3.40) and (16.24a) to write

$$\begin{aligned} \mathrm{div}\, (\mathbf{x} \times \mathbf{T}) &= (\mathbf{x} \times \mathbf{T}),_j \bullet \mathbf{e}_j = (\mathbf{x},_j \times \mathbf{T} + \mathbf{x} \times \mathbf{T},_j) \bullet \mathbf{e}_j \\ &= \mathbf{e}_j \times \mathbf{T}\mathbf{e}_j + \mathbf{x} \times (\mathbf{T},_j \bullet \mathbf{e}_j) = \mathbf{e}_j \times \mathbf{T}_j + \mathbf{x} \times \mathrm{div}\, \mathbf{T} \ . \end{aligned} \tag{17.3}$$

Now the balance of linear momentum (15.1) may be rewritten in the form

$$\int_P \left[\rho\, \dot{\mathbf{v}} - \rho\, \mathbf{b} - \mathrm{div}\, \mathbf{T}\right] dv = 0 \ . \tag{17.4}$$

Assuming that the integrand in (17.4) is a continuous function and assuming that (17.4) must hold for arbitrary regions P it follows from the results of Appendix B that

$$\rho\, \dot{\mathbf{v}} = \rho\, \mathbf{b} + \mathrm{div}\, \mathbf{T} \ , \tag{17.5}$$

must hold for each point of P. Letting v_i, b_i, T_{ij} be the Cartesian components of **v**,**b**,**T**, respectively, the component form of balance of linear momentum becomes

$$\rho\, \dot{v}_i = \rho\, b_i + T_{ij},_j \ . \tag{17.6}$$

Similarly, the balance of angular momentum (15.2) may be rewritten in the form

$$\int_P \left[\, \mathbf{x} \times \{\rho\, \dot{\mathbf{v}} - \rho\, \mathbf{b} - \mathrm{div}\, \mathbf{T}\} - \mathbf{e}_j \times \mathbf{T}_j \,\right] dv = 0 \ . \tag{17.7}$$

Assuming that the integrand in (17.7) is a continuous function, using the local form (17.5) of balance of linear momentum and assuming that (17.7) must hold for arbitrary regions P it follows from the results of Appendix B that

$$\mathbf{e}_j \times \mathbf{T}_j = 0 \ , \tag{17.7}$$

must hold for each point of P. Then, using (3.36) and (16.24a) equation (17.7) may be rewritten in the form

$$\mathbf{e}_j \times \mathbf{T}\, \mathbf{e}_j = \boldsymbol{\varepsilon} \bullet (\mathbf{e}_j \otimes \mathbf{T}\, \mathbf{e}_j) = \boldsymbol{\varepsilon} \bullet (\mathbf{e}_j \otimes \mathbf{e}_j\, \mathbf{T}^T) = \boldsymbol{\varepsilon} \bullet (\mathbf{I}\, \mathbf{T}^T) = \boldsymbol{\varepsilon} \bullet \mathbf{T}^T = 0 \ . \tag{17.8}$$

Since $\boldsymbol{\varepsilon}$ is skew-symmetric in any two of its indices we may conclude that the local form of angular momentum requires the stress tensor to be symmetric

$$\mathbf{T}^T = \mathbf{T} \ , \ T_{ij} = T_{ji} \ . \tag{17.9a,b}$$

18. Referential Forms of the Equations of Motion

In the previous sections we have defined the stress vector $\mathbf{t}$ as the force per unit area in the present configuration. This leads to a definition of stress which is sometimes referred to as the true stress. Alternatively, since the surface ∂P in the present configuration maps to the surface ∂P_0 in the reference configuration we can define another stress vector $\boldsymbol{\pi}$ as the force acting in the present configuration but measured per unit area in the reference configuration. This leads to a definition of stress which is sometimes referred to as engineering stress.

Recalling that the stress vector $\mathbf{t}$ depends on position $\mathbf{x}$, time t, and the unit outward normal $\mathbf{n}$, it follows that the stress vector $\boldsymbol{\pi}$ depends on position $\mathbf{X}$, time t, and the unit outward normal $\mathbf{N}$ to the surface ∂P_0. Thus, the force acting in the present configuration on an arbitrary material part S of the present surface P or the associated part S_0 of the reference surface P_0 of the body may be expressed in the equivalent forms

$$\int_S \mathbf{t}(\mathbf{n})\, da = \int_{S_0} \boldsymbol{\pi}(\mathbf{N})\, dA \quad , \tag{18.1}$$

where dA is the element of area in the reference configuration. Similarly, the quantities $\mathbf{v}$, $\mathbf{b}$, $\mathbf{x} \times \mathbf{v}$, $\mathbf{x} \times \mathbf{b}$ are measured per unit mass and represent the linear momentum, body force, angular momentum, and moment of body force, respectively. Therefore, since ρ_0 is the mass density per unit reference volume we have

$$\int_P \rho\, \mathbf{v}\, dv = \int_{P_0} \rho_0\, \mathbf{v}\, dV \quad , \tag{18.2a}$$

$$\int_P \rho\, \mathbf{b}\, dv = \int_{P_0} \rho_0\, \mathbf{b}\, dV \quad , \tag{18.2b}$$

$$\int_P \mathbf{x} \times \rho\, \mathbf{v}\, dv = \int_{P_0} \mathbf{x} \times \rho_0\, \mathbf{v}\, dV \quad , \tag{18.2c}$$

$$\int_P \mathbf{x} \times \rho\, \mathbf{b}\, dv = \int_{P_0} \mathbf{x} \times \rho_0\, \mathbf{b}\, dV \quad , \tag{18.2d}$$

where dV is the element of volume in the reference configuration.

Then with the help of the results (18.1) and (18.2) the balances of linear momentum (15.1) and angular momentum (15.2) may be rewritten in the forms

$$\frac{d}{dt}\int_{P_0} \rho_0 \, \mathbf{v} \, dV = \int_{P_0} \rho_0 \, \mathbf{b} \, dV + \int_{\partial P_0} \boldsymbol{\pi}(\mathbf{N}) \, dA \ , \tag{18.3a}$$

$$\frac{d}{dt}\int_{P_0} \mathbf{x} \times \rho_0 \, \mathbf{v} \, dV = \int_{P_0} \mathbf{x} \times \rho_0 \, \mathbf{b} \, dV + \int_{\partial P_0} \mathbf{x} \times \boldsymbol{\pi}(\mathbf{N}) \, dA \ . \tag{18.3b}$$

Following similar arguments as those in Sec 16 the stress vector $\boldsymbol{\pi}$ ($\mathbf{N}$) may be shown to be a linear function of $\mathbf{N}$ which may be represented in the form

$$\boldsymbol{\pi}(\mathbf{N}) = \boldsymbol{\Pi} \, \mathbf{N} \ , \ \pi_i \, (N_A) = \Pi_{iA} \, N_A \ , \ \boldsymbol{\Pi} = \Pi_{iA} \, \mathbf{e}_i \otimes \mathbf{e}_A \ , \tag{18.4a,b,c}$$

where π_i are the components of $\boldsymbol{\pi}$, and $\boldsymbol{\Pi}$, with components Π_{iA}, is a second order tensor called the first Piola-Kirchhoff stress tensor.

With the help of (18.4) the local form of balance of linear momentum becomes

$$\rho_0 \, \dot{\mathbf{v}} = \rho_0 \, \mathbf{b} + \mathrm{Div} \, \boldsymbol{\Pi} \ , \ \rho_0 \, \dot{v}_i = \rho_0 \, b_i + \Pi_{iA,A} \ , \tag{18.5a,b}$$

where Div denotes the divergence with respect to $\mathbf{X}$ and $(,_A)$ denotes partial differentiation with respect to X_A. To obtain the local form of angular momentum let us first consider

$$\mathrm{Div} \, (\mathbf{x} \times \boldsymbol{\Pi}) = (\mathbf{x} \times \boldsymbol{\Pi}),_A \bullet \mathbf{e}_A = (\mathbf{x},_A \times \boldsymbol{\Pi}) \bullet \mathbf{e}_A + \mathbf{x} \times (\boldsymbol{\Pi},_A \bullet \mathbf{e}_A)$$

$$= (\mathbf{F} \, \mathbf{e}_A) \times (\boldsymbol{\Pi} \, \mathbf{e}_A) + \mathbf{x} \times (\mathrm{Div} \, \boldsymbol{\Pi}) \ . \tag{18.6}$$

Thus, with the help of (18.5) the local form of balance of angular momentum yields

$$(\mathbf{F} \, \mathbf{e}_A) \times (\boldsymbol{\Pi} \, \mathbf{e}_A) = 0 \ . \tag{18.7}$$

Using (3.36) we may rewrite (18.7) in the form

$$0 = (\mathbf{F} \, \mathbf{e}_A) \times (\boldsymbol{\Pi} \, \mathbf{e}_A) = \boldsymbol{\varepsilon} \bullet (\mathbf{F} \, \mathbf{e}_A \otimes \boldsymbol{\Pi} \, \mathbf{e}_A) = \boldsymbol{\varepsilon} \bullet (\mathbf{F} \, \mathbf{e}_A \otimes \mathbf{e}_A \boldsymbol{\Pi}^T) = \boldsymbol{\varepsilon} \bullet (\mathbf{F} \, \mathbf{I} \, \boldsymbol{\Pi}^T) \ ,$$

$$\boldsymbol{\varepsilon} \bullet (\mathbf{F} \, \boldsymbol{\Pi}^T) = 0 \ . \tag{18.8}$$

Thus, since $\boldsymbol{\varepsilon}$ is skew-symmetric in any two of its indices we realize that the tensor $\mathbf{F} \, \boldsymbol{\Pi}^T$ must be symmetric

$$\mathbf{F} \, \boldsymbol{\Pi}^T = (\mathbf{F} \, \boldsymbol{\Pi}^T)^T = \boldsymbol{\Pi} \, \mathbf{F}^T \ , \ F_{iA} \, \Pi_{jA} = F_{jA} \, \Pi_{iA} \ . \tag{18.9a,b}$$

This means that the first Piola-Kirchhoff stress $\boldsymbol{\Pi}$ is not necessarily symmetric.

Since the stress vector $\mathbf{t}$ is related to the Cauchy stress $\mathbf{T}$ by the formula $\mathbf{t}(\mathbf{n}) = \mathbf{T}\,\mathbf{n}$ and since equation (18.1) relates the force acting on the part S of the surface ∂P to the force acting on the part S_0 of the surface ∂P_0, it should be possible to relate the Cauchy stress $\mathbf{T}$ to the first Piola-Kirchhoff stress $\mathbf{\Pi}$. To this end, we recall from (7.34) that

$$\boldsymbol{\pi}(\mathbf{N})\, dA = \mathbf{\Pi}\, \mathbf{N}\, dA = \mathbf{\Pi}\, \mathbf{F}^T\, \mathbf{n}\, J^{-1}\, da \ , \tag{18.10}$$

so equation (18.1) may be rewritten in the form

$$\int_S \left\{\mathbf{T} - J^{-1}\, \mathbf{\Pi}\, \mathbf{F}^T\right\} \mathbf{n}\, da = 0 \ . \tag{18.11}$$

Assuming that the integrand is continuous and that ∂P is arbitrary we obtain

$$\left\{\mathbf{T} - J^{-1}\, \mathbf{\Pi}\, \mathbf{F}^T\right\} \mathbf{n} = 0 \ . \tag{18.12}$$

However, the tensor in the brackets is independent of the normal $\mathbf{n}$ and $\mathbf{n}$ is arbitrary so we deduce that the Cauchy stress $\mathbf{T}$ is related to the first Piola-Kirchhoff stress $\mathbf{\Pi}$ by

$$\mathbf{T} = J^{-1}\, \mathbf{\Pi}\, \mathbf{F}^T \ , \ T_{ij} = J^{-1}\, \Pi_{iB}\, F_{jB} \ . \tag{18.13a,b}$$

Notice that (18.9a) and (18.13a) ensure that the Cauchy stress $\mathbf{T}$ is symmetric, which is the same result that we obtained from the balance of angular momentum referred to the present configuration.

The first Piola-Kirchhoff stress $\mathbf{\Pi}$, with components Π_{iA}, is referred to both the present configuration and the reference configuration and it is also called the nonsymmetric Piola-Kirchhoff stress. For many purposes it is convenient to introduce another stress $\mathbf{S}$, with components S_{AB}, which is referred to the reference configuration only and is defined by

$$\mathbf{\Pi} = \mathbf{F}\, \mathbf{S} \ , \ \Pi_{iB} = F_{iA}\, S_{AB} \ . \tag{18.14a,b}$$

It follows from the definition (18.14a) and the result (18.9a) that $\mathbf{S}$ is a symmetric tensor

$$\mathbf{S}^T = \mathbf{S} \ , \ S_{BA} = S_{AB} \ . \tag{18.15a,b}$$

For this reason $\mathbf{S}$ is also called the symmetric Piola-Kirchhoff stress. Finally, we note from (18.13a) and (18.14a) that the Cauchy stress $\mathbf{T}$ is related to $\mathbf{S}$ by the formula

$$\mathbf{T} = J^{-1}\, \mathbf{F}\mathbf{S}\mathbf{F}^T \ , \ T_{ij} = J^{-1}\, F_{iA}\, S_{AB}\, F_{jB} \ . \tag{18.16a,b}$$

Furthermore, recall that the Cauchy stress $\mathbf{T}$ can be separated into its spherical part $-p\mathbf{I}$ and its deviatoric part $\mathbf{T}'$, such that

$$\mathbf{T} = -p\mathbf{I} + \mathbf{T}' \;, \quad p = -\frac{1}{3}\,\mathbf{T} \bullet \mathbf{I} \;, \quad \mathbf{T}' \bullet \mathbf{I} = 0 \;, \qquad (18.17\text{a,b,c})$$

where p denotes the pressure. It follows from (18.16a) and (18.17) that the symmetric Piola-Kirchhoff stress $\mathbf{S}$ admits an analogous separation

$$\mathbf{S} = J\,\mathbf{F}^{-1}\,\mathbf{T}\,\mathbf{F}^{-T} \;, \qquad (18.18\text{a})$$

$$\mathbf{S} = -p\,J\,\mathbf{C}^{-1} + \mathbf{S}' \;, \qquad (18.18\text{b})$$

$$p = -\frac{1}{3}\,J^{-1}\,\mathbf{S} \bullet \mathbf{C} \;, \qquad (18.18\text{c})$$

$$\mathbf{S}' = J\,\mathbf{F}^{-1}\,\mathbf{T}'\,\mathbf{F}^{-T} \;, \quad \mathbf{S}' \bullet \mathbf{C} = 0 \;. \qquad (18.18\text{d,e})$$

It is important to emphasize that although $\mathbf{T}'$ is deviatoric (18.17c) the associated quantity $\mathbf{S}'$ is not (18.18e) even though $\mathbf{S}'$ is directly related to $\mathbf{T}'$ (18.18d).

19. Invariance Under Superposed Rigid Body Motions

From section 11 we recall that under superposed rigid body motion (SRBM) the point $\mathbf{x}$ at time t is moved to the point $\mathbf{x}^+$ at time $t^+=t+a$ such that $\mathbf{x}^+$ and $\mathbf{x}$ are related by the mapping

$$\mathbf{x}^+ = \mathbf{c}(t) + \mathbf{Q}(t)\,\mathbf{x} \;\;,\;\; \mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I} \;\;,\;\; \det \mathbf{Q} = +1 \;\;, \tag{19.1a,b,c}$$

where $\mathbf{c}$ is a vector, and $\mathbf{Q}$ is a second order tensor and both $\mathbf{c}$ and $\mathbf{Q}$ are functions of time only. Furthermore, we recall that in section 11 the mapping (19.1a) was used to derive a number of expressions for the values of various kinematic quantities associated with the superposed configuration P^+. In this section we will determine expressions for the superposed values of a number of kinetic quantities that include: the mass density ρ, the stress vector $\mathbf{t}$, the Cauchy stress tensor $\mathbf{T}$, and the body force $\mathbf{b}$. Consequently, we will derive expressions for the quantities

$$\{\, \rho^+ \,,\, \mathbf{t}^+ \,,\, \mathbf{T}^+ \,,\, \mathbf{b}^+ \,\} \;\; . \tag{19.2}$$

It is important to emphasize that the values of all kinematic and kinetic quantities in the superposed configuration P^+ must be consistent with the basic physical requirement that the balance laws remain form-invariant when expressed relative to P^+. Therefore the conservation of mass and balances of linear and angular momentum may be stated relative to P^+ in the forms

$$\int_{P^+} \rho^+ \, dv^+ = \int_{P_0} \rho_0 \, dV \;\;, \tag{19.3a}$$

$$\frac{d}{dt}\int_{P^+} \rho^+ \, \mathbf{v}^+ \, dv^+ = \int_{P^+} \rho^+ \, \mathbf{b}^+ \, dv^+ + \int_{\partial P^+} \mathbf{t}^+(\mathbf{n}^+) \, da^+ \;\;, \tag{19.3b}$$

$$\frac{d}{dt}\int_{P^+} \mathbf{x}^+ \times \rho^+ \, \mathbf{v}^+ \, dv^+ = \int_{P^+} \mathbf{x}^+ \times \rho^+ \, \mathbf{b}^+ \, dv^+ + \int_{\partial P^+} \mathbf{x}^+ \times \mathbf{t}^+(\mathbf{n}^+) \, da^+ \;\;, \tag{19.3c}$$

where ∂P^+ is the closed boundary of P^+. Using the arguments of section 16 it can be shown a Cauchy tensor $\mathbf{T}^+$ exists which is a function of position and time only, such that

$$\mathbf{t}^+(\mathbf{n}^+) = \mathbf{T}^+ \, \mathbf{n}^+ \;\; . \tag{19.4}$$

Then, with the help of the transport and divergence theorems and the result (11.19a) that $dv^+=J^+dV$, the local equations forms of (19.3) become

$$\rho^+ J^+ = \rho_0 \ , \tag{19.5a}$$

$$\rho^+ \dot{\mathbf{v}}^+ = \rho^+ \mathbf{b}^+ + \mathrm{div}^+ \mathbf{T}^+ \ , \ (\mathbf{T}^+)^T = \mathbf{T}^+ \ , \tag{19.5b,c}$$

where div^+ denotes the divergence operation with respect to $\mathbf{x}^+$.

Using the kinematic result (11.19a) that $J^+ = J$ it follows that the conservation of mass (19.5a) may be used to prove that the mass density remains unchanged by SRBM

$$\rho^+ = \rho \ . \tag{19.6}$$

In contrast, the balance of linear momentum (19.5b) contains two additional unknowns $\mathbf{b}^+$ and $\mathbf{T}^+$ [once $\rho^+ = \rho$ is used and $\dot{\mathbf{v}}^+$ is expressed in terms of derivatives of (19.1a)]. Therefore, to determine the forms for $\mathbf{b}^+$ and $\mathbf{T}^+$ it is essential to make a physical assumption. To this end, we assume that the component of the stress vector $\mathbf{t}^+$ in the direction of the outward normal $\mathbf{n}^+$ remains unchanged under SRBM so that

$$\mathbf{t}^+(\mathbf{n}^+) \bullet \mathbf{n}^+ = \mathbf{t}\,(\mathbf{n}) \bullet \mathbf{n} \ . \tag{19.7}$$

Recalling from (11.19) that

$$\mathbf{n}^+ = \mathbf{Q}\,\mathbf{n} \ , \ \mathbf{n} = \mathbf{Q}^T \mathbf{n}^+ \tag{19.8a,b}$$

it follows that (19.7) may be rewritten in the form

$$[\ \mathbf{t}^+ (\mathbf{n}^+) - \mathbf{Q}\,\mathbf{t}\,(\mathbf{Q}^T\mathbf{n}^+)\ ] \bullet \mathbf{n}^+ = 0 \ . \tag{19.9}$$

Although (19.9) must be valid for arbitrary $\mathbf{n}^+$ we cannot conclude that the coefficient of $\mathbf{n}^+$ vanishes because the coefficient also depends on $\mathbf{n}^+$. However, by using (19.4) and expressing the stress vector in terms of the stress tensor and the outward normal we have

$$[\ \mathbf{T}^+ - \mathbf{Q}\,\mathbf{T}\,\mathbf{Q}^T\ ] \bullet (\mathbf{n}^+ \otimes \mathbf{n}^+) = 0 \ . \tag{19.10}$$

Now, since (19.10) must hold for arbitrary unit vector $\mathbf{n}^+$ and the coefficient of $\mathbf{n}^+ \otimes \mathbf{n}^+$ is independent of $\mathbf{n}^+$ and is symmetric we may conclude that under SRBM the Cauchy stress tensor transforms by

$$\mathbf{T}^+ = \mathbf{Q}\,\mathbf{T}\,\mathbf{Q}^T \ . \tag{19.11}$$

It follows from (19.4),(19.8), and (19.11) we may deduce that under SRBM the stress vector transforms by

$$\mathbf{t}^+ = \mathbf{Q}\,\mathbf{t} \ . \tag{19.12}$$

In order to explain the physical consequences of the assumption (19.7) we use the results (19.11) and (19.12) to deduce that

$$\mathbf{t}^+ \bullet \mathbf{t}^+ = \mathbf{t} \bullet \mathbf{t} \;\;,\;\; \mathbf{T}^+ \bullet \mathbf{T}^+ = \mathbf{T} \bullet \mathbf{T} \;\; . \tag{19.13a,b}$$

This means that the magnitudes of both the stress vector and the Cauchy stress tensor remain unchanged by SRBM. Furthermore, in view of the assumption (19.7) this means that the angle between the stress vector **t** and the unit outward normal **n** also remains unchanged by SRBM. Consequently, the stress vector and stress tensor which characterize the response of the material are merely rotated by SRBM.

In view of the above results equation (19.5b) becomes an equation for determining $\mathbf{b}^+$. To this end, use (19.1a) and (19.11) to deduce that

$$\text{div}^+ \mathbf{T}^+ = \partial \mathbf{T}^+/\partial x_j^+ \bullet \mathbf{e}_j = \partial \mathbf{T}^+/\partial x_i \, (\partial x_i/\partial x_j^+) \, \mathbf{e}_j = (\mathbf{Q}\mathbf{T}\mathbf{Q}^T),_i \, (Q_{ji} \, \mathbf{e}_j) \;\; ,$$

$$\text{div}^+ \mathbf{T}^+ = \; (\mathbf{Q}\mathbf{T},_i\mathbf{Q}^T) \, (\mathbf{Q}\mathbf{e}_i) = \mathbf{Q}\mathbf{T},_i \, \mathbf{e}_i \;\;\; ,$$

$$\text{div}^+ \mathbf{T}^+ = \mathbf{Q} \, \text{div} \, \mathbf{T} \;\; . \tag{19.14}$$

Thus, with the help of (17.5),(19.6), and (19.14), equation (19.5b) demands that under SRBM the body force **b** transforms by

$$\mathbf{b}^+ = \dot{\mathbf{v}}^{\,+} + \mathbf{Q} \, (\mathbf{b} - \dot{\mathbf{v}} \,) \;\; . \tag{19.15}$$

For later convenience we summarize the transformation relations for kinetic quantities as follows

$$\rho^+ = \rho \;\;,\;\; \mathbf{T}^+ = \mathbf{Q} \, \mathbf{T} \, \mathbf{Q}^T \;\;,\;\; \mathbf{b}^+ = \dot{\mathbf{v}}^+ + \mathbf{Q} \, (\mathbf{b} - \dot{\mathbf{v}} \,) \;\; . \tag{19.16a,b,c}$$

Also, with the help of (18.4a),(18.14a),(18.16a), and (19.16) it can be shown that the Piola-Kirchhoff stress vector $\boldsymbol{\pi}$, nonsymmetric stress tensor $\boldsymbol{\Pi}$, and symmetric stress tensor **S** transform under SRBM by

$$\boldsymbol{\pi}^+ = \mathbf{Q} \, \boldsymbol{\pi} \;\;,\;\; \boldsymbol{\Pi}^+ = \mathbf{Q} \, \boldsymbol{\Pi} \;\;,\;\; \mathbf{S}^+ = \mathbf{S} \;\; . \tag{19.17a,b,c}$$

20. The Balance of Energy

In the previous sections we have focused attention on the purely mechanical theory. Although it is not our intention to discuss the complete thermodynamical theory it is desirable to introduce the balance of energy which is also called the first law of thermodynamics. In words the balance of energy connects notions of heat and work. In order to discuss this balance law it is necessary to introduce the concepts of internal energy, rate of heat supply, kinetic energy, and rate of work. To this end, we assume the existence of a scalar function $\varepsilon(\mathbf{x},t)$ called the specific (per unit mass) internal energy. Then the total internal energy $\mathcal{E}$ of the part P of the body is given by

$$\mathcal{E} = \int_P \rho\varepsilon \, dv \ . \tag{20.1}$$

Next we assume that heat can enter the body in two ways: either through an external specific rate of heat supply $r(\mathbf{x},t)$ that acts at each point $\mathbf{x}$ of P or through a heat flux $h(\mathbf{x},t\ ;\ \mathbf{n})$ per unit present area that acts at each point of the surface ∂P of P. Thus, the total rate of heat $\mathcal{H}$ supplied to P is given by

$$\mathcal{H} = \int_P \rho\, r \, dv \ - \int_{\partial P} h \, da \ . \tag{20.2}$$

We emphasize that the heat flux h is also a function of the unit outward normal $\mathbf{n}$ to ∂P and that the minus sign in (20.2) is introduced for later convenience. Furthermore, we note that the first term in (20.2) represents the rate of heat entering the body through radiation and the second term in (20.2) represents the rate of heat entering the body through heat conduction.

The total kinetic energy $\mathcal{K}$ of the part P is given by

$$\mathcal{K} = \int_P \frac{1}{2}\, \rho\, \mathbf{v} \bullet \mathbf{v} \, dv \ , \tag{20.3}$$

where $\mathbf{v}$ is the velocity. Also, the total rate of work $\mathcal{W}$ done <u>on</u> the part P is calculated by summing the rate of work supplied by the body force $\mathbf{b}$ and the stress vector (or surface traction) $\mathbf{t}$ so that

$$\mathcal{W} = \int_P \rho\, \mathbf{b} \bullet \mathbf{v} \, dv + \int_{\partial P} \mathbf{t} \bullet \mathbf{v} \, da \ . \tag{20.4}$$

Now the balance of energy may be expressed in words by the following statement:

> The rate of change of internal energy plus kinetic energy of an arbitrary part of a body equals the rate of supply of work and heat to that part.

The mathematical representation of this statement becomes

$$\dot{\mathcal{E}} + \dot{\mathcal{K}} = \mathcal{W} + \mathcal{H} \ . \tag{20.5}$$

Following analysis similar to that in section 16 it can be shown by applying the balance of energy (20.5) to an elemental tetrahedron that the heat flux h($\mathbf{x}$,t ; $\mathbf{n}$) must be a linear function of $\mathbf{n}$ and therefore may be expressed in the form

$$h(\mathbf{x},t\,;\mathbf{n}) = \mathbf{q}(\mathbf{x},t) \bullet \mathbf{n} \ , \tag{20.6}$$

where $\mathbf{q}(\mathbf{x},t)$ is a vector function of position and time only that is called the heat flux. Thus, $\mathcal{H}$ in (20.2) may be rewritten as

$$\mathcal{H} = \int_P \rho\, r\, dv \ - \int_{\partial P} \mathbf{q} \bullet \mathbf{n}\, da \ . \tag{20.7}$$

It follows from (20.7) that the vector $\mathbf{q}$ indicates the direction in which heat is flowing, since a positive value of $\mathbf{q} \bullet \mathbf{n}$ indicates that heat is flowing out of the part P.

In order to derive the local form of the balance of energy we use the transport and divergence theorems to obtain

$$\frac{d}{dt} \int_P \rho\, (\varepsilon + \frac{1}{2}\, \mathbf{v} \bullet \mathbf{v})\, dv$$

$$= \int_P \{\rho\, (\dot{\varepsilon} + \mathbf{v} \bullet \dot{\mathbf{v}}) + (\dot{\rho} + \rho \operatorname{div} \mathbf{v})\, (\varepsilon + \frac{1}{2} \mathbf{v} \bullet \mathbf{v})\} \ dv \ , \tag{20.8a}$$

$$\int_{\partial P} \mathbf{t} \bullet \mathbf{v}\, da = \int_P \{\, \mathbf{v} \bullet \operatorname{div} \mathbf{T} + \mathbf{T} \bullet \mathbf{L} \,\}\, dv \ , \tag{20.8b}$$

$$\int_{\partial P} \mathbf{q} \bullet \mathbf{n}\, da = \int_P \operatorname{div} \mathbf{q} \ dv \ . \tag{20.8c}$$

where we have used the result that

$$\operatorname{div} (\mathbf{v}\, \mathbf{T}) = (\mathbf{v}\, \mathbf{T})_{,i} \bullet \mathbf{e}_i = (\mathbf{v}\, \mathbf{T}_{,i}) \bullet \mathbf{e}_i + (\mathbf{v}_{,i}\, \mathbf{T}) \bullet \mathbf{e}_i = \mathbf{v} \bullet (\mathbf{T}_{,i}\, \mathbf{e}_i) + \mathbf{T} \bullet (\mathbf{v}_{,i} \otimes \mathbf{e}_i)$$

$$\operatorname{div} (\mathbf{v}\, \mathbf{T}) = \mathbf{v} \bullet \operatorname{div} \mathbf{T} + \mathbf{T} \bullet \mathbf{L} \ . \tag{20.9}$$

It follows from these results that the balance of energy may be rewritten in the form

$$\int_P \left\{ (\dot{\rho} + \rho \, \mathrm{div} \, \mathbf{v}) (\varepsilon + \frac{1}{2} \, \mathbf{v} \bullet \mathbf{v}) + \mathbf{v} \bullet (\rho \, \dot{\mathbf{v}} - \rho \, \mathbf{b} - \mathrm{div} \, \mathbf{T}) \right.$$

$$\left. + (\rho \, \dot{\varepsilon} - \rho \, r + \mathrm{div} \, \mathbf{q} - \mathbf{T} \bullet \mathbf{L}) \right\} dv = 0 \tag{20.10}$$

Assuming that the integrand in (20.10) is continuous and assuming that (20.10) must hold for all arbitrary parts P we may deduce the local equation

$$(\dot{\rho} + \rho \, \mathrm{div} \, \mathbf{v}) (\varepsilon + \frac{1}{2} \, \mathbf{v} \bullet \mathbf{v}) + \mathbf{v} \bullet (\rho \, \dot{\mathbf{v}} - \rho \, \mathbf{b} - \mathrm{div} \, \mathbf{T})$$

$$+ (\rho \, \dot{\varepsilon} - \rho \, r + \mathrm{div} \, \mathbf{q} - \mathbf{T} \bullet \mathbf{L}) = 0 \ . \tag{20.11}$$

However, in view of the conservation of mass (14.7), and the balance of linear momentum (17.5), equation (20.11) reduces to

$$\rho \, \dot{\varepsilon} - \rho \, r + \mathrm{div} \, \mathbf{q} - \mathbf{T} \bullet \mathbf{L} = 0 \ . \tag{20.12}$$

Furthermore, using the results of balance of angular momentum (17.9a) the term $\mathbf{T} \bullet \mathbf{L}$ becomes

$$\mathbf{T} \bullet \mathbf{L} = \mathbf{T} \bullet \mathbf{D} + \mathbf{T} \bullet \mathbf{W} = \mathbf{T} \bullet \mathbf{D} \ , \tag{20.13}$$

where we recall that the inner product of symmetric and skew-symmetric second order tensors vanishes. Thus, (20.12) finally reduces to

$$\rho \, \dot{\varepsilon} - \rho \, r + \mathrm{div} \, \mathbf{q} - \mathbf{T} \bullet \mathbf{D} = 0 \ . \tag{20.14}$$

Before closing this section we note that ε and r are assumed to remain unchanged by SRBM so that

$$\varepsilon^+ = \varepsilon \ , \ r^+ = r \ . \tag{20.15a,b}$$

Furthermore, we assume that $\mathbf{q} \bullet \mathbf{n}$ also remains unchanged by SRBM

$$\mathbf{q} \bullet \mathbf{n} = \mathbf{q}^+ \bullet \mathbf{n}^+ \ . \tag{20.16}$$

Thus, with the help of (11.19) we may deduce that under SRBM

$$\mathbf{q}^+ = \mathbf{Q} \, \mathbf{q} \ . \tag{20.17}$$

21. Derivation of Balance Laws From Energy and Invariance Requirements

In this section we show that the conservation of mass and the balances of linear and angular momentum can be derived directly from the balance of energy and invariance requirements under SRBM. This unique inter relationship shows how fundamental the invariance requirements are in the general theory of a continuum. Specifically , we start with the assumption that the balance of energy remains uninfluenced by SRBM so that it can be stated relative to the superposed configuration P^+ in the form

$$\dot{\mathcal{E}}^+ + \dot{\mathcal{K}}^+ = \mathcal{W}^+ + \mathcal{H}^+ \ , \tag{21.1}$$

where the total internal energy $\mathcal{E}^+$, kinetic energy $\mathcal{K}^+$, rate of work $\mathcal{W}^+$ and rate of heat supply $\mathcal{H}^+$, referred to the superposed configuration are given by

$$\mathcal{E}^+ = \int_{P^+} \rho^+\varepsilon^+ \, dv^+ \ , \ \mathcal{K}^+ = \int_{P^+} \frac{1}{2}\rho^+ \mathbf{v}^+ \bullet \mathbf{v}^+ \, dv^+ \ , \tag{21.2a,b}$$

$$\mathcal{W}^+ = \int_{P^+} \rho^+ \mathbf{b}^+ \bullet \mathbf{v}^+ \, dv^+ + \int_{\partial P^+} \mathbf{t}^+ \bullet \mathbf{v}^+ \, da^+ \ , \tag{21.2c}$$

$$\mathcal{H}^+ = \int_{P^+} \rho^+ r^+ \, dv^+ - \int_{\partial P^+} \mathbf{q}^+ \bullet \mathbf{n}^+ \, da^+ \ . \tag{21.2d}$$

Using the transport theorem, (19.4), the divergence theorem and continuity, it follows the local form of (21.1) becomes

$$(\dot{\rho}^+ + \rho^+ \,\mathrm{div}^+ \mathbf{v}^+)(\varepsilon^+ + \frac{1}{2}\mathbf{v}^+ \bullet \mathbf{v}^+) + \mathbf{v}^+ \bullet (\rho^+ \dot{\mathbf{v}}^+ - \rho^+ \mathbf{b}^+ - \mathrm{div}^+ \mathbf{T}^+)$$
$$+ (\rho^+ \dot{\varepsilon}^+ - \rho^+ r^+ + \mathrm{div}^+ \mathbf{q}^+ - \mathbf{T}^+ \bullet \mathbf{L}^+) = 0 \ , \tag{21.3}$$

where div^+ is the divergence operator relative to $\mathbf{x}^+$.

Now, with the help of the invariance conditions (11.22b),(11.23a),(19.14), (19.16), (20.15),(20.17), as well as the results

$$\mathrm{div}^+ \mathbf{v}^+ = \partial\mathbf{v}^+/\partial x_j^+ \bullet \mathbf{e}_j = \partial\mathbf{v}^+/\partial x_i \,(\partial x_i/\partial x_j^+) \bullet \mathbf{e}_j = (\dot{\mathbf{c}} + \mathbf{\Omega}\,\mathbf{Q}\,\mathbf{x} + \mathbf{Q}\,\mathbf{v})_{,i} \bullet (Q_{ji}\,\mathbf{e}_j)$$
$$= (\mathbf{\Omega}\,\mathbf{Q}\,\mathbf{e}_i + \mathbf{Q}\,\mathbf{v}_{,i}) \bullet (\mathbf{Q}\,\mathbf{e}_i) = \mathbf{\Omega} \bullet (\mathbf{Q}\mathbf{e}_i \otimes \mathbf{Q}\mathbf{e}_i) + \mathbf{v}_{,i} \bullet \mathbf{e}_i$$
$$= \mathbf{\Omega} \bullet (\mathbf{Q}\,\mathbf{e}_i \otimes \mathbf{e}_i\,\mathbf{Q}^T) + \mathrm{div}\,\mathbf{v} = \mathbf{\Omega} \bullet \mathbf{I} + \mathrm{div}\,\mathbf{v} = \mathrm{div}\,\mathbf{v} \ , \tag{21.4a}$$

$$\mathrm{div}^+ \mathbf{q}^+ = \mathrm{div}\,\mathbf{q} \ , \tag{21.4b}$$

$$\mathbf{T}^+ \bullet \mathbf{L}^+ = \mathbf{QTQ}^T \bullet (\mathbf{QLQ}^T + \mathbf{\Omega}) = \mathbf{T} \bullet \mathbf{L} + \mathbf{T} \bullet \mathbf{Q}^T\mathbf{\Omega}\, \mathbf{Q} \ , \tag{21.4c}$$

equation (21.3) reduces to

$$(\dot{\rho} + \rho \, \mathrm{div}\, \mathbf{v}) (\varepsilon + \frac{1}{2} \mathbf{v}^+ \bullet \mathbf{v}^+) + \mathbf{v}^+ \bullet \mathbf{Q}(\rho \dot{\mathbf{v}} - \rho \, \mathbf{b} - \mathrm{div}\, \mathbf{T})$$

$$+ (\rho \dot{\varepsilon} - \rho \, r + \mathrm{div}\, \mathbf{q} - \mathbf{T} \bullet \mathbf{L} - \mathbf{T} \bullet \mathbf{Q}^T\mathbf{\Omega}\, \mathbf{Q}) = 0 \ . \tag{21.5}$$

Equation (21.5) is assumed to be valid for all SRBM. In the following we use two specific SRBM to derive results that are necessary consequences of (21.5). To this end, let us first consider the case of a trivial SRBM for which

$$\mathbf{c} = 0 \ , \ \dot{\mathbf{c}} = 0 \ , \ \mathbf{Q} = \mathbf{I} \ , \ \dot{\mathbf{Q}} = \mathbf{\Omega} = 0 \ , \tag{21.6a,b,c,d}$$

$$\mathbf{x}^+ = \mathbf{x} \ , \ \mathbf{v}^+ = \mathbf{v} \ . \tag{21.6e,f}$$

Substitution of (21.6) into (21.5) we obtain the equation (20.11). Next, consider the special SRBM that represents a constant velocity translation, which is obtained from (11.16) and (11.22a) by taking

$$\mathbf{c} = 0 \ , \ \dot{\mathbf{c}} = u\, \mathbf{u} \ , \ \mathbf{u} \bullet \mathbf{u} = 1, \tag{21.7a,b,c}$$

$$\mathbf{Q} = \mathbf{I} \ , \ \dot{\mathbf{Q}} = 0 \ , \tag{21.7d,e}$$

where (21.7) represent the instantaneous values at a specified but arbitrary time t. It follows from (11.16) and (11.22a) that at this time

$$\mathbf{x}^+ = \mathbf{x} \ , \ \mathbf{v}^+ = \mathbf{v} + u\, \mathbf{u} \ , \ \mathbf{\Omega} = 0 \ . \tag{21.8a,b,c}$$

The conditions (21.8) indicate that at time t the body occupies the same position as in P, but that a translation (without rotation) in the constant direction **u** with constant velocity u has been superimposed on the motion. Substituting (21.7d) and (21.8) into (21.5) and subtracting (20.11) from the result we deduce that

$$(\dot{\rho} + \rho \, \mathrm{div}\, \mathbf{v}) [u\, \mathbf{u} \bullet \mathbf{v} + \frac{1}{2} u^2] + u\, \mathbf{u} \bullet (\rho \dot{\mathbf{v}} - \rho \, \mathbf{b} - \mathrm{div}\, \mathbf{T}) = 0 \ , \tag{21.9}$$

must hold for arbitrary u and **u**. Since the coefficients of u and u^2 in (21.9) are independent of u, each of these coefficients must vanish, so we obtain the local form of conservation of mass

$$\dot{\rho} + \rho \, \mathrm{div}\, \mathbf{v} = 0 \ , \tag{21.10}$$

and the condition that

$$\mathbf{u} \bullet (\rho \dot{\mathbf{v}} - \rho \mathbf{b} - \text{div } \mathbf{T}) = 0 \quad . \tag{21.11}$$

Furthermore, since the direction $\mathbf{u}$ is arbitrary and the coefficient of $\mathbf{u}$ is independent of $\mathbf{u}$ we also obtain the local form of balance of linear momentum

$$\rho \dot{\mathbf{v}} = \rho \mathbf{b} + \text{div } \mathbf{T} \quad . \tag{21.12}$$

Now with the help of the results (21.10) and (21.12), equation (20.11) reduces to

$$\rho \dot{\varepsilon} - \rho r + \text{div } \mathbf{q} - \mathbf{T} \bullet \mathbf{L} = 0 \quad , \tag{21.13}$$

so that (21.5) yields the equation

$$\mathbf{T} \bullet \mathbf{Q}^T \mathbf{\Omega} \, \mathbf{Q} = 0 \quad . \tag{21.14}$$

However, since $\mathbf{\Omega}$ is an arbitrary skew-symmetric tensor, $\mathbf{Q}^T \mathbf{\Omega} \, \mathbf{Q}$ is also an arbitrary skew-symmetric tensor. Thus, since $\mathbf{T}$ does not depend on $\mathbf{\Omega}$ the Cauchy stress $\mathbf{T}$ must be symmetric

$$\mathbf{T}^T = \mathbf{T} \quad , \tag{21.15}$$

which is the consequence of balance of angular momentum. Finally, substitution of (21.15) into (21.13) and using (20.13) we obtain the reduced energy equation

$$\rho \dot{\varepsilon} - \rho r + \text{div } \mathbf{q} - \mathbf{T} \bullet \mathbf{D} = 0 \quad . \tag{21.16}$$

In the above analysis we have proved that the conservation of mass, the balances of linear and angular momentum, and the balance of energy, all referred to the present configuration P, are necessary consequences of the balance of energy and invariance under SRBM. Although these results were obtained using special simple SRBM it is easy to see using the invariance conditions (19.16),(20.15) and (20.17) that these balance laws remain form-invariant under arbitrary SRBM.

22. Boundary and Initial Conditions

In this section we confine attention to the discussion of initial and boundary conditions for the purely mechanical theory. In general, the number of initial conditions required and the type of boundary conditions required will depend on the specific type of material under consideration. However, it is possible to make some general observations that apply to all materials.

To this end, we recall that the local forms of conservation of mass (14.7) and balance of linear momentum (17.5) are partial differential equations which require both initial and boundary conditions. Specifically, the conservation of mass (14.7) is first order in time with respect to density ρ so it is necessary to specify the initial value of density at each point of the body

$$\rho(\mathbf{x},0) = \bar{\rho}(\mathbf{x}) \quad \text{on P for } t = 0\ . \tag{22.1}$$

Also the balance of linear momentum (17.5) is second order in time with respect to position $\mathbf{x}$ so that it is necessary to specify the initial value of $\mathbf{x}$ and the initial value of the velocity $\mathbf{v}$ at each point of the body

$$\hat{\mathbf{x}}(\mathbf{X},0) = \bar{\mathbf{x}}(\mathbf{X}) \quad \text{on P for } t = 0, \tag{22.2a}$$

$$\hat{\mathbf{v}}(\mathbf{X},0) = \tilde{\mathbf{v}}(\mathbf{x},0) = \bar{\mathbf{v}}(\mathbf{x}) \quad \text{on P for } t = 0\ . \tag{22.2b}$$

Guidance for determining the appropriate form of boundary conditions is usually obtained by considering the rate of work done by the stress vector. From (20.4) we observe that $\mathbf{t} \bullet \mathbf{v}$ is the rate of work per unit present area done by the stress vector. At each point of the surface ∂P we can define a right-handed orthogonal coordinate system with base vectors { $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{n}$ }, where $\mathbf{n}$ is the unit outward normal to ∂P and $\mathbf{s}_1$ and $\mathbf{s}_2$ are orthogonal vectors tangent to ∂P. Then with reference to this coordinate system we may write

$$\mathbf{t} \bullet \mathbf{v} = (\mathbf{t} \bullet \mathbf{s}_1)(\mathbf{v} \bullet \mathbf{s}_1) + (\mathbf{t} \bullet \mathbf{s}_2)(\mathbf{v} \bullet \mathbf{s}_2) + (\mathbf{t} \bullet \mathbf{n})(\mathbf{v} \bullet \mathbf{n}) \quad \text{on } \partial\text{P}\ . \tag{22.3}$$

Using this representation we define three types of boundary conditions

<u>Kinematic</u>: All three components of the velocity are specified

$$(\mathbf{v} \bullet \mathbf{s}_1)\ ,\ (\mathbf{v} \bullet \mathbf{s}_2)\ ,\ (\mathbf{v} \bullet \mathbf{n}) \ \text{ specified on } \partial\text{P for all } t \ge 0\ , \tag{22.4}$$

Kinetic: All three components of the stress vector are specified

$$(\mathbf{t} \bullet \mathbf{s}_1)\ ,\ (\mathbf{t} \bullet \mathbf{s}_2)\ ,\ (\mathbf{t} \bullet \mathbf{n}) \text{ specified on } \partial P \text{ for all } t \geq 0\ , \tag{22.5}$$

Mixed: Complementary components of both the velocity and the stress vector are specified

$$(\mathbf{v} \bullet \mathbf{s}_1) \text{ or } (\mathbf{t} \bullet \mathbf{s}_1) \text{ specified on } \partial P \text{ for all } t \geq 0, \tag{22.6}$$

$$(\mathbf{v} \bullet \mathbf{s}_2) \text{ or } (\mathbf{t} \bullet \mathbf{s}_2) \text{ specified on } \partial P \text{ for all } t \geq 0\ , \tag{22.6}$$

$$(\mathbf{v} \bullet \mathbf{n}) \text{ or } (\mathbf{t} \bullet \mathbf{n}) \text{ specified on } \partial P \text{ for all } t \geq 0\ . \tag{22.6}$$

Essentially, the complementary components $(\mathbf{t} \bullet \mathbf{s}_1),(\mathbf{t} \bullet \mathbf{s}_2),(\mathbf{t} \bullet \mathbf{n})$ are the responses to the motions $(\mathbf{v} \bullet \mathbf{s}_1),(\mathbf{v} \bullet \mathbf{s}_2),(\mathbf{v} \bullet \mathbf{n})$, respectively. Therefore, it is important to emphasize that for example both $(\mathbf{v} \bullet \mathbf{n})$ and $(\mathbf{t} \bullet \mathbf{n})$ cannot be specified at the same point of ∂P because this would mean that both the motion and the stress response can be specified independently of the material properties and geometry of the body. Notice also, that since the initial position of points on the boundary ∂P are specified by the initial condition (22.2a), the velocity boundary conditions (22.4) can be used to determine the position of the boundary for all time. This means that the kinematic boundary conditions (22.4) could also be characterized by specifying the position of points on the boundary for all time.

23. Linearization

In the previous sections we have considered the exact formulation of the theory of simple continua. The resulting equations are nonlinear so they are quite difficult to solve analytically. However, often it is possible to obtain relevant physical insight about the solution of a problem by considering a simpler approximate theory. In this section we will develop the linearized equations associated with this nonlinear theory.

We first note that a tensor **u** is said to be of order ε^n [$O(\varepsilon^n)$] if there exists a real finite number C, independent of ε, such that

$$|\mathbf{u}| < C\,\varepsilon^n \quad \text{as} \quad \varepsilon \to 0 \ . \tag{23.1}$$

In what follows we will linearize various kinematical quantities as well as the conservation of mass and the balance of linear momentum and boundary conditions by considering small deviations from a reference configuration in which the body is stress-free, at rest, and free of body force. To this end, we assume that the density ρ is of order 1 [$O(\varepsilon^0)$]

$$\rho = \rho_0 + O(\varepsilon) \ , \tag{23.2}$$

and that the displacement **u**, body force **b**, Cauchy stress **T**, nonsymmetric Piola-Kirchhoff stress $\mathbf{\Pi}$, and symmetric Piola-Kirchhoff stress **S** are of order ε

$$\{ \mathbf{u} , \mathbf{b} , \mathbf{T} , \mathbf{\Pi} , \mathbf{S} \} = O(\varepsilon) \ . \tag{23.3}$$

The resulting theory will be a linear theory if ε is infinitesimal

$$\varepsilon << 1 \ . \tag{23.4}$$

<u>Kinematics</u>: Recalling from (7.30b) that the position **x** of a material point in the present configuration may be represented by

$$\mathbf{x} = \mathbf{X} + \mathbf{u} \ , \tag{23.5}$$

the deformation gradient **F** becomes

$$\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X} = \mathbf{I} + \partial\mathbf{u}/\partial\mathbf{X} \ . \tag{23.6}$$

In what follows we use (23.6) to derive a number of kinematical results. For this purpose it is convenient to separate the displacement gradient into its symmetric part $\boldsymbol{\varepsilon}$ and its skew-symmetric part $\boldsymbol{\omega}$, such that

$$\partial\mathbf{u}/\partial\mathbf{X} = \boldsymbol{\varepsilon} + \boldsymbol{\omega} \ , \tag{23.7a}$$

$$\boldsymbol{\varepsilon} = \frac{1}{2}\,[\partial\mathbf{u}/\partial\mathbf{X} + (\partial\mathbf{u}/\partial\mathbf{X})^T] = \boldsymbol{\varepsilon}^T \;,\; \boldsymbol{\omega} = \frac{1}{2}\,[\partial\mathbf{u}/\partial\mathbf{X} - (\partial\mathbf{u}/\partial\mathbf{X})^T] = -\,\boldsymbol{\omega}^T \;, \quad (23.7\text{b,c})$$

where we note that both $\boldsymbol{\varepsilon}$ and $\boldsymbol{\omega}$ are of order ε. Now with the help of (23.6) and (23.7) it follows that

$$\mathbf{F} = \mathbf{I} + \partial\mathbf{u}/\partial\mathbf{X} = \mathbf{I} + \boldsymbol{\varepsilon} + \boldsymbol{\omega} \;, \quad (23.8\text{a})$$

$$\mathbf{F}^{-1} = \mathbf{I} - \partial\mathbf{u}/\partial\mathbf{X} + O(\varepsilon^2) = \mathbf{I} - \boldsymbol{\varepsilon} - \boldsymbol{\omega} + O(\varepsilon^2) \;, \quad (23.8\text{b})$$

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{I} + 2\,\boldsymbol{\varepsilon} + O(\varepsilon^2) \;, \quad (23.8\text{c})$$

$$\mathbf{E} = \frac{1}{2}\,(\mathbf{C} - \mathbf{I}) = \boldsymbol{\varepsilon} + O(\varepsilon^2) \;, \quad (23.8\text{d})$$

$$\mathbf{M} = \mathbf{C}^{1/2} = \mathbf{I} + \boldsymbol{\varepsilon} + O(\varepsilon^2) \;, \quad (23.8\text{e})$$

$$\mathbf{M}^{-1} = \mathbf{I} - \boldsymbol{\varepsilon} + O(\varepsilon^2) \;, \quad (23.8\text{f})$$

$$\mathbf{R} = \mathbf{F}\mathbf{M}^{-1} = \mathbf{I} + \boldsymbol{\omega} + O(\varepsilon^2) \;, \quad (23.8\text{g})$$

which indicates that $\boldsymbol{\varepsilon}$ is the linearized strain measure and $\boldsymbol{\omega}$ is the linearized rotation measure. Furthermore, we may use (23.8) to deduce that

$$\partial\mathbf{u}/\partial\mathbf{x} = (\partial\mathbf{u}/\partial\mathbf{X})\,\mathbf{F}^{-1} = \partial\mathbf{u}/\partial\mathbf{X} + O(\varepsilon^2) \;, \quad (23.9\text{a})$$

$$\mathbf{e} = \frac{1}{2}\,(\mathbf{I} - \mathbf{F}^{-T}\mathbf{F}^{-1}) = \boldsymbol{\varepsilon} + O(\varepsilon^2) \;, \quad (23.9\text{b})$$

so that for the linear theory where terms of order ε^2 are neglected there is no distinction between the strain measures $\mathbf{E}$ and $\mathbf{e}$. To determine the linearized expression for the change in volume we recall the Cayley-Hamilton theorem for $\mathbf{C}$ which states that $\mathbf{C}$ satisfies its own characteristic equation and write

$$-\,\mathbf{C}^3 + (\mathbf{C} \bullet \mathbf{I})\,\mathbf{C}^2 - \frac{1}{2}\,[(\mathbf{C} \bullet \mathbf{I})^2 - \mathbf{C}^2 \bullet \mathbf{I}]\,\mathbf{C} + I_3\,\mathbf{I} = 0 \;. \quad (23.10)$$

Now, taking the inner product of (23.10) with $\mathbf{I}$ we have

$$I_3 = \det\mathbf{C} = \frac{1}{3}\Big[\,\mathbf{C}^3 \bullet \mathbf{I} - \frac{3}{2}\,(\mathbf{C} \bullet \mathbf{I})\,(\mathbf{C}^2 \bullet \mathbf{I}) + \frac{1}{2}\,(\mathbf{C} \bullet \mathbf{I})^3\Big] \;. \quad (23.11)$$

However, with the help of (23.8c) we may deduce that

$$\mathbf{C} \bullet \mathbf{I} = 3 + 2\,\boldsymbol{\varepsilon} \bullet \mathbf{I} + O(\varepsilon^2) \;, \quad (23.12\text{a})$$

$$\mathbf{C}^2 \bullet \mathbf{I} = 3 + 4\,\boldsymbol{\varepsilon} \bullet \mathbf{I} + O(\varepsilon^2) \;, \quad (23.12\text{b})$$

$$\mathbf{C}^3 \bullet \mathbf{I} = 3 + 6\,\boldsymbol{\varepsilon} \bullet \mathbf{I} + O(\varepsilon^2) \;, \quad (23.12\text{c})$$

so that (23.11) yields

$$I_3 = 1 + 2\,\boldsymbol{\varepsilon} \bullet \mathbf{I} + O(\varepsilon^2) \;\; , \;\; J = I_3^{1/2} = 1 + \boldsymbol{\varepsilon} \bullet \mathbf{I} + O(\varepsilon^2) \;\; . \qquad (23.13a,b)$$

Thus, the trace of the linearized strain $\boldsymbol{\varepsilon}$ is the relative increase in volume

$$\frac{dv}{dV} - 1 = \frac{dv - dV}{dV} = \boldsymbol{\varepsilon} \bullet \mathbf{I} \;\; , \qquad (23.14)$$

and

$$\dot{J}/J = \text{div}\,\mathbf{v} = \mathbf{D} \bullet \mathbf{I} = \dot{\boldsymbol{\varepsilon}} \bullet \mathbf{I} + O(\varepsilon^2) \;\; . \qquad (23.15)$$

<u>Kinetics</u>: It follows from (23.2) and (23.15) that the conservation of mass (14.7) for the linear theory becomes

$$\dot{\rho} + \rho_0\,(\dot{\boldsymbol{\varepsilon}} \bullet \mathbf{I}) = 0 \;\; , \qquad (23.16)$$

where we have neglected terms of order ε^2. Also, since the stresses $\mathbf{T},\boldsymbol{\Pi},\mathbf{S}$ are related by the equations (18.13a),(18.14a) and (18.16a) it follows that

$$\boldsymbol{\Pi} = \mathbf{F}\mathbf{S} = \mathbf{S} + O(\varepsilon^2) \;\; , \;\; \mathbf{T} = J^{-1}\,\boldsymbol{\Pi}\,\mathbf{F}^T = \mathbf{S} + O(\varepsilon^2) \;\; . \qquad (23.17a,b)$$

This means that for the linear theory where we neglect terms of order ε^2 there is no distinction between the three types of stresses

$$\mathbf{T} \approx \boldsymbol{\Pi} \approx \mathbf{S} \;\; . \qquad (23.18)$$

This is of course consistent with the fact that for the linear theory the geometry of the present configuration is only slightly different from the geometry of the reference configuration. Further in this regard we note that

$$\text{div}\,\mathbf{T} = \partial\mathbf{T}/\partial x_i \bullet \mathbf{e}_i = (\partial\mathbf{T}/\partial X_A)\,(\partial X_A/\partial x_i) \bullet \mathbf{e}_i = \partial\mathbf{T}/\partial X_A\,(\delta_{iA}) \bullet \mathbf{e}_i + O(\varepsilon^2)$$

$$= \partial\mathbf{T}/\partial X_A \bullet \mathbf{e}_A + O(\varepsilon^2) = \text{Div}\,\mathbf{T} + O(\varepsilon^2) \;, \qquad (23.19a)$$

$$\text{Div}\,\boldsymbol{\Pi} = \text{Div}\,\mathbf{S} + O(\varepsilon^2) \;\; , \qquad (23.19b)$$

so that the balance of linear momentum (17.5) or (18.5a) yield

$$\rho_0\,\ddot{\mathbf{u}} = \rho_0\,\mathbf{b} + \text{Div}\,\mathbf{T} \;\; , \qquad (23.20)$$

where again we have neglected terms of order ε^2.

<u>Boundary Conditions</u>: The boundary conditions (22.4)-(22.6) are expressed in terms of values of functions of order ε that are evaluated at points on the boundary ∂P of the

surface in the present configuration. The linearized form of these boundary conditions can be determined by considering an arbitrary function f of order ε and using a Taylor series expansion to deduce that

$$f(\mathbf{x},t) = f(\mathbf{X} + \mathbf{u},t) = f(\mathbf{X},t) + \partial f/\partial \mathbf{x} \bullet \mathbf{u} + O(\varepsilon^3) = f(\mathbf{X},t) + O(\varepsilon^2) \quad . \tag{23.21}$$

This means that for the linear theory the distinction between the Lagrangian and Eulerian representations of any function of order ε vanishes. Thus, to within the order of accuracy of the linear theory the boundary conditions can be evaluated at points on the reference boundary ∂P_0 instead of on the present boundary ∂P.

Finally, we emphasize that the linear theory derived from a given nonlinear theory is unique but not vice versa. This means that an infinite number of nonlinear theories exist which when linearized yield the same linear theory. Consequently, a linear theory provides little guidance for developing an appropriate nonlinear theory.

24. Material dissipation

Within the context of the purely mechanical theory it is possible to define the rate of material dissipation $\mathcal{D}$ by the equation

$$\int_P \mathcal{D}\, dv = \mathcal{W} - \dot{\mathcal{K}} - \dot{\mathcal{U}} \ , \tag{24.1}$$

where the kinetic energy $\mathcal{K}$ and the rate of work $\mathcal{W}$ done on the body are defined by (20.3) and (20.4), respectively, and $\mathcal{U}$ represents the total energy associated with the strain energy function Σ

$$\mathcal{U} = \int_P \rho\, \Sigma\, dv \ . \tag{24.2}$$

Next, using (14.9), (20.3), (20.4) and (20.8b) and the local forms of the conservation of mass and the balances of linear and angular momentum, it follows that the rate of work $\mathcal{W}$ can be rewritten in the form

$$\mathcal{W} = \int_P \rho\, \mathbf{b} \bullet \mathbf{v}\, dv + \int_{\partial P} \mathbf{t} \bullet \mathbf{v}\, da \ = \int_P [\, \rho\, \mathbf{b} + \mathrm{div}\, \mathbf{T}\,] \bullet \mathbf{v}\, dv + \int_P \mathbf{T} \bullet \mathbf{L}\, dv \ ,$$

$$\mathcal{W} = \int_P \rho\, \dot{\mathbf{v}} \bullet \mathbf{v}\, dv + \int_P \mathbf{T} \bullet \mathbf{D}\, dv \ ,$$

$$\mathcal{W} = \dot{\mathcal{K}} + \int_P \mathbf{T} \bullet \mathbf{D}\, dv \ . \tag{24.3}$$

Also, it can be shown that

$$\dot{\mathcal{U}} = \int_P \rho\, \dot{\Sigma}\, dv \ . \tag{24.4}$$

and that that local form of the rate of dissipation becomes

$$\mathcal{D} = \mathbf{T} \bullet \mathbf{D} - \rho\, \dot{\Sigma} \geq 0 \ , \tag{24.6}$$

where the condition has been imposed that for general material response the rate of material dissipation must be nonnegative. Moreover, it is observed form (24.1) that for a dissipative material ($\mathcal{D} > 0$) the rate of work supplied to the body is greater than the rates of change of kinetic and strain energies that can be stored in the body.

25. Nonlinear Elastic Solids

In this section we derive constitutive equations for a nonlinear elastic solid within the context of the purely mechanical theory. In general a constitutive equation is an equation that characterizes the response of a given material to deformations or deformation rates. An elastic material is a very special material because is exhibits ideal behavior in the sense that it has no material dissipation. One of the most important features of an elastic material is that it is characterized by a strain energy function.

An elastic material is characterized by the following four assumptions:

Assumption 1: The rate of material dissipation (24.6) vanishes ($\mathcal{D} = 0$)

$$\mathcal{D} = 0 \;\Rightarrow\; \rho \dot{\Sigma} = \mathbf{T} \bullet \mathbf{D} \quad . \tag{25.1}$$

Assumption 2: The strain energy Σ is a function of the deformation gradient $\mathbf{F}$ and reference position $\mathbf{X}$ only

$$\Sigma = \tilde{\tilde{\Sigma}}(\mathbf{F};\mathbf{X}) \quad , \tag{25.2}$$

where we have included dependence on the reference position $\mathbf{X}$ to allow for the possibility that the material may be inhomogeneous in the reference configuration.

Assumption 3: The strain energy Σ is invariant under superposed rigid body motions (SRBM)

$$\Sigma^+ = \Sigma \quad . \tag{25.3}$$

Assumption 4: The Cauchy stress $\mathbf{T}$ is independent of the rate of deformation $\mathbf{L}$.

In order to explore the physical consequences of the assumption 1 we define the total strain energy $\mathcal{U}$ by

$$\mathcal{U} = \int_P \rho\, \Sigma \, dv \quad , \tag{25.4}$$

and use the transport theorem, the conservation of mass, and (25.1) to deduce that

$$\dot{\mathcal{U}} = \int_P \rho\, \dot{\Sigma} \, dv = \int_P \mathbf{T} \bullet \mathbf{D} \, dv \, . \tag{25.5}$$

Thus, using (24.1) and (25.1) it is possible to derive the following theorem:

$$\mathcal{W} = \dot{\mathcal{K}} + \dot{\mathcal{U}} \, , \tag{25.6}$$

which states that for an elastic material the rate of work done on the body due to body forces and contact forces equals the rate of change of kinetic and strain energies. Since Σ depends on the present configuration through the present value of $\mathbf{F}$ only the value of the strain energy Σ is independent of the particular loading path which caused $\mathbf{F}$. Consequently, the total work done on the body vanishes for any closed cycle in which the values of velocity $\mathbf{v}$ and deformation gradient $\mathbf{F}$ are the same at the beginning and end of the cycle.

The assumption (25.3) places restrictions on the functional form (25.2). To see this we recall that under SRBM $\mathbf{F}^+=\mathbf{QF}$ so that (25.3) requires

$$\Sigma^+ = \tilde{\Sigma}(\mathbf{F}^+;\mathbf{X}) = \tilde{\Sigma}(\mathbf{QF};\mathbf{X}) = \tilde{\Sigma}(\mathbf{F};\mathbf{X}) \quad , \tag{25.7}$$

to hold for arbitrary proper orthogonal $\mathbf{Q}$. However, with the help of the polar decomposition theorem $\mathbf{F}=\mathbf{RM}$ we deduce the restriction that

$$\tilde{\Sigma}(\mathbf{F};\mathbf{X}) = \tilde{\Sigma}(\mathbf{QF};\mathbf{X}) = \tilde{\Sigma}(\mathbf{QRM};\mathbf{X}) \quad , \tag{25.8}$$

must hold for arbitrary values of the proper orthogonal tensor $\mathbf{Q}$. Since the deformation may be inhomogeneous the rotation tensor $\mathbf{R}$ may be a function of position $\mathbf{X}$. However, for a given value of $\mathbf{X}$, say $\mathbf{X}_1$ we may choose $\mathbf{Q}=\mathbf{R}^T(\mathbf{X}_1)$ so that (25.8) yields

$$\tilde{\Sigma}(\mathbf{F};\mathbf{X}) = \tilde{\Sigma}\big(\mathbf{R}^T(\mathbf{X}_1)\mathbf{RM};\mathbf{X}\big) \ . \tag{25.9}$$

Now, evaluating (25.9) at $\mathbf{X}_1$ we deduce that locally

$$\tilde{\Sigma}(\mathbf{F};\mathbf{X}_1) = \tilde{\Sigma}(\mathbf{M};\mathbf{X}_1) = \hat{\Sigma}(\mathbf{C};\mathbf{X}_1) \ . \tag{25.10}$$

Thus, a necessary condition for the strain energy Σ to be locally invariant under SRBM is that the strain energy function Σ can depend on the deformation gradient $\mathbf{F}$ only through its dependence on the deformation tensor $\mathbf{C}$. It is easy to see that this condition is also a sufficient condition because under SRBM $\mathbf{C}^+=\mathbf{C}$. However, since $\mathbf{X}_1$ is an arbitrary material point we conclude that for each point $\mathbf{X}$ the strain energy Σ can depend on $\mathbf{F}$ only through its dependence on $\mathbf{C}$

$$\tilde{\Sigma}(\mathbf{F};\mathbf{X}) = \hat{\Sigma}(\mathbf{C};\mathbf{X}) \ . \tag{25.11}$$

Now, with the help of (25.8) equation (25.1) yields

$$\mathbf{T} \bullet \mathbf{D} = \rho \frac{\partial \Sigma}{\partial \mathbf{C}} \bullet \dot{\mathbf{C}} = \rho \frac{\partial \Sigma}{\partial \mathbf{C}} \bullet 2\, \mathbf{F}^T \mathbf{D} \mathbf{F} = 2\rho\, \mathbf{F} \frac{\partial \Sigma}{\partial \mathbf{C}}\, \mathbf{F}^T \bullet \mathbf{D} \ , \tag{25.12a}$$

$$(\mathbf{T} - 2\rho\, \mathbf{F} \frac{\partial \Sigma}{\partial \mathbf{C}}\, \mathbf{F}^T) \bullet \mathbf{D} = 0 \ . \tag{25.12b}$$

However, since the coefficient of **D** in (25.12b) is independent of the rate **D** and is symmetric it follows that for any fixed values of **F**,**X** the coefficient of **D** is fixed and yet the **D** can be an arbitrary symmetric tensor. Therefore, the necessary condition that (25.12b) be valid for arbitrary motions is that the Cauchy stress be given by a derivative of the strain energy

$$\mathbf{T} = 2\rho\, \mathbf{F} \frac{\partial \Sigma}{\partial \mathbf{C}}\, \mathbf{F}^T \ . \tag{25.13}$$

Using the conservation of mass (14.5) and the relationship (18.16a) the symmetric Piola-Kirchhoff stress **S** becomes

$$\mathbf{S} = 2\rho_0 \frac{\partial \Sigma}{\partial \mathbf{C}} \ . \tag{25.14}$$

Notice that the results (25.13) and (25.14) are automatically properly invariant under SRBM. Also, notice that the result (25.14) is similar to the result for a simple spring that the force is equal to a derivative of the potential (strain) energy.

26. Material Symmetry

In order to continue our discussion of an elastic material it is desirable to first consider the notion of material symmetry. To this end, consider a general elastic material which is referred to a reference configuration with base vectors $\mathbf{e}_A$. Then, with reference to another orthogonal set of base vectors $\mathbf{e}_A^*$ let us machine a tension specimen from the material that is oriented in the $\mathbf{e}_1^*$ direction. In general the response of this tension specimen will be different for different choices of the direction $\mathbf{e}_1^*$. If this is true then the material is called anisotropic. On the other hand if the response of the material is the same for all choices of the direction $\mathbf{e}_1^*$ then the material is called isotropic.

In order to make these notions more precise let us consider an arbitrary deformation $\mathbf{C}$ from the reference configuration which is defined by its components C_{AB} relative to the reference axes such that

$$\mathbf{C} = C_{AB}\, \mathbf{e}_A \otimes \mathbf{e}_B \quad . \tag{26.1}$$

Now consider another deformation $\mathbf{C}^*$ relative to the same reference configuration which is related to $\mathbf{C}$ and is defined by

$$\mathbf{C}^* = C_{AB}\, \mathbf{e}_A^* \otimes \mathbf{e}_B^* \quad . \tag{26.2}$$

Since the components of $\mathbf{C}^*$ relative to the axis $\mathbf{e}_A^*$ are the same as the components of $\mathbf{C}$ relative to the axes $\mathbf{e}_A$, $\mathbf{C}^*$ represents the same deformation as $\mathbf{C}$ but relative to a different set of material axes. This means that by comparing the responses to $\mathbf{C}$ and $\mathbf{C}^*$ we can compare the responses associated with different material orientations. More specifically we say that the response to the deformations $\mathbf{C}$ and $\mathbf{C}^*$ is the same if the value of the strain energy Σ is the same for all values of C_{AB}

$$\hat{\Sigma}(\mathbf{C}) = \hat{\Sigma}(\mathbf{C}^*) \quad . \tag{26.3}$$

In general we can define the orientation of the material axes $\mathbf{e}_A^*$ relative to $\mathbf{e}_A$ by the orthogonal transformation $\mathbf{H}$ defined by

$$\mathbf{H} = \mathbf{e}_A \otimes \mathbf{e}_A^* \quad , \quad \mathbf{H}\mathbf{H}^T = \mathbf{H}^T\mathbf{H} = \mathbf{I} \quad . \tag{26.4a,b}$$

It follows from (26.4) that

$$\mathbf{e}_A = \mathbf{H}\,\mathbf{e}_A^* \ , \ \mathbf{e}_A^* = \mathbf{H}^T\,\mathbf{e}_A \ , \tag{26.5a,b}$$

so that $\mathbf{C}^*$ is related to $\mathbf{C}$ and $\mathbf{H}$ by the formula

$$\mathbf{C}^* = C_{AB}\,\mathbf{H}^T\,\mathbf{e}_A \otimes \mathbf{H}^T\,\mathbf{e}_B = \mathbf{H}^T\,(C_{AB}\,\mathbf{e}_A \otimes \mathbf{e}_B)\,\mathbf{H} = \mathbf{H}^T\,\mathbf{C}\,\mathbf{H} \ . \tag{26.6}$$

Now with the help of (26.3) and (26.6) it follows that the response of the material to arbitrary deformations associated with different material orientations will be the same provided that

$$\hat{\Sigma}\,(\mathbf{C}) = \hat{\Sigma}\,(\mathbf{H}^T\mathbf{C}\mathbf{H}) \ . \tag{26.9}$$

In other words, the functional form of the strain energy Σ remains form-invariant to a group of orthogonal transformations **H** which characterize the material symmetries exhibited by a given material.For the case of crystalline materials these symmetry groups can be related to the different crystal structures.

For the most general anisotropic elastic material the material has no symmetry so the group of **H** contains only the identity **I**, whereas an isotropic elastic material has complete symmetry so the group of **H** is the full orthogonal group. Furthermore it is important to emphasize that the notion of material symmetry is necessarily connected with the chosen reference configuration because **H** in (26.4a) is defined relative to fixed material directions in this reference configuration.

27. Isotropic Nonlinear Elastic Material

If an elastic material is isotropic in its reference configuration then the strain energy Σ must remain form-invariant for the full orthogonal group of **H** in (26.9). It follows that Σ must be an isotropic function of **C**, which in turn means that Σ must depend on **C** only through its invariants. Recalling that the principal invariants of **C** are the same as the invariants of **B** and are given by

$$I_1 = \mathbf{C} \bullet \mathbf{I} = \mathbf{B} \bullet \mathbf{I} \,, \tag{27.1a}$$

$$I_2 = \frac{1}{2} \left[(\mathbf{C} \bullet \mathbf{I})^2 - \mathbf{C} \bullet \mathbf{C} \right] = \frac{1}{2} \left[(\mathbf{B} \bullet \mathbf{I})^2 - \mathbf{B} \bullet \mathbf{B} \right] , \tag{27.1b}$$

$$J^2 = I_3 = \det \mathbf{C} = \det \mathbf{B} \,, \tag{27.1c}$$

it follows that for an isotropic elastic material the strain energy Σ can be an arbitrary function of the invariants I_1, I_2, J

$$\Sigma = \Sigma\, (I_1, I_2, J) \ . \tag{27.2}$$

Furthermore, from (27.1) we may deduce that

$$\dot{I}_1 = \mathbf{I} \bullet \dot{\mathbf{C}} \ \ , \ \dot{I}_2 = \left[\ (\mathbf{C} \bullet \mathbf{I})\, \mathbf{I} - \mathbf{C} \right] \bullet \dot{\mathbf{C}} \ \ , \ \dot{J} = J\, \mathbf{I} \bullet \mathbf{D} = \frac{1}{2} J\, \mathbf{C}^{-1} \bullet \dot{\mathbf{C}} \,. \tag{27.3a,b,c}$$

Thus, (25.14) and (25.13) yield

$$\mathbf{S} = 2\rho_0 \left[\frac{\partial \Sigma}{\partial I_1} + (\mathbf{C} \bullet \mathbf{I}) \frac{\partial \Sigma}{\partial I_2} \right] \mathbf{I} - 2\rho_0 \left[\frac{\partial \Sigma}{\partial I_2} \right] \mathbf{C} + \rho_0 \left[\frac{\partial \Sigma}{\partial J} \right] J\, \mathbf{C}^{-1} \,, \tag{27.4a}$$

$$\mathbf{T} = 2\rho_0 J^{-1} \left[\frac{\partial \Sigma}{\partial I_1} + (\mathbf{B} \bullet \mathbf{I}) \frac{\partial \Sigma}{\partial I_2} \right] \mathbf{B} - 2\rho_0 J^{-1} \left[\frac{\partial \Sigma}{\partial I_2} \right] \mathbf{B}^2 + \rho_0 \left[\frac{\partial \Sigma}{\partial J} \right] \mathbf{I} \,. \tag{27.4b}$$

Also, with the help of (18.17) and (18.18) we may deduce that the pressure p, the deviator **T**', and the tensor **S**' are given by

$$p = -\frac{2}{3} \rho_0 J^{-1} \left[\frac{\partial \Sigma}{\partial I_1} + (\mathbf{B} \bullet \mathbf{I}) \frac{\partial \Sigma}{\partial I_2} \right] \mathbf{B} \bullet \mathbf{I} + \frac{2}{3} \rho_0 J^{-1} \left[\frac{\partial \Sigma}{\partial I_2} \right] \mathbf{B}^2 \bullet \mathbf{I} - \rho_0 \left[\frac{\partial \Sigma}{\partial J} \right] \,, \tag{27.5a}$$

$$\mathbf{S}' = 2\rho_0 \left[\frac{\partial \Sigma}{\partial I_1} + (\mathbf{C} \bullet \mathbf{I}) \frac{\partial \Sigma}{\partial I_2} \right] \left[\mathbf{I} - \frac{1}{3} (\mathbf{C} \bullet \mathbf{I})\, \mathbf{C}^{-1} \right]$$

$$- 2\rho_0 \left[\frac{\partial \Sigma}{\partial I_2} \right] \left[\mathbf{C} - \frac{1}{3} (\mathbf{C}^2 \bullet \mathbf{I})\, \mathbf{C}^{-1} \right] , \tag{27.5b}$$

$$\mathbf{T}' = 2\rho_0 J^{-1}\left[\frac{\partial\Sigma}{\partial I_1} + (\mathbf{B}\bullet\mathbf{I})\frac{\partial\Sigma}{\partial I_2}\right]\left[\mathbf{B} - \frac{1}{3}(\mathbf{B}\bullet\mathbf{I})\,\mathbf{I}\right]$$

$$- 2\rho_0 J^{-1}\left[\frac{\partial\Sigma}{\partial I_2}\right]\left[\mathbf{B}^2 - \frac{1}{3}(\mathbf{B}^2\bullet\mathbf{I})\,\mathbf{I}\right] \ . \tag{27.5c}$$

Notice from (27.5a) that all three invariants I_1, I_2, J contribute to the determination of the pressure. This is because the invariants I_1 and I_2 are not pure measures of distortional deformation. However, recalling from (7.23) that $\mathbf{C}'$ is a pure measure of distortional deformation [det $\mathbf{C}'=1$] it follows that $\mathbf{C}'$ has only two nontrivial invariants which can be written in the forms

$$\alpha_1 = \mathbf{C}'\bullet\mathbf{I} = \mathbf{B}'\bullet\mathbf{I} \ , \ \ \alpha_2 = \mathbf{C}'\bullet\mathbf{C}' = \mathbf{B}'\bullet\mathbf{B}' \ . \tag{27.6a,b}$$

where $\mathbf{B}'$ is the distortional part of $\mathbf{B}$ defined by

$$\mathbf{B}' = J^{-2/3}\,\mathbf{B} \ \ , \ \ \det\mathbf{B}' = 1 \ \ . \tag{27.7a,b}$$

Thus, a general isotropic elastic material can be characterized by the alternative assumption that the strain energy function Σ depends on the invariants $\{\alpha_1,\alpha_2,J\}$ in stead of on $\{I_1,I_2,J\}$ so that

$$\Sigma = \hat{\Sigma}\,(\alpha_1,\alpha_2,J) \ . \tag{27.8}$$

Now in order to determine expressions similar to (27.5) associated with the assumption (27.8) we note that

$$\dot{\alpha}_1 = J^{-2/3}\left[\mathbf{I} - \frac{1}{3}(\mathbf{C}\bullet\mathbf{I})\,\mathbf{C}^{-1}\right]\bullet\dot{\mathbf{C}} \ \ , \tag{27.9a}$$

$$\dot{\alpha}_2 = 2\,J^{-4/3}\left[\mathbf{C} - \frac{1}{3}(\mathbf{C}^2\bullet\mathbf{I})\,\mathbf{C}^{-1}\right]\bullet\dot{\mathbf{C}} \ \ . \tag{27.9b}$$

Then, using these results we have

$$p = -\,\rho_0\,\frac{\partial\hat{\Sigma}}{\partial J} \ , \tag{27.10a}$$

$$\mathbf{S}' = 2\,J^{-2/3}\,\rho_0\left[\frac{\partial\hat{\Sigma}}{\partial\alpha_1}\right]\left[\mathbf{I} - \frac{1}{3}(\mathbf{C}\bullet\mathbf{I})\,\mathbf{C}^{-1}\right]$$

$$+ 4\,J^{-4/3}\rho_0\left[\frac{\partial\hat{\Sigma}}{\partial\alpha_2}\right]\left[\mathbf{C} - \frac{1}{3}(\mathbf{C}^2\bullet\mathbf{I})\,\mathbf{C}^{-1}\right] \ , \tag{27.10b}$$

$$\mathbf{T}' = 2\, J^{-2/3}\, \rho_0 J^{-1} \left[\frac{\partial \hat{\Sigma}}{\partial \alpha_1} \right] \left[\mathbf{B} - \frac{1}{3} (\mathbf{B} \bullet \mathbf{I})\, \mathbf{I} \right]$$

$$+ 4\, J^{-4/3}\, \rho_0 J^{-1} \left[\frac{\partial \hat{\Sigma}}{\partial \alpha_2} \right] \left[\mathbf{B}^2 - \frac{1}{3} (\mathbf{B}^2 \bullet \mathbf{I})\, \mathbf{I} \right] . \tag{27.10c}$$

Now, notice that the pressure is related to the derivative of Σ with respect to the dilatation J and the deviatoric stress $\mathbf{T}'$ is related to derivatives of Σ with respect to the distortional measures of deformation α_1 and α_2, but also depends on the dilatation J. However, this does not mean that the pressure is independent of (α_1,α_2) because the derivative $\partial\hat{\Sigma}/\partial J$ may retain dependence on (α_1,α_2).

Significant advances in the theory of finite elasticity were made studying the response of natural rubber and modeling the material by a Neo-Hookean strain energy

$$\rho_0\, \Sigma = C_1\, (I_1 - 3) \ , \tag{27.11}$$

or a Mooney-Rivlin strain energy

$$\rho_0\, \Sigma = C_1\, (I_1 - 3) + C_2\, (I_2 - 3) \ , \tag{27.12}$$

where C_1 and C_2 are material constants. Also, in these studies rubber was modeled as an incompressible material using the constraint that

$$J = 1 \ . \tag{27.11}$$

In general for a constrained theory the stress $\mathbf{T}$ is separated additively into a part $\hat{\mathbf{T}}$ determined by constitutive equations of the type (27.5) or (27.10) and another part $\bar{\mathbf{T}}$, called a constraint response which is assumed to do not work and is determined by the equations of motion and boundary conditions. Thus, in a constrained theory the Cauchy stress $\mathbf{T}$ is given by

$$\mathbf{T} = \hat{\mathbf{T}} + \bar{\mathbf{T}} \ , \ \bar{\mathbf{T}} \bullet \mathbf{D} = 0 \ . \tag{27.12a,b}$$

For the specific case of incompressibility it may be shown that the constraint response becomes

$$\bar{\mathbf{T}} = -\bar{p}\ \mathbf{I} \ . \tag{27.13}$$

Consequently, since $\overline{p}$ is not determined by a constitutive equation the total pressure p is also not determined by a constitutive equation.

28. Linear Elastic Material

For a linear elastic material the stress **S** is a linear function of the strain **E**. Consequently, from the result (25.14) we observe that the strain energy is a quadratic function of the strain **E**. For example, let **K** be a constant fourth order tensor that characterizes the elastic properties of the material such that the strain energy is defined by

$$\rho_0 \, \Sigma = \frac{1}{2} \, \mathbf{K} \bullet (\mathbf{E} \otimes \mathbf{E}) \ . \tag{28.1}$$

It follows from (28.1) that since the strain **E** is symmetric and the fourth order tensor **E**⊗**E** is symmetric $\left[\ (\mathbf{E}\otimes\mathbf{E})^{T(2)} = {}^{LT(2)}(\mathbf{E}\otimes\mathbf{E}) = (\mathbf{E}\otimes\mathbf{E}) \ \right]$, the tensor **K** has the following symmetries

$$\mathbf{K} = \mathbf{K}^T = {}^{LT}\mathbf{K} = \mathbf{K}^{T(2)} \ . \tag{28.2}$$

Thus, it follows from (25.14), (28.1), and (28.2) that

$$\mathbf{S} = \rho_0 \frac{\partial \Sigma}{\partial \mathbf{E}} = \mathbf{K} \bullet \mathbf{E} \ . \tag{28.3}$$

Letting E_{AB}, S_{AB} , K_{ABCD} be the Cartesian components of **E**, **S**, **K**, respectively, equations (28.1),(28.2) and (28.3) may be written in the component forms

$$\rho_0 \, \Sigma = \frac{1}{2} K_{ABCD} \, E_{AB} \, E_{CD} \ , \tag{28.4a}$$

$$K_{ABCD} = K_{ABDC} = K_{BACD} = K_{CDAB} \ , \tag{28.4b}$$

$$S_{AB} = K_{ABCD} \, E_{CD} \ . \tag{28.4c}$$

Material symmetry considerations of the functional form (28.1) define a group of orthogonal transformations **H** for which

$$\mathbf{K} \bullet (\mathbf{E} \otimes \mathbf{E}) = \mathbf{K} \bullet (\mathbf{H}^T\mathbf{E}\mathbf{H} \otimes \mathbf{H}^T\mathbf{E}\mathbf{H}) \ , \tag{28.5a}$$

$$K_{ABCD} \, E_{AB} \, E_{CD} = K_{MNRS} \, (H_{AM}E_{AB}H_{BN}) \, (H_{CR}E_{CD}H_{DS}) \ . \tag{28.5b}$$

However, since (28.5b) must be valid for arbitrary values of the strain E_{AB} and since K_{ABCD} and H_{AB} are independent of the strain we deduce that

$$K_{ABCD} = H_{AM} \, H_{BN} \, H_{CR} \, H_{DS} \, K_{MNRS} \ . \tag{28.6}$$

In the following we consider four Cases of materials:

<u>Case I (General Anisotropic)</u>: If the material posseses no symmetry then the symmetry group consists only of **H=I** and the 81 constants K_{ABCD} are restricted only by the

symmetries (28.2) and the number of independent constants reduces to 21 which are given by

$$\begin{pmatrix} K_{1111} & K_{1112} & K_{1113} & K_{1122} & K_{1123} & K_{1133} & K_{1212} \\ K_{1213} & K_{1222} & K_{1223} & K_{1233} & K_{1313} & K_{1322} & K_{1323} \\ K_{1333} & K_{2222} & K_{2223} & K_{2233} & K_{2323} & K_{2333} & K_{3333} \end{pmatrix} \tag{28.7}$$

<u>Case II</u>: If the material possesses symmetry about the X_3=0 plane then we may take

$$H_{AB} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \ , \tag{28.8}$$

so that from (28.6) and (28.7) it follows that any component in which the index 3 appears an odd number of times must vanish

$$K_{1113} = K_{1123} = K_{1213} = K_{1223} = K_{1322} = K_{1333} = K_{2223} = K_{2333} = 0 \ . \tag{28.9}$$

Thus, the remaining 13 independent constants are given by

$$\begin{pmatrix} K_{1111} & K_{1112} & K_{1122} & K_{1133} & K_{1212} & K_{1222} & K_{1233} \\ K_{1313} & K_{1323} & K_{2222} & K_{2233} & K_{2323} & K_{3333} & \end{pmatrix} \tag{28.10}$$

<u>Case III</u>: If the material possesses symmetry about the X_3=0 and X_2=0 planes then in addition to (28.8) we may take

$$H_{AB} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \tag{28.11}$$

so that from (28.6) and (28.10) it follows that any component in which the index 2 appears an odd number of times must vanish

$$K_{1112} = K_{1222} = K_{1233} = K_{1323} = 0 \ . \tag{28.12}$$

Thus, the remaining 9 independent constants are given by

$$\begin{pmatrix} K_{1111} & K_{1122} & K_{1133} & K_{1212} & K_{1313} & K_{2222} & K_{2233} \\ K_{2323} & K_{3333} & & & & & \end{pmatrix} \ . \tag{28.13}$$

Notice from (28.13) that the index 1 only appears an even number of times so that the material also possesses symmetry about the X_1=0 plane. This material is called <u>orthotropic</u>.

<u>Case IV</u>: If the material possesses symmetry with respect to the full orthogonal group then the material is called <u>isotropic</u> with a center of symmetry. Using the results of

Appendix E it follows that the material is characterized by only two independent constants λ and μ, called Lame's constants, such that

$$K_{1111} = K_{2222} = K_{3333} = \lambda + 2\mu \ , \ K_{1122} = K_{1133} = K_{2233} = \lambda \ , \quad (28.14a,b)$$

$$K_{1212} = K_{1313} = K_{2323} = \mu \ . \quad (28.14c)$$

Thus, the fourth order tensor **K** may be expressed in the forms

$$K_{ABCD} = \lambda \, \delta_{AB} \, \delta_{CD} + \mu \, [\, \delta_{AC} \, \delta_{BD} + \delta_{AD} \, \delta_{BC} \,] \ , \quad (28.15a)$$

$$\mathbf{K} = \lambda \, \mathbf{I} \otimes \mathbf{I} + \mu \, [\, \mathbf{e}_M \otimes \mathbf{e}_N \otimes \mathbf{e}_M \otimes \mathbf{e}_N + \mathbf{e}_M \otimes \mathbf{e}_N \otimes \mathbf{e}_N \otimes \mathbf{e}_M \,] \ . \quad (28.15b)$$

It follows that the strain energy (28.1) and the stress (28.3) may be written in the forms

$$\rho_0 \, \Sigma = \frac{1}{2} \ \lambda \, (\mathbf{E} \bullet \mathbf{I})^2 + \mu \, \mathbf{E} \bullet \mathbf{E} \ , \quad (28.16a)$$

$$\mathbf{S} = \lambda \, (\mathbf{E} \bullet \mathbf{I}) \, \mathbf{I} + 2\mu \, \mathbf{E} \ . \quad (28.16b)$$

Notice that the strain energy (28.16a) is a function of the invariants of **E** as it should be for an isotropic material.

In the above we have characterized an elastic material which has a strain energy that is a quadratic function of strain **E** and a stress **S** that is a linear function of strain **E**. If **E** is the exact Lagrangian strain then the formulation is exact and in particular is valid for large rotations. Of course it represents a special constitutive equation that should be expected to be reasonably accurate for small values of strain. In order to obtain the fully linearized theory we restrict the displacement **u** to be small. This forces the rotations also to remain small and the strain **E** to be approximated by the linear strain $\boldsymbol{\varepsilon}$.

From physical considerations we expect that any strain should cause an increase in strain energy. Mathematically, this means that the strain energy function is positive definite

$$\Sigma > 0 \quad \text{for any} \ \ \mathbf{E} \neq 0 \ . \quad (28.17)$$

Recalling that the strain **E** may be separated into its spherical and deviatoric parts

$$\mathbf{E} = \frac{1}{3} (\mathbf{E} \bullet \mathbf{I}) \ \mathbf{I} + \mathbf{E}' \ , \ \mathbf{E}' \bullet \mathbf{I} = 0 \ , \quad (28.18a,b)$$

it follows that the strain energy may be rewritten in the form

$$\rho_0 \, \Sigma = \frac{1}{2} \left(\frac{3\lambda + 2\mu}{3} \right) (\mathbf{E} \bullet \mathbf{I})^{\,2} + \mu \, \mathbf{E}' \bullet \mathbf{E}' \ . \quad (28.19)$$

Since the terms ($\mathbf{E} \bullet \mathbf{I}$) and $\mathbf{E}' \bullet \mathbf{E}'$ are independent of each other we may deduce that the strain energy will be positive definite whenever

$$3\lambda + 2\mu > 0 \ , \ \mu > 0 \ . \qquad (28.20a,b)$$

Finally, we note that an isotropic elastic material can be characterized by any two of the following material constants: λ (Lame's constant); μ (shear modulus); E (Young's modulus); ν (Poisson's ratio); or k (bulk modulus), which are related in Table 28.1. Furthermore, using Table 27.1 it may be shown that the restrictions (28.20) also require that

$$k > 0 \ , \ E > 0 \ , \ -1 < \nu < \frac{1}{2} \ . \qquad (28.21a,b,c)$$

	λ	μ	E	ν	k
λ , μ			$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\frac{\lambda}{2(\lambda+\mu)}$	$\frac{3\lambda+2\mu}{3}$
λ , ν		$\frac{\lambda(1-2\nu)}{2\nu}$	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$		$\frac{\lambda(1+\nu)}{3\nu}$
λ , k		$\frac{3(k-\lambda)}{2}$	$\frac{9k(k-\lambda)}{3k-\lambda}$	$\frac{\lambda}{3k-\lambda}$	
μ , E	$\frac{\mu(2\mu-E)}{E-3\mu}$			$\frac{E-2\mu}{2\mu}$	$\frac{\mu E}{3(3\mu-E)}$
μ , ν	$\frac{2\mu\nu}{1-2\nu}$		$2\mu(1+\nu)$		$\frac{2\mu(1+\nu)}{3(1-2\nu)}$
μ , k	$\frac{3k-2\mu}{3}$		$\frac{9k\mu}{3k+\mu}$	$\frac{3k-2\mu}{2(3k+\mu)}$	
E , ν	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$			$\frac{E}{3(1-2\nu)}$
E , k	$\frac{3k(3k-E)}{9k-E}$	$\frac{3Ek}{9k-E}$		$\frac{3k-E}{6k}$	
ν , k	$\frac{3k\nu}{1+\nu}$	$\frac{3k(1-2\nu)}{2(1+\nu)}$	$3k(1-2\nu)$		

$$\mu = \frac{(E-3\lambda)+\sqrt{(E-3\lambda)^2+8\lambda E}}{4} , \qquad \nu = \frac{-(E+\lambda)+\sqrt{(E+\lambda)^2+8\lambda^2}}{4\lambda} ,$$

$$k = \frac{(3\lambda+E)+\sqrt{(3\lambda+E)^2-4\lambda E}}{6}$$

Table 28.1

29. Viscous and Inviscid Fluids

Within the context of the purely mechanical theory a general viscous fluid is characterized by the constitutive assumption that the Cauchy stress **T** is a function of the dilatation J, the velocity **v**, and the velocity gradient **L**. However, for convenience it is desirable to separate **L** into its symmetric part **D** and its skew-symmetric part **W** and write

$$\mathbf{T} = \tilde{\mathbf{T}}(J, \mathbf{v}, \mathbf{D}, \mathbf{W}). \tag{29.1}$$

In the following we will use invariance under superposed rigid body motions (SRBM) to develop restrictions on the functional form (29.1). To this end, recall that since (29.1) must hold for all motions it must also hold for SRBM so that

$$\mathbf{T}^+ = \tilde{\mathbf{T}}(J^+, \mathbf{v}^+, \mathbf{D}^+, \mathbf{W}^+) \ . \tag{29.2}$$

However, under SRBM the Cauchy stress **T** transforms by

$$\mathbf{T}^+ = \mathbf{Q}\mathbf{T}\mathbf{Q}^T \ , \tag{29.3}$$

where **Q** is a proper orthogonal tensor function of time only. Thus, the functional form (29.1) must satisfy the restrictions

$$\tilde{\mathbf{T}}(J^+, \mathbf{v}^+, \mathbf{D}^+, \mathbf{W}^+) = \mathbf{Q}\tilde{\mathbf{T}}(J, \mathbf{v}, \mathbf{D}, \mathbf{W})\mathbf{Q}^T \ . \tag{29.4}$$

Recalling that under SRBM

$$J^+ = J \ , \ \mathbf{v}^+ = \dot{\mathbf{c}} + \boldsymbol{\Omega}\,\mathbf{Q}\mathbf{x} + \mathbf{Q}\mathbf{v} \ , \ \dot{\mathbf{Q}} = \boldsymbol{\Omega}\mathbf{Q} \ , \tag{29.5a,b,c}$$

$$\mathbf{D}^+ = \mathbf{Q}\mathbf{D}\mathbf{Q}^T \ , \ \mathbf{W}^+ = \mathbf{Q}\mathbf{W}\mathbf{Q}^T + \boldsymbol{\Omega} \ , \tag{29.5d,e}$$

equation (29.4) becomes

$$\tilde{\mathbf{T}}(J, \dot{\mathbf{c}} + \boldsymbol{\Omega}\,\mathbf{Q}\mathbf{x} + \mathbf{Q}\mathbf{v}, \mathbf{Q}\mathbf{D}\mathbf{Q}^T, \mathbf{Q}\mathbf{W}\mathbf{Q}^T + \boldsymbol{\Omega}) = \mathbf{Q}\tilde{\mathbf{T}}(J, \mathbf{v}, \mathbf{D}, \mathbf{W})\mathbf{Q}^T \ . \tag{29.6}$$

Since (29.6) must hold for all motions we can obtain necessary restrictions on the functional form $\tilde{\mathbf{T}}$ by considering special SRBM. Specifically, consider the simple SRBM characterized by a superposed rigid body translational velocity for which

$$\dot{\mathbf{c}} \neq 0 \ , \ \mathbf{Q} = \mathbf{I} \ , \ \boldsymbol{\Omega} = 0 \ . \tag{29.7a,b,c}$$

Substituting (29.7) into (29.6) we have

$$\tilde{\mathbf{T}}(J, \dot{\mathbf{c}} + \mathbf{v}, \mathbf{D}, \mathbf{W}) = \tilde{\mathbf{T}}(J, \mathbf{v}, \mathbf{D}, \mathbf{W}) \ . \tag{29.8}$$

However, since we can choose the value of $\dot{\mathbf{c}}$ arbitrarily and the right hand side of (29.8) is independent of $\dot{\mathbf{c}}$ it follows that the Cauchy stress cannot depend on the velocity **v**. Thus, **T** must be expressed as another function $\bar{\mathbf{T}}$ of J,**D**,**W** only

$$\mathbf{T} = \bar{\mathbf{T}}(J, \mathbf{D}, \mathbf{W}) \ , \tag{29.9}$$

and the restriction (29.6) becomes

$$\bar{\mathbf{T}}(J, \mathbf{QDQ}^T, \mathbf{QWQ}^T + \mathbf{\Omega}) = \mathbf{Q}\bar{\mathbf{T}}(J, \mathbf{D}, \mathbf{W})\mathbf{Q}^T \ . \tag{29.10}$$

Next consider the special case of rigid body rotation for which at time t we specify

$$\mathbf{Q} = \mathbf{I} \ , \ \dot{\mathbf{Q}} = \mathbf{\Omega} \ . \tag{29.11}$$

Substituting (29.11) into (29.10) we require

$$\bar{\mathbf{T}}(J, \mathbf{D}, \mathbf{W} + \mathbf{\Omega}) = \bar{\mathbf{T}}(J, \mathbf{D}, \mathbf{W}). \tag{29.12}$$

However, $\mathbf{\Omega}$ can be an arbitrary skew-symmetric tensor and the right hand side of (29.12) is independent of $\mathbf{\Omega}$ so we may conclude that the Cauchy stress cannot depend on the spin tensor **W**. This means that the most general viscous fluid is characterized by the constitutive equation

$$\mathbf{T} = \hat{\mathbf{T}}(J) + \check{\mathbf{T}}(J, \mathbf{D}) \ , \tag{29.13}$$

where $\hat{\mathbf{T}}(J)$ characterizes the elastic response due to dilatation and $\check{\mathbf{T}}(J, \mathbf{D})$ characterizes the viscous response. Also, these constitutive equations must satisfy the restrictions that

$$\hat{\mathbf{T}}(J) = \mathbf{Q}\hat{\mathbf{T}}(J)\mathbf{Q}^T \ , \ \check{\mathbf{T}}(J, \mathbf{QDQ}^T) = \mathbf{Q}\check{\mathbf{T}}(J, \mathbf{D})\mathbf{Q}^T \ . \tag{29.14a,b}$$

<u>Reiner-Rivlin Fluid</u>: Since the restrictions (29.14b) must hold for all proper orthogonal **Q** the function $\check{\mathbf{T}}$ is called an isotropic tensor function of its argument **D**. This notion of an isotropic tensor function should not be confused with the notion of an isotropic tensor as discussed in appendix E. Furthermore, since the restriction (29.14b) is unaltered by the interchange of **Q** with **–Q** it follows that $\check{\mathbf{T}}$ is a hemotropic function of **D** (isotropic

with a center of symmetry). Now, using a result from the theory of invariants it follows that the most general form of $\check{\mathbf{T}}$ can be expressed as

$$\check{\mathbf{T}} = d_0\,\mathbf{I} + d_1\,\mathbf{D} + d_2\,\mathbf{D}^2 \quad , \tag{29.15}$$

where d_0, d_1, d_2 are scalar functions of J and the three invariants of **D**. This functional form characterizes what is called a Reiner-Rivlin fluid. Moreover, the strain energy is taken to be a function of the dilatation

$$\Sigma = \hat{\Sigma}(J) \quad , \tag{29.16}$$

and the elastic stress $\hat{\mathbf{T}}$ and the rate of material dissipation (24.6) are given by

$$\hat{\mathbf{T}}(J) = -\,\hat{p}(J)\,\mathbf{I} \;\; , \;\; \hat{p}(J) = -\,\rho_0\,\frac{\partial \hat{\Sigma}}{\partial J} \;\; , \;\; \mathcal{D} = \check{\mathbf{T}} \bullet \mathbf{D} \geq 0 \quad , \tag{29.17a,b,c}$$

which places restrictions on the functional form for the viscous stress $\check{\mathbf{T}}$.

<u>Newtonian Viscous Fluid</u>: A Newtonian viscous fluid is a special case of a Reiner-Rivlin fluid in which the viscous stress $\check{\mathbf{T}}$ is a linear function of the rate of deformation **D**. For this case, $\check{\mathbf{T}}$ reduces to

$$\check{\mathbf{T}} = \lambda\,(\mathbf{D} \bullet \mathbf{I})\,\mathbf{I} + 2\mu\,\mathbf{D} \quad , \tag{29.18}$$

where λ and μ are scalar functions of J only. It follows that $\check{\mathbf{T}}$ can be rewritten in the alternative form

$$\mathbf{T} = -\,\hat{p}(J)\,\mathbf{I} + \check{\mathbf{T}} \;\; , \;\; \check{\mathbf{T}} = -\,\check{p}\,\mathbf{I} + 2\mu\,\mathbf{D}' \quad , \tag{29.19a}$$

$$p = -\frac{1}{3}\,\mathbf{T} \bullet \mathbf{I} = \hat{p} + \check{p} \;\; , \;\; \check{p} = -\frac{1}{3}\,\check{\mathbf{T}} \bullet \mathbf{I} = -\,(\lambda + \frac{2}{3}\,\mu)\,(\mathbf{D} \bullet \mathbf{I}) \;, \tag{29.19b,c}$$

$$\mathbf{D}' = \mathbf{D} - \frac{1}{3}\,(\mathbf{D} \bullet \mathbf{I})\;\mathbf{I} \tag{29.19d}$$

which shows that the total pressure p has an elastic part $\hat{p}$ and a viscous part $\check{p}$ that depends on the rate of volume expansion ($\mathbf{D} \bullet \mathbf{I}$). Moreover, the rate of material dissipation (29.17c) is satisfied provided that

$$(\lambda + \frac{2}{3}\mu) \geq 0 \;,\; \mu \geq 0 \;. \qquad (29.20)$$

Inviscid Fluid: For an inviscid fluid the Cauchy stress is independent of the rate of deformation $\mathbf{D}$ so that $\check{\mathbf{T}}$ vanishes and the Cauchy stress is given by

$$\mathbf{T} = \hat{\mathbf{T}}(J) = -\,p(J)\,\mathbf{I} \;. \qquad (29.21)$$

This means for an inviscid fluid the stress vector $\mathbf{t}$ always acts normal to the surface on which it is applied

$$\mathbf{t} = \mathbf{T}\,\mathbf{n} = -\,p\,\mathbf{n} \;. \qquad (29.22)$$

30. Elastic-Plastic Materials

In this section we summarize the main features of constitutive equations which model the rate-independent elastic-plastic response of a typical metal. A good review of the linear theory for elastic-plastic materials may be found in an article by Naghdi (1960). Here, we consider the nonlinear theory and use the strain space formulation of plasticity which was proposed by Naghdi and Trapp (1975).

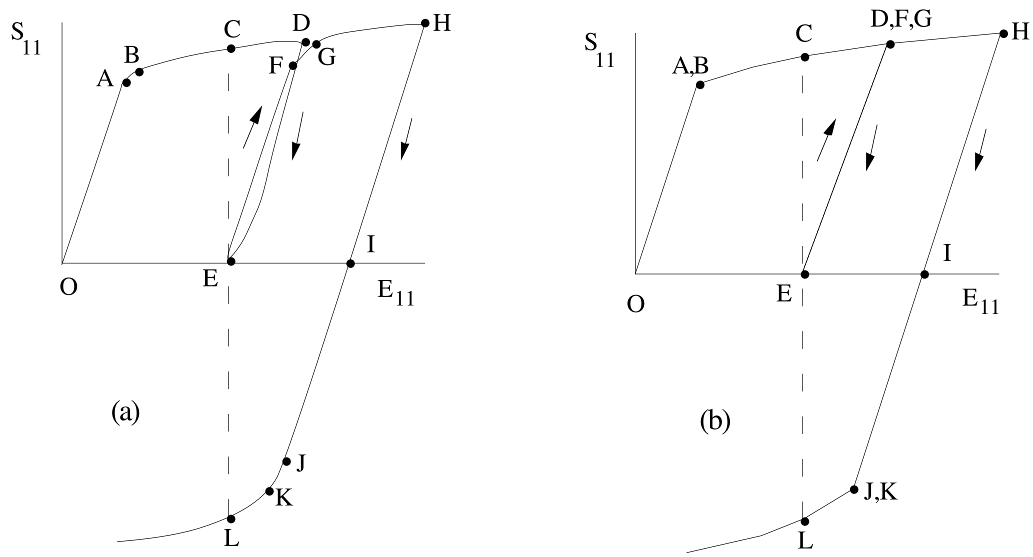

Fig. 30.1: (a) Stress-Strain Response Of A Typical Metal To Uniaxial Stress; (b) Idealization Of The Stress-Strain Response Of A Metal To Uniaxial Stress

Fig. 30.1a shows that stress-strain response of a typical metal to uniaxial stress loading. The quantity S_{11} is the (11) component of the symmetric Piola-Kirchhoff stress **S** and the quantity E_{11} is the (11) component of the Lagrangian strain **E**. The material is loaded in tension along the path OABCD, unloaded along DE, reloaded along EFGH, unloaded along HI, and reloaded in compression along IJKL. Inspection of the points C,E, and L in Fig. 30.1a reveals that the stress in an elastic-plastic material can have significantly different values for the same value of strain E_{11}. This means that the response of an elastic-plastic material depends on the past history of deformation (i.e. the responses to the deformation histories OABC, OAB–E, and OAB–L are different).

The points A,F, and J in Fig. 30.1a represent points on the loading paths beyond which the stress-strain relationship becomes nonlinear. These points are called the proportional limits. Although the curve OABCD is nonlinear we cannot determine whether the response is elastic or elastic-plastic until we considering unloading. Since the response shown in Fig. 1a does not unload along the same loading path we know that the response is not elastic but rather is elastic-plastic. Consequently, the points B,G, and K represent the points on the loading paths beyond which some detectable value of strain (normally taken to be 0.2%) remains when the material is unloaded to zero stress. These points are called the yield points and deformation beyond them causes permanent changes in the response of the material. It is also important to mention that the paths BCD, GH, and KL represent strain hardening paths where the stress increases with increasing strain.

To model the material response shown in Fig. 30.1a it is common to separate the response into two parts: elastic response which is reversible and plastic response which is irreversible. Also, we idealize the material response as shown in Fig. 30.1b by making the following assumptions:

(a) There is a distinct yield point that forms the boundary between elastic and plastic response.

(b) Unloading along DE and and reloading along EF follow the same path.

For the constitutive model we introduce a symmetric positive definite second order tensor $\mathbf{C}_p$ called the plastic deformation, and a scalar measure of work hardening κ, both of which are functions of the material point $\mathbf{X}$ and time t. Furthermore, we assume that the boundary between elastic and plastic response is characterized by a yield function $g(\mathbf{C},\mathbf{C}_p,\kappa)$, which depends on the variables

$$\{ \mathbf{C} , \mathbf{C}_p , \kappa \} \ . \tag{30.1}$$

The yield function is also assumed to be continuously differentiable with respect to its arguments and at yield it is assumed to satisfy the equation

$$g(\mathbf{C},\mathbf{C}_p,\kappa) = 0 \ . \tag{30.2}$$

Since g=0 determines the boundary between elastic and plastic response, we can without loss in generality take g to be negative for elastic response.

The plastic deformation is specified by a flow rule which is an equation for the rate of change of plastic deformation of the form

$$\dot{\mathbf{C}}_p = \Gamma \mathbf{A} \ , \tag{30.3}$$

and the hardening is specified by an evolution equation of the form

$$\dot{\kappa} = \Gamma \mathrm{K} \ . \tag{30.4}$$

In (30.3) and (30.4), $\mathbf{A}$ is a symmetric tensor, K is a scalar, and both are functions of the variables (30.1). Also, Γ is a scalar function of the variables (30.1) and the rate $\dot{\mathbf{C}}$ which characterizes loading and unloading. To motivate the form for Γ we differentiate the yield function (30.2) to deduce that

$$\dot{g} = \hat{g} - \Gamma \bar{g} \ , \tag{30.5a}$$

$$\hat{g} = \frac{\partial g}{\partial \mathbf{C}} \bullet \dot{\mathbf{C}} \ , \ \bar{g} = -\left[\frac{\partial g}{\partial \mathbf{C}_p} \bullet \mathbf{A} + \frac{\partial g}{\partial \kappa} \mathrm{K} \right] . \tag{30.5b,c}$$

Notice that when Γ vanishes plastic deformation rate and hardening rate also vanish so the response should be elastic. Under these conditions the sign of $\hat{g}$ indicates whether the yield surface tends to grow or shrink. Consequently, when g=0 and the material is at its elastic-plastic boundary, the sign of $\hat{g}$ determines whether the material response will correspond to loading into the plastic region or unloading into the elastic region. Furthermore, since we require the material response to be rate-independent the scalar Γ must be homogeneous function of order one in the time rate of change of tensors.

Therefore, with this background in mind we specify the loading and unloading conditions by taking Γ in the form

$$
\Gamma = \begin{cases} 0 & \text{during elastic response} \quad (g < 0) \\ 0 & \text{during unloading} \quad (g = 0 \text{ and } \hat{g} < 0) \\ 0 & \text{during neutral loading} \quad (g = 0 \text{ and } \hat{g} = 0) \\ \bar{\Gamma}\,\hat{g} & \text{during loading} \quad (g = 0 \text{ and } \hat{g} > 0) \end{cases} \tag{30.6c}
$$

where $\bar{\Gamma}$ is a function of the variables (30.1) which is determined by the consistency condition that g remains zero during plastic loading. Thus, substituting (30.6d) into (30.5a) and requiring that during plastic loading g=0 and $\dot{g}$ =0, we obtain the result that

$$
\bar{\Gamma} = \frac{1}{\bar{g}} \quad . \tag{30.7}
$$

The constitutive equations (30.2)-(30.7) are called rate type constitutive equations because the evolution of the quantities $\mathbf{C}_p$ and κ are specified by constitutive equations for their time rate of change instead of for the variables themselves. As mentioned above, the specification (30.6d) causes the equations (30.3) and (30.4) to be homogeneous in time so that the response is to any specified deformation path is insensitive to the rate that the path is traversed. Also, we mention that $\mathbf{C}_p$ and κ are assumed to be unaltered by superposed rigid body motions

$$
\mathbf{C}_p^+ = \mathbf{C}_p \quad , \quad \kappa^+ = \kappa \quad , \tag{30.8}
$$

so the quantities {g,**A**,Γ,K} are also unaltered by superposed rigid body motions.

There is a general consensus that the constitutive equations (30.2)-(30.4) cannot be specified totally arbitrarily. However, there is no consensus about the specific form of appropriate constitutive restrictions. For example, within the context of the thermodynamical theory it is necessary to ensure that the elastic response is consistent with the notion that a strain energy exists and the plastic response is dissipative. Using the work of Green and Naghdi (1965,1966) it can be shown that for a strain energy function Σ of the form

$$\Sigma = \Sigma\,(\mathbf{C},\mathbf{C}_p,\kappa) \ , \tag{30.9}$$

that the symmetric Piola-Kirchhoff stress **S** must be related to Σ by the equation

$$\mathbf{S} = 2\rho_0 \frac{\partial \Sigma}{\partial \mathbf{C}} \ , \tag{30.10}$$

and plastic deformation will be dissipative whenever the material dissipation (24.6) is nonnegative

$$\mathcal{D} = -\,\rho \left(\frac{\partial \Sigma}{\partial \mathbf{C}_p} \bullet \dot{\mathbf{C}}_p + \frac{\partial \Sigma}{\partial \kappa}\,\dot{\kappa} \right) > 0 \ . \tag{30.11}$$

A simple specific set of constitutive equations that is valid for large deformations can be characterized by the assumptions

$$g = \frac{\sigma_e}{\kappa} - 1 \ , \ \ \sigma_e^2 = \frac{3}{2} \ \mathbf{T}' \bullet \mathbf{T}' \ , \tag{30.12a,b}$$

$$\mathbf{A} = \left[\left(\frac{3}{\mathbf{C}_p^{-1} \bullet \mathbf{C}} \right) \mathbf{C} - \mathbf{C}_p \right] \ , \ \ K = m_1\,(Z_1 - \kappa) \ , \tag{30.12c,d}$$

$$2\rho_0 \Sigma = 2\,f(J) + \mu_0\,(\alpha_1 - 3) \ , \ \ \alpha_1 = \mathbf{C}_p^{-1} \bullet \mathbf{C}' \ , \tag{30.12e,f}$$

where σ_e is the Von Mises stress; m_1, Z_1, μ_0 are material constants; and f(J) is a function of the dilatation J. The specification (30.12c) is consistent with the notion of plastic incompressibility because

$$I_{3p} = \det \mathbf{C}_p = 1 \ , \ \ \dot{I}_{3p} = I_{3p}\,\mathbf{C}_p^{-1} \bullet \dot{\mathbf{C}}_p = 0 \ . \tag{30.13a,b}$$

Consequently, the scalar α_1 defined by (30.12f) is a pure measure of elastic distortion. Also, the functional form (30.12d) indicates that hardening tends to saturate when κ attains the value Z_1. Now, using (30.10) and (30.11) we obtain

$$\mathbf{S} = -\,p\,J\,\mathbf{C}^{-1} + \mathbf{S}' \ , \ \ p = -\frac{df}{dJ} \ , \tag{30.14a,b}$$

$$\mathbf{S}' = \mu_0 \, J^{-1/3} \left[\mathbf{C}_p^{-1} - \frac{1}{3} (\mathbf{C} \bullet \mathbf{C}_p^{-1}) \; \mathbf{C}^{-1} \right] \;, \tag{30.14c}$$

$$\mathcal{D} = - J^{-1} \rho_0 \frac{\partial \Sigma}{\partial \mathbf{C}_p} \bullet \dot{\mathbf{C}}_p = \frac{1}{2} J^{-1} \mu_0 \, \mathbf{C}_p^{-1} \mathbf{C}' \mathbf{C}_p^{-1} \bullet \Gamma \, \mathbf{A} \;. \tag{30.14d}$$

Furthermore, introducing a tensorial measure $\mathbf{B}_e'$ of elastic distortional deformation

$$\mathbf{B}_e' = \mathbf{F}' \mathbf{C}_p^{-1} \mathbf{F}'^T \;\; , \;\; \det \mathbf{B}_e' = 1 \;\; , \tag{30.15a,b}$$

the deviatoric part $\mathbf{T}'$ of the Cauchy stress and the rate of plastic dissipation become

$$\mathbf{T}' = J^{-1} \mu_0 \left[\mathbf{B}_e' - \frac{1}{3} (\mathbf{B}_e' \bullet \mathbf{I}) \, \mathbf{I} \right] \;, \tag{30.16a}$$

$$\mathcal{D} = - J^{-1} \rho_0 \frac{\partial \Sigma}{\partial \mathbf{C}_p} \bullet \dot{\mathbf{C}}_p = \frac{1}{2} J^{-2/3} \mu_0 \, \Gamma \left[\left(\frac{3}{\mathbf{B}_e' \bullet \mathbf{I}} \right) (\mathbf{B}_e' \bullet \mathbf{B}_e') - (\mathbf{B}_e' \bullet \mathbf{I}) \right] \;. \tag{30.16b}$$

An alternative approach to plasticity has been developed by Eckart (1948) and Leonov (1976) for elastically isotropic elastic-plastic materials, and by Besseling (1966) for elastically anisotropic materials. Also, a plasticity theory formulated in terms of physically based microstructural variables, which is motivated by these previous works, has been proposed by Rubin (1994). In this alternative approach, there is no need to introduce a measure of plastic deformation. Instead, attention is focused on the evolution of an elastic deformation tensor and the effects of plasticity are introduced only through the rate of relaxation that plasticity causes on the evolution of elastic deformation.

For the simple case of elastically isotropic elastic-plastic response the elastic deformation is characterized by the scalar measure J of dilatation and a unimodular tensor $\mathbf{B}_e'$ which is a measure of elastic distortional deformation. Also, for generality, a measure of isotropic hardening κ is introduced. These quantities are determined by the following evolution equations

$$\dot{J} = J \, \mathbf{D} \bullet \mathbf{I} \; , \;\; \dot{\kappa} = \Gamma K \;\; , \tag{30.17a,b}$$

$$\dot{\mathbf{B}}_e' = \mathbf{L} \mathbf{B}_e' + \mathbf{B}_e' \mathbf{L}^T - \frac{2}{3} (\mathbf{D} \bullet \mathbf{I}) \, \mathbf{B}_e' \; - \Gamma \, \mathbf{A}_p \;\; , \tag{30.17c}$$

where Γ is a scalar to be determined. The scalar function K and tensor $\mathbf{A}_p$ require constitutive equations. First of all it is noted that when Γ vanishes the evolution equation (27.17c) can be integrated to obtain

$$\mathbf{B}'_e = \mathbf{B}' = \mathbf{F}'\mathbf{F}'^T \ , \ \mathbf{F}' = J^{-1/3}\,\mathbf{F} \ . \tag{30.18a,b}$$

This indicates that $\mathbf{B}'_e$ becomes the usual measure of elastic distortional deformation that is used in describing isotropic nonlinear elastic materials. This also means that the term $\Gamma\mathbf{A}_p$ determines the relaxation effects of plasticity on the evolution of elastic deformation. Moreover, since $\mathbf{B}'_e$ must remain a unimodular tensor it follows that

$$\dot{\mathbf{B}}'_e \bullet \mathbf{B}'^{-1}_e = 0 \ , \tag{30.19}$$

so that $\mathbf{A}_p$ must be restricted by the condition

$$\mathbf{A}_p \bullet \ \mathbf{B}'^{-1}_e = 0 \ . \tag{30.20}$$

Now, the stress response is determined by a strain energy function of the form

$$\Sigma = \Sigma(J,\alpha_1,\alpha_2,\kappa) \ , \ \alpha_1 = \mathbf{B}'_e \bullet \mathbf{I} \ , \ \alpha_2 = \mathbf{B}'_e \bullet \mathbf{B}'_e \ , \tag{30.21a,b,c}$$

where α_1 and α_2 are the two nontrivial invariants of $\mathbf{B}'_e$. Using the fact that

$$\dot{\alpha}_1 = 2\,[\mathbf{B}'_e - \tfrac{1}{3}(\mathbf{B}'_e \bullet \mathbf{I})\,\mathbf{I}\,] \bullet \mathbf{D} - \Gamma\,\mathbf{A}_p \bullet \mathbf{I} \ , \tag{30.22a}$$

$$\dot{\alpha}_2 = 4\,[\mathbf{B}'^2_e - \tfrac{1}{3}(\mathbf{B}'^2_e \bullet \mathbf{I})\,\mathbf{I}\,] \bullet \mathbf{D} - 2\,\Gamma\,\mathbf{A}_p \bullet \mathbf{B}'_e \ , \tag{30.22b}$$

it can be shown that

$$\dot{\Sigma} = [J\frac{\partial\Sigma}{\partial J}\ \mathbf{I} + 2\frac{\partial\Sigma}{\partial\alpha_1}\{\mathbf{B}'_e - \tfrac{1}{3}(\mathbf{B}'_e \bullet \mathbf{I})\,\mathbf{I}\} + 4\frac{\partial\Sigma}{\partial\alpha_2}\{\mathbf{B}'^2_e - \tfrac{1}{3}(\mathbf{B}'^2_e \bullet \mathbf{I})\,\mathbf{I}\}] \bullet \mathbf{D}$$

$$+\ \Gamma\,[\frac{\partial\Sigma}{\partial\kappa}K - \ \{\frac{\partial\Sigma}{\partial\alpha_1}\mathbf{I} + 2\frac{\partial\Sigma}{\partial\alpha_2}\ \mathbf{B}'_e\} \bullet \mathbf{A}_p\,] \ . \tag{30.23}$$

Also, for this theory the Cauchy stress $\mathbf{T}$ is given by

$$\mathbf{T} = -\,p\,\mathbf{I} + \mathbf{T}' \ , \ p = -\,\rho_0\frac{\partial\Sigma}{\partial J} \ , \tag{30.24a,b}$$

$$\mathbf{T}' = 2\rho\frac{\partial\Sigma}{\partial\alpha_1}[\mathbf{B}'_e - \tfrac{1}{3}(\mathbf{B}'_e \bullet \mathbf{I})\,\mathbf{I}] + 4\rho\frac{\partial\Sigma}{\partial\alpha_2}[\mathbf{B}'^2_e - \tfrac{1}{3}(\mathbf{B}'^2_e \bullet \mathbf{I})\,\mathbf{I}] \ , \tag{30.24c}$$

and the rate of dissipation (24.6) due to plastic deformation must satisfy the inequality

$$\mathcal{D} = -\,\Gamma\,\rho\,\Big[\frac{\partial\Sigma}{\partial\kappa}\,K - \Big\{\frac{\partial\Sigma}{\partial\alpha_1}\,\mathbf{I} + 2\,\frac{\partial\Sigma}{\partial\alpha_2}\,\mathbf{B}'_e\Big\} \bullet \mathbf{A}_p\,\Big] \geq 0 \quad . \tag{30.25}$$

In general, the yield surface for rate-independent plasticity is assumed to be a function of the same variables as the strain energy function

$$g = g(J,\alpha_1,\alpha_2,\kappa) \quad . \tag{30.26}$$

Thus, it can be shown that

$$\dot{g} = \hat{g} - \Gamma\,\bar{g}, \tag{30.27a}$$

$$\hat{g} = \Big[J\,\frac{\partial g}{\partial J}\,\mathbf{I} + 2\,\frac{\partial g}{\partial\alpha_1}\Big\{\mathbf{B}'_e - \tfrac{1}{3}\,(\mathbf{B}'_e \bullet \mathbf{I})\,\mathbf{I}\Big\} + 4\,\frac{\partial g}{\partial\alpha_2}\Big\{\mathbf{B}'^2_e - \tfrac{1}{3}\,(\mathbf{B}'^2_e \bullet \mathbf{I})\,\mathbf{I}\Big\}\Big] \bullet \mathbf{D}\ , \tag{30.27b}$$

$$\bar{g} = -\,\frac{\partial g}{\partial\kappa}\,K + \Big[\frac{\partial g}{\partial\alpha_1}\,\mathbf{I} + 2\,\frac{\partial g}{\partial\alpha_1}\,\mathbf{B}'_e\,\Big] \bullet \mathbf{A}_p \quad . \tag{30.27c}$$

Moreover, the yield function is chosen so that $\bar{g}$ is positive whenever the material state is at the onset of plasticity g=0

$$\bar{g} > 0 \quad \text{whenever} \quad g = 0 \quad . \tag{30.28}$$

Then, the function Γ is determined by the expressions (30.6) and the consistency condition (30.7).

In particular, notice that the deviatoric stress $\mathbf{T}'$ vanishes when the elastic distortional deformation $\mathbf{B}'_e$ equals $\mathbf{I}$. This suggests that the relaxation effects of plasticity cause the elastic distortional deformation to evolve toward the unity tensor. Thus, the tensor $\mathbf{A}_p$ is taken in the form

$$\mathbf{A}_p = \mathbf{B}'_e - \Big[\frac{3}{\mathbf{B}'^{-1}_e \bullet \mathbf{I}}\Big]\,\mathbf{I} \quad , \tag{30.29}$$

where the coefficient of $\mathbf{I}$ has been chosen so that the restriction (30.20) is satisfied.

As a simple special case, the strain energy function Σ and the rate of hardening function K are specified by

$$2\rho_0\Sigma = 2\,f(J) + \mu_0\,(\alpha_1 - 3) \quad , \quad K = m_1\,(Z_1 - \kappa) \quad , \tag{30.30}$$

where the function f(J) determines the response to dilatation, μ_0 is the positive reference value of the shear modulus, m_1 is a positive constant controlling the rate of hardening and Z_1 is the saturated value of hardening. Then, using this strain energy function it follows that the stress if given by

$$p = -\frac{df}{dJ} \ , \quad \mathbf{T}' = J^{-1} \mu_0 \left[\mathbf{B}_e' - \tfrac{1}{3} (\mathbf{B}_e' \bullet \mathbf{I})\, \mathbf{I}\right] \ , \tag{30.31}$$

and the restriction (30.25) on the rate of plastic dissipation requires

$$\mathcal{D} = -\,\Gamma\, \rho \;= \frac{1}{2} J^{-1} \mu_0\, \mathbf{I} \bullet \Gamma \mathbf{A}_p = \frac{1}{2} J^{-1} \mu_0\, \Gamma \left[\mathbf{B}_e' \bullet \mathbf{I} - \left\{\frac{9}{\mathbf{B}_e'^{-1} \bullet \mathbf{I}}\right\}\right] \geq 0 \ . \tag{30.32}$$

Since $\mathbf{B}_e'$ is a symmetric unimodular tensor it can be shown by expressing it in its spectral form in terms of its positive eigenvalues $\{\beta_1, \beta_2, 1/\beta_1\beta_2\}$ that

$$\mathbf{B}_e' \bullet \mathbf{I} = \beta_1 + \beta_2 + \frac{1}{\beta_1 \beta_2} \geq 3 \ , \tag{30.33a}$$

$$\mathbf{B}_e'^{-1} \bullet \mathbf{I} = \frac{1}{\beta_1} + \frac{1}{\beta_2} + \beta_1 \beta_2 \geq 3 \ , \tag{30.33b}$$

so that the dissipation inequality (30.32) is automatically satisfied.

The main advantage of this alternative approach to plasticity theory is that the initial values of $\{J, \mathbf{B}_e', \kappa\}$ required to integrate the evolution equations (30.17) can be measured in the present configuration. This means that all relevant information about the past history of deformation can be measured in the present state of the material. This is important from a physical point of view because knowledge of the state of the material in the present configuration does not reveal sufficient information to determine the value of plastic strain that has been measured relative to a reference configuration, which itself cannot be determined in the present configuration. In other words, the initial condition on plastic deformation $\mathbf{C}_p$ required to integrate the evolution equation (30.3) cannot be determined from knowledge of the present configuration only. This fact causes an arbitrariness to be introduced into the more classical theory of plasticity that is not present in this alternative theory.

References

Besseling, J.F. (1966). A Thermodynamic Approach To Rheology, Proc. of the IUTAM Symposium on Irreversible Aspects of Continuum Mechanics and Transfer of Physical Characteristics in Moving Fluids, Vienna. (Eds. H. Parkus and L. I. Sedov), Springer-Verlag, Wein, 1968, pp. 16-53.

Eckart, C. (1948). The Thermodynamics Of Irreversible Processes. IV. The Theory Of Elasticity And Anelasticity, *Physical Review* **73**, pp. 373-382.

Flory, P. J., (1961). Thermodynamic Relations for High Elastic Materials. *Trans. Faraday Soc*. **57**, 829-838.

Leonov, A.I. (1976). Nonequilibrium Thermodynamics And Rheology Of Viscoelastic Polymer Media, *Rheologica Acta* **15**, pp. 85-98.

Naghdi, P. M. (1960). Stress-Strain Relations In Plasticity And Thermoplasticity. Proc. Second Symp. Naval Structural Mech. (Edited by E. H. Lee and P. S. Symonds), Pergamon Press, 121-167.

Naghdi, P. M., and Trapp, J. A. (1975). The Significance Of Formulating Plasticity Theory With Reference To Loading Surfaces In Strain Space, *Int. J. Engng. Sci.* **13**, 785-797.

Rubin, M.B. (1994). Plasticity Theory Formulated In Terms Of Physically Based Microstructural Variables – Part I: Theory, *Int. J. Solids Structures* **31**, pp. 2615-2634.

Appendix A: **Eigenvalues**, **Eigenvectors**, **and Principal Invariants of a Tensor**

In this appendix we briefly review some basic properties of eigenvalues and eigenvectors. The vector **v** is said to be an eigenvector of a real second order symmetric tensor **T** with the associated eigenvalue σ if

$$\mathbf{T}\,\mathbf{v} = \sigma\,\mathbf{v} \;\;,\;\; T_{ij}\,v_j = \sigma\,v_i \;\;. \tag{A1a,b}$$

It follows that the characteristic equation for determining the three values of the eigenvalue σ is given by

$$\det(\mathbf{T} - \sigma\mathbf{I}) = -\sigma^3 + \sigma^2 I_1 - \sigma I_2 + I_3 = 0 \;\;, \tag{A2}$$

where I_1,I_2,I_3 are the principal invariants of an arbitrary real tensor **T**

$$I_1(\mathbf{T}) = \mathbf{T} \bullet \mathbf{I} = \mathrm{tr}\,\mathbf{T} = T_{mm} \;\;, \tag{A3a}$$

$$I_2(\mathbf{T}) = \frac{1}{2}\;[(\mathbf{T} \bullet \mathbf{I})^2 - (\mathbf{T} \bullet \mathbf{T}^T)] = \frac{1}{2}\;[(T_{mm})^2 - T_{mn}T_{nm}] \;\;, \tag{A3b}$$

$$I_3(\mathbf{T}) = \det \mathbf{T} = \frac{1}{6}\;\varepsilon_{ijk}\varepsilon_{lmn}T_{il}T_{jm}T_{kn} \;. \tag{A3c}$$

It can be shown that since **T** is a real symmetric tensor the three roots of the cubic equation (A2) are real. Also, it can be shown that the three independent eigenvectors **v** obtained by solving (A1) can be chosen to form an orthonormal set of vectors.

Recalling that **T** can be separated into its spherical part T **I** and its deviatoric part **T**' such that

$$\mathbf{T} = T\,\mathbf{I} + \mathbf{T}' \;\;,\;\; T_{ij} = T\,\delta_{ij} + T_{ij}' \;\;, \tag{A4a,b}$$

$$T = \frac{1}{3}\,(\mathbf{T} \bullet \mathbf{I}) \;=\frac{1}{3}\,(T_{mm}) \;\;,\;\; \mathbf{T}' \bullet \mathbf{I} = T_{mm}' = 0 \;\;, \tag{A4c,d}$$

it follows that when **v** is an eigenvector of **T** it is also an eigenvector of **T**'

$$\mathbf{T}'\,\mathbf{v} = (\mathbf{T} - T\,\mathbf{I})\,\mathbf{v} = (\sigma - T)\,\mathbf{v} = \sigma'\,\mathbf{v} \;\;, \tag{A5}$$

with the associated eigenvalue σ' related to σ by

$$\sigma = \sigma' + T \;\;. \tag{A6}$$

However since the first principal invariant of **T**' vanishes we may write the characteristic equation for σ' in the form

$$\det(\mathbf{T}' - \sigma'\mathbf{I}) = -(\sigma')^3 \;+ \sigma'(\frac{\sigma_e^2}{3}) + J_3 = 0 \;\;, \tag{A7}$$

where we have defined the alternative invariants σ_e and J_3 by

$$\sigma_e^2 = \frac{3}{2}\,\mathbf{T}' \bullet \mathbf{T}' = -\,3\,I_2(\mathbf{T}')\ \ ,\ \ J_3 = \det \mathbf{T}' = I_3(\mathbf{T}')\ \ . \qquad \text{(A8a,b)}$$

Note that if σ_e vanishes then $\mathbf{T}'$ vanishes so that from (A7) σ' vanishes and from (A6) it follows that there is only one distinct eigenvalue

$$\sigma = T\ \ . \qquad \text{(A9)}$$

On the other hand, if σ_e does not vanish we may divide (A7) by $(\sigma_e/3)^3$ to obtain

$$\left(\frac{3\sigma'}{\sigma_e}\right)^3 - 3\left(\frac{3\sigma'}{\sigma_e}\right) - 2\,\hat{J}_3 = 0\ \ , \qquad \text{(A10)}$$

where the invariant $\hat{J}_3$ is defined by

$$\hat{J}_3 = \frac{27\,J_3}{2\,\sigma_e^{\,3}}\ \ . \qquad \text{(A11)}$$

Since (A10) is in the standard form for a cubic, the solution can be obtained easily using the trigonometric form

$$\sin 3\beta = -\,\hat{J}_3\ \ ,\ \ -\frac{\pi}{6} \le \beta \le \frac{\pi}{6}\ \ , \qquad \text{(A12a)}$$

$$\sigma_1' = \frac{2\sigma_e}{3}\cos\left(\frac{\pi}{6} + \beta\right)\ \ , \qquad \text{(A12b)}$$

$$\sigma_2' = \frac{2\sigma_e}{3}\sin(\beta)\ \ , \qquad \text{(A12c)}$$

$$\sigma_3' = -\frac{2\sigma_e}{3}\cos\left(\frac{\pi}{6} - \beta\right)\ \ , \qquad \text{(A12d)}$$

where the eigenvalues $\sigma_1', \sigma_2', \sigma_3'$ are ordered so that

$$\sigma_1' \ge \sigma_2' \ge \sigma_3'\ \ . \qquad \text{(A13)}$$

Once these values have been determined the three values of σ may be calculated using (A6).

Furthermore, we note that the value of β or $\hat{J}_3$ may be used to identify three states of deviatoric stress denoted by: triaxial compression (TXC); torsion (TOR); and triaxial extension (TXE); and defined by

$$\beta = \frac{\pi}{6} \ , \ \hat{J}_3 = -1 \ , \ \text{(TXC)} \ , \tag{A14a}$$

$$\beta = 0 \ , \ \hat{J}_3 = 0 \ , \ \text{(TOR)} \ , \tag{A14b}$$

$$\beta = -\frac{\pi}{6} \ , \ \hat{J}_3 = 1 \ , \ \text{(TXE)} \ . \tag{A14c}$$

Appendix B: **Consequences of Continuity**

A function $\phi(\mathbf{x},t)$ is said to be continuous with respect to position $\mathbf{x}$ in a region R if for every $\mathbf{y}$ in R and every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$| \phi(\mathbf{x},t) - \phi(\mathbf{y},t) | < \varepsilon \quad \text{whenever} \quad | \mathbf{x} - \mathbf{y} | < \delta \ . \tag{B1}$$

<u>Theorem</u>: If $\phi(\mathbf{x},t)$ is <u>continuous</u> in R and

$$\int_P \phi \, dv = 0 \ , \tag{B2}$$

for <u>every</u> part P in R, then the necessary and sufficient condition for the validity of (B1) is that ϕ vanishes at every point in R

$$\phi = 0 \quad \text{in R} \ . \tag{B3}$$

<u>Proof (Sufficiency)</u>: If ϕ=0 in R then (B2) is trivially satisfied.

<u>Proof (Necessity)</u>: (Proof by contradiction). Suppose that a point $\mathbf{y}$ in R exits for which $\phi(\mathbf{y},t)>0$. Then, by continuity of ϕ there exists a region P_δ defined by the delta sphere such that

$$| \phi(\mathbf{x},t) - \phi(\mathbf{y},t) | < \frac{1}{2} \phi(\mathbf{y},t) \quad \text{whenever} \quad | \mathbf{x} - \mathbf{y} | < \delta \ . \tag{B4}$$

Alternatively, (B4) may be written as

$$-\frac{1}{2} \phi(\mathbf{y},t) < \phi(\mathbf{x},t) - \phi(\mathbf{y},t) < \frac{1}{2} \phi(\mathbf{y},t) \quad \text{whenever} \quad | \mathbf{x} - \mathbf{y} | < \delta \ , \tag{B5a}$$

$$\frac{1}{2} \phi(\mathbf{y},t) < \phi(\mathbf{x},t) < \frac{3}{2} \phi(\mathbf{y},t) \quad \text{whenever} \quad | \mathbf{x} - \mathbf{y} | < \delta \ . \tag{B5b}$$

Since the volume V_δ of the region P_δ is positive

$$V_\delta = \int_{P_\delta} dv > 0 \ , \tag{B6}$$

it follows from (B5b) and (B6) that

$$\int_{P_\delta} \phi \, dv > \int_{P_\delta} \frac{1}{2} \phi(\mathbf{y},t) \, dv = \frac{1}{2} \phi(\mathbf{y},t) \, V_\delta > 0 \ , \tag{B7}$$

which contradicts the condition (B2) so that ϕ in R cannot be positive. Similarly, we realize that if $\phi(\mathbf{y},t)<0$, then by continuity of ϕ there exists a region P_δ defined by the delta sphere such that

$$| \phi(\mathbf{x},t) - \phi(\mathbf{y},t) | < -\frac{1}{2} \phi(\mathbf{y},t) \text{ whenever } | \mathbf{x} - \mathbf{y} | < \delta , \tag{B8a}$$

$$\frac{1}{2} \phi(\mathbf{y},t) < \phi(\mathbf{x},t) - \phi(\mathbf{y},t) < -\frac{1}{2} \phi(\mathbf{y},t) \text{ whenever } | \mathbf{x} - \mathbf{y} | < \delta , \tag{B8b}$$

$$\frac{3}{2} \phi(\mathbf{y},t) < \phi(\mathbf{x},t) < \frac{1}{2} \phi(\mathbf{y},t) \text{ whenever } | \mathbf{x} - \mathbf{y} | < \delta . \tag{B8c}$$

Hence,

$$\int_{P_\delta} \phi \, dv < \int_{P_\delta} \frac{1}{2} \phi(\mathbf{y},t) \, dv = \frac{1}{2} \phi(\mathbf{y},t) \, V_\delta < 0 , \tag{B9}$$

which contradicts the condition (B2) so that ϕ in R cannot be negative. Combining the results of (B7) and (B9) we deduce that ϕ must vanish at each point of R, which proves the necessity of (B3).

Appendix C: **Lagrange Multipliers**

Special Case: Let $f=f(x_1,x_2,x_3)$ be a real valued function of the three variables x_i and assume that f is continuously differentiable. We say that f has a stationary value (extrememum) at the point $\mathbf{x}_0$ if

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 = 0 \text{ at } \mathbf{x}_0 \ . \tag{C1}$$

If the variables x_i are independent of each other then from (C1) we may conclude that

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} = 0 \text{ at } \mathbf{x}_0 \ . \tag{C2}$$

Let us now consider the problem of finding the points $\mathbf{x}_0$ which make f stationary and which satisfy the constraint condition that

$$\phi(x_1,x_2,x_3) = 0 \ . \tag{C3}$$

In other words, from the set of all points which satisfy the constraint (C3) we search for those $\mathbf{x}_0$ which also make f stationary. To this end, we differentiate (C3) along paths on the constraint surface to obtain

$$d\phi = \frac{\partial \phi}{\partial x_1} dx_1 + \frac{\partial \phi}{\partial x_2} dx_2 + \frac{\partial \phi}{\partial x_3} dx_3 = 0 \ . \tag{C4}$$

The condition for f to be stationary is again given by (C1) but now we can no longer conclude the results (C2) because x_i are dependent and must satisfy (C3).

The method of Lagrange multipliers suggests that we multiply (C4) by an arbitrary scalar λ and then subtract the result from (C1) to obtain

$$\left(\frac{\partial f}{\partial x_1} - \lambda \frac{\partial \phi}{\partial x_1}\right) dx_1 + \left(\frac{\partial f}{\partial x_2} - \lambda \frac{\partial \phi}{\partial x_2}\right) dx_2 + \left(\frac{\partial f}{\partial x_3} - \lambda \frac{\partial \phi}{\partial x_3}\right) dx_3 = 0 \quad \text{at } \mathbf{x}_0 \ . \tag{C5}$$

In order for the constraint (C3) to be active we require that at each point at least one of the partial derivatives $\partial\phi/\partial x_i \neq 0$. For simplicity we assume that

$$\frac{\partial \phi}{\partial x_3} \neq 0 \ . \tag{C6}$$

Next we can choose λ so that the coefficient of dx_3 in (C5) vanishes

$$\frac{\partial \phi}{\partial x_3}\lambda = \frac{\partial f}{\partial x_3} \ , \tag{C7}$$

so that equation (C5) reduces to

$$\left(\frac{\partial f}{\partial x_1} - \lambda \frac{\partial \phi}{\partial x_1}\right) dx_1 + \left(\frac{\partial f}{\partial x_2} - \lambda \frac{\partial \phi}{\partial x_1}\right) dx_2 = 0 \quad \text{at } \mathbf{x}_0 \, . \tag{C8}$$

Now since $\partial\phi/\partial x_3 \neq 0$ we can choose dx_3 so that equation (C4) is satisfied for arbitrary choice of dx_1 and dx_2. Hence, the values of dx_1 and dx_2 may be specified independently in (C8) so we may conclude that

$$\frac{\partial f}{\partial x_i} = \lambda \frac{\partial \phi}{\partial x_i} \ \text{ at } \mathbf{x}_0 \ , \tag{C9}$$

where we have also used the specification (C7). In summary, we say that of all the points satisfying the constraint (C3), the ones that correspond to stationary values of f are the ones for which $\mathbf{x}_0$ and λ are determined by the four equations (C3) and (C9).

Another way of examining the same problem is to write the function f and the constraint ϕ in the forms

$$f = f(x_\alpha, x_3) \ , \ \phi = \phi(x_\alpha, x_3) = 0 \ , \tag{C10a,b}$$

where a Greek index is assumed to take only the values 1,2. Since $\partial\phi/\partial x_3 \neq 0$, the implicit function theorem states that a function $g(x_\alpha)$ exists such that when $x_3 = g(x_\alpha)$ the constraint (C10b) is satisfied

$$\phi(x_\alpha, g(x_\alpha)) = 0 \ \text{ for all } x_\alpha \ . \tag{C11}$$

Substituting the value $x_3 = g(x_\alpha)$ into (C10a) we obtain a function of x_α only which determines the value of f only for those points that satisfy the constraint condition (C10b)

$$f = f(x_\alpha \, , \, g(x_\alpha)) \ . \tag{C12}$$

Since x_α are independent variables in (C12) it follows that the stationary values are determined by the equation

$$df = \left(\frac{\partial f}{\partial x_\alpha} + \frac{\partial f}{\partial x_3}\frac{\partial g}{\partial x_\alpha}\right) dx_\alpha = 0 \ . \tag{C13}$$

Thus, for stationary points

$$\frac{\partial f}{\partial x_\alpha} = -\frac{\partial f}{\partial x_3}\,\frac{\partial g}{\partial x_\alpha}\ . \tag{C14}$$

However, since the constraint (C10b) is satisfied for all values of x_α we have

$$d\phi = \left(\frac{\partial \phi}{\partial x_\alpha} + \frac{\partial \phi}{\partial x_3}\frac{\partial g}{\partial x_\alpha}\right)dx_\alpha = 0 \ \Rightarrow\ -\frac{\partial g}{\partial x_\alpha} = \left(\frac{\partial \phi}{\partial x_\alpha}\right)\Big/\left(\frac{\partial \phi}{\partial x_3}\right). \tag{C15a,b}$$

Substituting (C15b) into (C14) we obtain

$$\frac{\partial f}{\partial x_\alpha} = \left(\frac{\dfrac{\partial f}{\partial x_3}}{\dfrac{\partial \phi}{\partial x_3}}\right)\frac{\partial \phi}{\partial x_\alpha} = \lambda\,\frac{\partial \phi}{\partial x_\alpha}\ ,\ \ \lambda = \left(\frac{\dfrac{\partial f}{\partial x_3}}{\dfrac{\partial \phi}{\partial x_3}}\right), \tag{C16a,b}$$

so that the conditions (C14) and (C16a) may be summarized by the conditions

$$\frac{\partial f}{\partial x_i} = \lambda\,\frac{\partial \phi}{\partial x_i}\ , \tag{C17}$$

which are seen to be the same conditions as (C9). Note that geometrically this means that the gradient of f is parallel to the gradient of ϕ at a stationary point of f which satisfies the constraint (C3).

<u>General Case:</u> For the general case let f be a real valued function of m+n variables

$$f = f(x_i,y_j)\ \ ,\ \ i=1,2,...,m\ \ ,\ \ j=1,2,...,n \tag{C18}$$

and consider n constraint equations of the form

$$\phi_r = \phi_r\,(x_i,y_j) = 0\quad r=1,2,...,n\ . \tag{C19}$$

Furthermore, assume that f and ϕ_r are continuously differentiable and that all the constraints ϕ_r are active so that

$$\det\left(\frac{\partial \phi_r}{\partial y_j}\right) \neq 0\ \ r = 1,2,...,n\ . \tag{C20}$$

Now form the auxiliary function h defined by

$$h = f - \lambda_r\,\phi_r\ , \tag{C21}$$

where λ_r are scalars called Lagrange multipliers that are independent of x_i and y_j and summation is implied over the repeated index r. The method of Lagrange multipliers suggests that the points which satisfy the n constraints (C19) and which make the function f stationary are determined by solving the m+n equations

$$\frac{\partial h}{\partial x_i} = \frac{\partial f}{\partial x_i} - \lambda_r \frac{\partial \phi_r}{\partial x_i} = 0 \quad i=1,2,...,m \ , \tag{C22a}$$

$$\frac{\partial h}{\partial y_j} = \frac{\partial f}{\partial y_j} - \lambda_r \frac{\partial \phi_r}{\partial y_j} = 0 \quad j=1,2,...,n \ , \tag{C22b}$$

together with the n constraints (C19) for the m+2n unknowns x_i, y_j, λ_r. This method produces a <u>necessary</u> condition for f to have a stationary value. However, each stationary point must be checked individually to determine if it is a maximum, minimum or point of inflection.

Appendix D: **Stationary Values of Normal And Shear Stresses**

Stationary Values of Normal Stress:Letting T_{ij} be the components of the Cauchy stress **T** relative to a fixed rectangular Cartesian coordinate system, recall that the normal stress σ acting on the plane defined by the unit outward normal n_j is given by

$$\sigma = \mathbf{t} \bullet \mathbf{n} = \mathbf{T} \bullet (\mathbf{n} \otimes \mathbf{n}) = T_{ij}\, n_i n_j \quad . \tag{D1}$$

For a given value of stress T_{ij} at a point we want to find the planes n_j for which σ is stationary. Since **n** is a unit vector we require n_j to satisfy the constraint equation

$$\phi = n_j n_j - 1 = 0 \quad . \tag{D2}$$

Using the method of Lagrange multipliers described in Appendix C we form the function h

$$h = \sigma - \lambda\, \phi = T_{ij}\, n_i n_j - \lambda\, (n_j n_j - 1) \quad , \tag{D3}$$

and solve for n_j and λ using the constraint (D2) and the three equations

$$\frac{\partial h}{\partial n_k} = 2\, (T_{kj} - \lambda\, \delta_{kj})\, n_j = 0 \quad . \tag{D4}$$

It follows from (D2) and (D4) that the stationary values of σ occur when

$$\mathbf{T\, n} = \lambda\, \mathbf{n} \quad , \quad \mathbf{n} \bullet \mathbf{n} = 1 \quad . \tag{D5a,b}$$

This means that σ attains its stationary values on the three planes that are defined by **n** parallel to the principal directions of the stress tensor **T**. The associated stationary values of σ are the principal values of the stress tensor **T**. Since **T** is a real and symmetric tensor these principal values and directions are real so the principal values σ_i may be ordered with

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \quad . \tag{D6}$$

For later convenience we take the base vectors $\mathbf{p}_i$ of the Cartesian coordinate system to be parallel to the principal directions of **T** so that **T** may be represented in the diagonal form

$$\mathbf{T} = \sigma_1\, \mathbf{p}_1 \otimes \mathbf{p}_1 + \sigma_2\, \mathbf{p}_2 \otimes \mathbf{p}_2 + \sigma_3\, \mathbf{p}_3 \otimes \mathbf{p}_3 \quad , \quad T_{ij} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \quad . \tag{D7a,b}$$

It then follows that for this choice and a arbitrary value of **n** we have

$$\mathbf{t} = \mathbf{T}\,\mathbf{n} = \sigma_1\, n_1\, \mathbf{p}_1 + \sigma_2\, n_2\, \mathbf{p}_2 + \sigma_3\, n_3\, \mathbf{p}_3 \ , \tag{D8a}$$

$$\sigma(n_j) = \sigma_1\, n_1^2 + \sigma_2\, n_2^2 + \sigma_3\, n_3^2 \ . \tag{D8b}$$

Thus, from (D6) and (D8b) we may deduce that

$$\sigma_1 = \sigma_1(n_1^2 + n_2^2 + n_3^2) \geq \sigma_1\, n_1^2 + \sigma_2\, n_2^2 + \sigma_3\, n_3^2 = \sigma(n_j) \ , \tag{D9a}$$

$$\sigma(n_j) = \sigma_1\, n_1^2 + \sigma_2\, n_2^2 + \sigma_3\, n_3^2 \geq \sigma_3(n_1^2 + n_2^2 + n_3^2) = \sigma_3 \ , \tag{D9b}$$

$$\sigma_1 \geq \sigma(n_j) \geq \sigma_3 \ . \tag{D9c}$$

This means that the normal stress σ assumes its maximum value σ_1 on the plane defined by the principal direction $\mathbf{p}_1$ and its minimum value on the plane defined by the principal direction $\mathbf{p}_3$. The value σ_2 is called a minimax and is assumed by σ on the plane defined by the principal direction $\mathbf{p}_2$.

Stationary Values of Shear Stress: Recalling that the shear stress $\mathbf{t}_s$ with magnitude τ is defined such that

$$\mathbf{t}_s = \mathbf{t} - (\mathbf{t} \bullet \mathbf{n})\,\mathbf{n} \ , \ \tau^2 = \mathbf{t}_s \bullet \mathbf{t}_s = \mathbf{t} \bullet \mathbf{t} - \sigma^2 \ , \tag{D10a,b}$$

we may use the representation (D8) to deduce that

$$\tau^2 = \sigma_1^2\, n_1^2 + \sigma_2^2\, n_2^2 + \sigma_3^2\, n_3^2 - (\sigma_1\, n_1^2 + \sigma_2\, n_2^2 + \sigma_3\, n_3^2)^2 \ . \tag{D11}$$

In order to determine stationary values of τ subject to the constraint (D2) we use the method of Lagrange multipliers and form the function h

$$h = \tau^2 - \lambda\,(n_j n_j - 1) \ . \tag{D12}$$

Then the stationary values are found by solving the constraint (D2) and the three equations

$$\frac{\partial h}{\partial n_1} = 2n_1\,[\,\sigma_1^2 - 2\,\sigma_1(\sigma_1\, n_1^2 + \sigma_2\, n_2^2 + \sigma_3\, n_3^2) - \lambda\,] = 0 \ , \tag{D13a}$$

$$\frac{\partial h}{\partial n_2} = 2n_2\,[\,\sigma_2^2 - 2\,\sigma_2(\sigma_1\, n_1^2 + \sigma_2\, n_2^2 + \sigma_3\, n_3^2) - \lambda\,] = 0 \ , \tag{D13b}$$

$$\frac{\partial h}{\partial n_3} = 2n_3\,[\,\sigma_3^2 - 2\,\sigma_3(\sigma_1\, n_1^2 + \sigma_2\, n_2^2 + \sigma_3\, n_3^2) - \lambda\,] = 0 \ . \tag{D13c}$$

One solution of (D2) and (D13) is given by

$$\mathbf{n} = \pm\,\mathbf{p}_1 \;\;,\;\; \tau = 0 \;\;,\;\; \sigma = \sigma_1 \;\;, \tag{D14a}$$

$$\mathbf{n} = \pm\,\mathbf{p}_2 \;\;,\;\; \tau = 0 \;\;,\;\; \sigma = \sigma_2 \;\;, \tag{D14b}$$

$$\mathbf{n} = \pm\,\mathbf{p}_3 \;\;,\;\; \tau = 0 \;\;,\;\; \sigma = \sigma_3 \;\;. \tag{D14c}$$

Hence, the shear stress τ assumes its absolute minimum value of zero on the planes whose normals are in the principal directions of stress. Furthermore, we note that on these same planes the normal stress σ assumes its stationary values. A second solution of (D2) and (D13) is given by

$$\mathbf{n} = \pm\frac{1}{\sqrt{2}}(\mathbf{p}_1 \pm \mathbf{p}_3) \;\;,\;\; \tau = \frac{\sigma_1 - \sigma_3}{2} \;\;,\;\; \sigma = \frac{\sigma_1 + \sigma_3}{2} \;\;, \tag{D15a}$$

$$\mathbf{n} = \pm\frac{1}{\sqrt{2}}(\mathbf{p}_1 \pm \mathbf{p}_2) \;\;,\;\; \tau = \frac{\sigma_1 - \sigma_2}{2} \;\;,\;\; \sigma = \frac{\sigma_1 + \sigma_2}{2} \;\;, \tag{D15b}$$

$$\mathbf{n} = \pm\frac{1}{\sqrt{2}}(\mathbf{p}_2 \pm \mathbf{p}_3) \;\;,\;\; \tau = \frac{\sigma_2 - \sigma_3}{2} \;\;,\;\; \sigma = \frac{\sigma_2 + \sigma_3}{2} \;\;. \tag{D15c}$$

Note that the maximum value of shear stress is equal to one half the difference of the maximum and minimum values of normal stress and it occurs on the plane whose normal bisects the angle between the normals to the planes of maximum and minimum normal stress.

Appendix E: Isotropic Tensors

Let $\mathbf{e}_i$ and $\mathbf{e}_i'$ be two sets of orthonormal base vectors that are connected by the orthogonal transformation $\mathbf{A}$

$$\mathbf{A} = \mathbf{e}_m \otimes \mathbf{e}_m' \ , \tag{E1a}$$

$$A_{ij} = \mathbf{A} \bullet (\mathbf{e}_i \otimes \mathbf{e}_j) = \mathbf{e}_i' \bullet \mathbf{e}_j \ , \ A_{ij}' = \mathbf{A} \bullet (\mathbf{e}_i' \otimes \mathbf{e}_j') = \mathbf{e}_i' \bullet \mathbf{e}_j \ . \tag{E1b,c}$$

Furthermore, let $\mathbf{T}$ be a tensor of any order whose components referred to $\mathbf{e}_i$ are $T_{ij \cdots m}$ and whose components referred to $\mathbf{e}_i'$ are $T_{ij \cdots m}'$. Since $\mathbf{T}$ is a tensor, its components $T_{ij \cdots m}$ and $T_{ij \cdots m}'$ are connected by the transformation relations

$$T_{ij \cdots m}' = A_{ir} A_{js} \ldots A_{mt} \, T_{rs \cdots t} \ . \tag{E2}$$

<u>Isotropic Tensor</u>: A tensor is said to be isotropic if its components relative to any two right-handed orthonormal coordinate systems are equal. Mathematically, this means that

$$T_{ij \cdots m}' = T_{ij \cdots m} \ , \tag{E3}$$

holds for all proper orthogonal transformations $\mathbf{A}$ ($\det \mathbf{A} = +1$). If (E3) holds for all orthogonal transformation (i.e. including those with $\det \mathbf{A} = -1$) then the tensor is said to be isotropic with a center of symmetry.

<u>Zero Order Isotropic Tensor</u>: By definition, scalar invariants satisfy the restriction (E3) so they are zero order isotropic tensors.

<u>First Order Isotropic Tensor</u>: The only first order isotropic tensor is the zero vector

$$T_i = 0 \ . \tag{E4}$$

Proof: For a first order tensor (E2) becomes

$$T_i' = A_{ir} T_r \ . \tag{E5}$$

Taking A_{ij} to be

$$A_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \ , \tag{E6a,b}$$

we obtain the restrictions

$$T_1 = -T_1 \ , \ T_2 = -T_2 \ , \ T_3 = -T_3 \ , \tag{E7a,b,c}$$

so that the only solution is (E4).

Second Order Isotropic Tensor: The most general second order isotropic tensor has the form

$$T_{ij} = \lambda\, \delta_{ij} \ , \tag{E8}$$

where λ is a scalar invariant.

Proof: For a second order tensor (E2) becomes

$$T'_{ij} = A_{ir}A_{js}T_{rs} \ . \tag{E9}$$

Taking A_{ij} to be

$$A_{ij} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \ , \tag{E10}$$

we obtain the restrictions

$$\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} = \begin{pmatrix} T_{33} & T_{31} & T_{32} \\ T_{13} & T_{11} & T_{12} \\ T_{23} & T_{21} & T_{22} \end{pmatrix} \ . \tag{E11}$$

Also, taking A_{ij} to be

$$A_{ij} = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \ , \tag{E12}$$

we obtain the additional restrictions

$$\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} = \begin{pmatrix} T_{33} & T_{31} & -T_{32} \\ T_{13} & T_{11} & -T_{12} \\ T_{23} & -T_{21} & T_{22} \end{pmatrix} \ . \tag{E13}$$

Thus, from (E11) and (E13) we have

$$T_{11} = T_{22} = T_{33} = \lambda \ , \ \text{all other } T_{ij} = 0 \ , \tag{E14a,b}$$

which may be rewritten in the form (E8).

Third Order Isotropic Tensor: The most general third order isotropic tensor has the form

$$T_{ijk} = \lambda\, \varepsilon_{ijk} \ , \tag{E15}$$

where λ is a scalar invariant.

Proof: For a third order tensor (E2) becomes

$$T'_{ijk} = A_{ir}A_{js}A_{kt}T_{rst} \ . \tag{E16}$$

Denoting T_{ijk} by

$$T_{ijk} = \begin{pmatrix} T_{111} & T_{112} & T_{113} & T_{121} & T_{122} & T_{123} & T_{131} & T_{132} & T_{133} \\ T_{211} & T_{212} & T_{213} & T_{221} & T_{222} & T_{223} & T_{231} & T_{232} & T_{233} \\ T_{311} & T_{312} & T_{313} & T_{321} & T_{322} & T_{323} & T_{331} & T_{332} & T_{333} \end{pmatrix} , \qquad \text{(E17)}$$

and specifying A_{ij} by (E10) we obtain

$$T_{ijk} = \begin{pmatrix} T_{333} & T_{331} & T_{332} & T_{313} & T_{311} & T_{312} & T_{323} & T_{321} & T_{322} \\ T_{133} & T_{131} & T_{132} & T_{113} & T_{111} & T_{112} & T_{123} & T_{121} & T_{122} \\ T_{233} & T_{231} & T_{232} & T_{213} & T_{211} & T_{212} & T_{223} & T_{221} & T_{222} \end{pmatrix} . \qquad \text{(E18)}$$

Also, specifying A_{ij} by (E12) we obtain

$$T_{ijk} = \begin{pmatrix} -T_{333} & -T_{331} & T_{332} & -T_{313} & -T_{311} & T_{312} & T_{323} & T_{321} & -T_{322} \\ -T_{133} & -T_{131} & T_{132} & -T_{113} & -T_{111} & T_{112} & T_{123} & T_{121} & -T_{122} \\ T_{233} & T_{231} & -T_{232} & T_{213} & T_{211} & -T_{212} & -T_{223} & -T_{221} & -T_{222} \end{pmatrix} . \qquad \text{(E19)}$$

Then, using (E17)-(E19) we deduce that

$$T_{ijk} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & T_{123} & 0 & T_{132} & 0 \\ 0 & 0 & T_{213} & 0 & 0 & 0 & T_{231} & 0 & 0 \\ 0 & T_{312} & 0 & T_{321} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} , \qquad \text{(E20a)}$$

$$T_{123} = T_{312} = T_{231} \ , \ T_{132} = T_{321} = T_{213} \ . \qquad \text{(E20b,c)}$$

Next we specify A_{ij} by

$$A_{ij} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} , \qquad \text{(E21)}$$

to deduce that

$$T_{123} = -T_{321} \ . \qquad \text{(E22)}$$

Thus, T_{ijk} may be rewritten in the form (E15).

<u>Fourth Order Isotropic Tensor</u>: The most general fourth order isotropic tensor has the form

$$T_{ijkl} = \lambda\, \delta_{ij}\, \delta_{kl} + \mu\, \delta_{ik}\, \delta_{jl} + \gamma\, \delta_{il}\, \delta_{jk} \ , \qquad \text{(E23)}$$

where λ, μ, γ, are scalar invariants.

Proof: For a fourth order tensor (E2) becomes

$$T_{ijkl} = A_{ir}A_{js}A_{kt}A_{lu}T_{rstu} \ . \qquad \text{(E24)}$$

By specifying A_{ij} in the forms (E6a,b) it can be shown that the 81 components of T_{ijkl} reduce to only 21 nonzero components which are denoted by $\bar{T}_{ijkl}$ with

$$\bar{T}_{ijkl} = \begin{pmatrix} T_{1111} & T_{1122} & T_{1133} & T_{1212} & T_{1221} & T_{1313} & T_{1331} \\ T_{2112} & T_{2121} & T_{2211} & T_{2222} & T_{2233} & T_{2323} & T_{2332} \\ T_{3113} & T_{3131} & T_{3223} & T_{3232} & T_{3311} & T_{3322} & T_{3333} \end{pmatrix} . \tag{E25}$$

Specifying A_{ij} by (E10) we obtain the restrictions

$$\bar{T}_{ijkl} = \begin{pmatrix} T_{3333} & T_{3311} & T_{3322} & T_{3131} & T_{3113} & T_{3232} & T_{3223} \\ T_{1331} & T_{1313} & T_{1133} & T_{1111} & T_{1122} & T_{1212} & T_{1221} \\ T_{2332} & T_{2323} & T_{2112} & T_{2121} & T_{2233} & T_{2211} & T_{2222} \end{pmatrix} . \tag{E26}$$

Also, specifying A_{ij} by

$$A_{ij} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \tag{E27}$$

we obtain the additional restrictions

$$\bar{T}_{ijkl} = \begin{pmatrix} T_{3333} & T_{3322} & T_{3311} & T_{3232} & T_{3223} & T_{3131} & T_{3113} \\ T_{2332} & T_{2323} & T_{2233} & T_{2222} & T_{2211} & T_{2121} & T_{2112} \\ T_{1331} & T_{1313} & T_{1221} & T_{1212} & T_{1133} & T_{1122} & T_{1111} \end{pmatrix} . \tag{E28}$$

Then, from (E25),(E26) and (E28) we have

$$T_{1111} = T_{2222} = T_{3333} , \tag{E29a}$$

$$T_{1122} = T_{3311} = T_{2233} = T_{3322} = T_{2211} = T_{1133} = \lambda , \tag{E29b}$$

$$T_{1212} = T_{3311} = T_{2323} = T_{3232} = T_{2121} = T_{1313} = \mu , \tag{E29c}$$

$$T_{1331} = T_{3223} = T_{2112} = T_{3113} = T_{2332} = T_{1221} = \gamma . \tag{E29d}$$

Next, specifying A_{ij} by

$$A_{ij} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{E30}$$

we obtain

$$T_{1111} = A_{1r}A_{1s}A_{1t}A_{1u}\, T_{rstu} , \tag{E31a}$$

$$T_{1111} = \frac{1}{4}(T_{1111} + T_{1122} + T_{1212} + T_{1221} + T_{2112} + T_{2121} + T_{2211} + T_{2222}) , \tag{E31b}$$

so that using (E29) and (E31) we may deduce that

$$T_{1111} = T_{2222} = T_{3333} = \lambda + \mu + \gamma \ . \tag{E32}$$

Thus, with the help of (E29) and (E32) we may rewrite T_{ijkl} in the form (E23). Notice that T_{ijkl} in (E23) automatically has the symmetries

$$T_{ijkl} = T_{klij} \ , \ \mathbf{T}^{T(2)} = \mathbf{T} \ . \tag{E33a,b}$$

<u>Special Case</u>: As a special case, if we further restrict the isotropic tensor T_{ijkl} to be symmetric in its first two indices

$$T_{ijkl} = T_{jikl} \ , \ {}^{LT}\mathbf{T} = \mathbf{T} \ , \tag{E34a,b}$$

then we may deduce that

$$\gamma = \mu \ , \tag{E35}$$

so that T_{ijkl} becomes

$$T_{ijkl} = \lambda \, \delta_{ij} \, \delta_{kl} + \mu \, (\delta_{ik} \, \delta_{jl} + \delta_{il}\delta_{jk}) \ . \tag{E36}$$

Then it can be seen from (E36) that T_{ijkl} has the additional symmetries

$$T_{ijlk} = T_{ijkl} \ , \ \mathbf{T}^{T} = \mathbf{T} \ . \tag{E37}$$

HOMEWORK PROBLEM SETS

PROBLEM SET 1

Problem 1.1: Expand the following equations for an index range of three:

(a) $$a_i + b_i = c_i \ ,$$

(b) $$t_i = T_{ij}\, n_j \ ,$$

(c) $$I = c_{ij}\, x_i\, x_j \ ,$$

(d) $$\phi = A_{jj}\, B_{kk} \ ,$$

(e) $$A = A_{ij}\, A_{ij} \ ,$$

(f) How many distinct equations are there in cases (a), (b), (c), respectively?

(g) How many terms are there on the right-hand sides of (c) and (d)?

Problem 1.2: Expand and simplify the following expressions:

(a) $$\delta_{ij}\, a_j \ ,$$

(b) $$\delta_{ij}\, x_i\, x_j \ ,$$

(c) $$a_{ij}\, x_i\, x_j \ \text{ with } a_{ij} = a_{ji} \ \text{(symmetric)} \ ,$$

(d) $$a_{ij}\, x_i\, x_j \ \text{ with } a_{ij} = -\, a_{ji} \ \text{(skew-symmetric)} \ ,$$

(e) $$t_i = -\, p\, \delta_{ij}\, n_j$$

Problem 1.3: Verify the identities

(a) $$\delta_{ii} = 3 \ ,$$

(b) $$\delta_{ij}\, \delta_{ij} = \delta_{ii} \ ,$$

(c) $$\delta_{ij}\, a_{jk} = a_{ik} \ .$$

Problem 1.4: Letting a comma denote partial differentiation with respect to position such that

$$f_{,j} = \frac{\partial f}{\partial x_j} \ , \qquad \text{(P1.4)}$$

(a) Verify that $x_{i,j} = \delta_{ij}$.

(b) Using the result of part (a) write a simplified indicial expression for $(x_i\, x_i)_{,j}$.

(c) Using the result of part (a) write a simplified indicial expression for $(x_i\, x_i)_{,jj}$.

Problem 1.5: Simplify the following expression without expanding the indices

$$(\delta_{ij} + c_{ij})\,(\delta_{ik} + c_{ik}) - \delta_{jk} - \delta_{mn}\, c_{mj}\, c_{nk} \ . \tag{P1.5}$$

Problem 1.6: Let **A** be a second order tensor with components A_{ij} which is represented by

$$\mathbf{A} = A_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j \ . \tag{P1.6a}$$

Using the formula (3.27b) show that the components A^T_{ij} of $\mathbf{A}^T$ are given by

$$A^T_{ij} = \mathbf{A}^T \bullet (\mathbf{e}_i \otimes \mathbf{e}_j) = A_{ji} \ . \tag{P1.6b}$$

Problem 1.7: Let **A** and **B** be second order tensors with components A_{ij} and B_{ij}, respectively. Using the representation

$$\mathbf{AB} = A_{im} B_{mj}\, \mathbf{e}_i \otimes \mathbf{e}_j \ , \tag{P1.7a}$$

Prove that

$$(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T \ . \tag{P1.7b}$$

Problem 1.8: Using (3.31) prove the validity of (3.32).

PROBLEM SET 2

Problem 2.1: Let

$$A_{(ij)} = \frac{1}{2}(A_{ij} + A_{ji}) \quad , \quad A_{[ij]} = \frac{1}{2}(A_{ij} - A_{ji}) \quad . \qquad \text{(P2.1a,b)}$$

(a) Demonstrate that $A_{(ij)}$ is symmetric and hence $A_{(ij)} = A_{(ji)}$.

(b) Demonstrate that $A_{[ij]}$ is skew-symmetric and hence $A_{[ij]} = -A_{[ji]}$.

(c) Show that an arbitrary square array A_{ij} can always be expressed as the <u>sum</u> of its symmetric and skew-symmetric parts, i.e.,

$$A_{ij} = A_{(ij)} + A_{[ij]} \quad . \qquad \text{(P2.1c)}$$

(d) With the help of (P2.1a,b) above show that

$$A_{ii} = A_{(ii)} \quad . \qquad \text{(P2.1d)}$$

(e) Given arbitrary square arrays A_{ij} and B_{ij}, show that

$$\text{(i)} \;\; A_{(ij)}\, B_{ij} = A_{(ij)}\, B_{(ij)} \quad , \;\; \text{(ii)} \;\; A_{[ij]}\, B_{ij} = A_{[ij]}\, B_{[ij]} \quad , \qquad \text{(P2.1e,f)}$$

$$\text{(iii)} \;\; A_{ij}\, B_{ij} = A_{(ij)}\, B_{(ij)} + A_{[ij]}\, B_{[ij]} \quad . \qquad \text{(P2.1g)}$$

Problem 2.2: Suppose that B_{ij} is skew-symmetric and A_{ij} is symmetric. Show that

$$A_{ij}\, B_{ij} = 0 \quad . \qquad \text{(P2.2)}$$

Problem 2.3: Let **T** be a third order tensor and let **a**, **b**, **c** be arbitrary vectors. Prove that

$$\mathbf{T} \bullet (\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}) = \mathbf{a} \bullet [\mathbf{T} \bullet (\mathbf{b}\otimes\mathbf{c})] = [(\mathbf{a}\otimes\mathbf{b}) \bullet \mathbf{T}] \bullet \mathbf{c} = (\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}) \bullet \mathbf{T} \quad . \qquad \text{(P2.3)}$$

Problem 2.4: Using (P2.3) and the definition (3.34) of the permutation tensor $\boldsymbol{\varepsilon}$ show that the components of $\boldsymbol{\varepsilon}$ are given by

$$\boldsymbol{\varepsilon} \bullet (\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k) = \mathbf{e}_i \times \mathbf{e}_j \bullet \mathbf{e}_k = \varepsilon_{ijk} \;. \qquad \text{(P2.4)}$$

Problem 2.5: Using the properties (3.34) and (3.36) of the permutation tensor $\boldsymbol{\varepsilon}$ and the result (P2.3) with **T** replaced by $\boldsymbol{\varepsilon}$, prove the permutation property of the scalar triple product of three vectors

$$\boldsymbol{\varepsilon} \bullet (\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}) = \mathbf{a} \bullet \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} \bullet \mathbf{c} \quad . \qquad \text{(P2.5)}$$

Problem 2.6: Let **a** and **b** be two vectors and define **c** by the vector product

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} \ . \qquad \text{(P2.6a)}$$

(a) Show that when **a** and **b** are referred to the rectangular Cartesian basis $\mathbf{e}_i$ then the components c_i of **c** are related to the components a_i, b_i of **a** and **b**, respectively, by the expression

$$c_i = \varepsilon_{ijk}\, a_j\, b_k \ . \qquad \text{(P2.6b)}$$

(b) What is the indicial counterpart of the vector product $\mathbf{a} \times \mathbf{a} = 0$? (Show your work).

Problem 2.7: Let **a**, **b**, **c** be three vectors and recall that the vector triple product may be expanded in the form

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{c})\, \mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\, \mathbf{c} \ . \qquad \text{(P2.7a)}$$

Using this result and the properties (3.34) and (3.36) of the permutation tensor $\boldsymbol{\varepsilon}$ prove that

(a) $$[\boldsymbol{\varepsilon} \bullet (\mathbf{e}_i \otimes \mathbf{e}_j)] \bullet [\boldsymbol{\varepsilon} \bullet (\mathbf{e}_r \otimes \mathbf{e}_s)] = \varepsilon_{mij}\, \varepsilon_{mrs} = \delta_{ir}\, \delta_{js} - \delta_{is}\, \delta_{jr} \ . \qquad \text{(P2.7b)}$$

(b) $$(\boldsymbol{\varepsilon}\, \mathbf{e}_j) \bullet (\boldsymbol{\varepsilon}\, \mathbf{e}_s) = \varepsilon_{mij}\, \varepsilon_{mis} = 2\, \delta_{js} \ . \qquad \text{(P2.7c)}$$

(c) $$\boldsymbol{\varepsilon} \bullet \boldsymbol{\varepsilon} = \varepsilon_{mij}\, \varepsilon_{mij} = 6 \ . \qquad \text{(P2.7d)}$$

Problem 2.8: Prove that for an arbitrary vector **a**

$$\boldsymbol{\varepsilon} \bullet (\boldsymbol{\varepsilon}\, \mathbf{a}) = 2\, \mathbf{a} \ . \qquad \text{(P2.8)}$$

Problem 2.9: Show that

$$\boldsymbol{\varepsilon}^T = -\, \boldsymbol{\varepsilon} \ , \ {}^{LT}\boldsymbol{\varepsilon} = -\, \boldsymbol{\varepsilon} \ , \ {}^{LT}(\boldsymbol{\varepsilon}^T) = \boldsymbol{\varepsilon} \ . \qquad \text{(P2.9a,b)}$$

Problem 2.10: Let **W** be a second order tensor defined by the vector $\boldsymbol{\omega}$ through the equation

$$\mathbf{W} = -\, \boldsymbol{\varepsilon}\, \boldsymbol{\omega} \ . \qquad \text{(P2.10a)}$$

(a) Using (3.17) and (P2.9b) show that **W** is a skew-symmetric tensor

$$\mathbf{W}^T = - \mathbf{W} \ . \qquad \text{(P2.10b)}$$

(b) Using (3.34) and (P2.10a) show that for an arbitrary vector **a**

$$\mathbf{W\ a} = \boldsymbol{\omega} \times \mathbf{a} \ . \qquad \text{(P2.10c)}$$

(c) Using (P2.8) show that (P2.10a) may be solved for $\boldsymbol{\omega}$ to obtain

$$\boldsymbol{\omega} = -\frac{1}{2} \ \boldsymbol{\varepsilon} \bullet \mathbf{W} \ . \qquad \text{(P2.10d)}$$

Note that $\boldsymbol{\omega}$ is called the axial vector of the skew-symmetric tensor **W**.

Problem 2.11: Prove the validity of (3.44).

Problem 2.12: Let **T** be a second order tensor. Determine the restrictions on the components T_{ij} of **T** imposed by the vector equation

$$\mathbf{e}_j \times \mathbf{T}\mathbf{e}_j = 0 \ . \qquad \text{(P2.12)}$$

Problem 2.13: Prove the validity of (4.12a,b,c,d).

Problem 2.14: Let **v** and **w** be vectors and **A** and **B** be second order tensors. Prove that

$$(\mathbf{A\ v}) \bullet (\mathbf{B\ w}) = \mathbf{v} \bullet (\mathbf{A}^T\mathbf{B\ w}) = (\mathbf{B}^T\mathbf{A\ v}) \bullet \mathbf{w} \ . \qquad \text{(P2.14)}$$

Problem 2.15: Recall that the determinant det **T** of the second order tensor **T** with components T_{ij} may be expressed in the form

$$\det \mathbf{T} = \frac{1}{6} \ \varepsilon_{ijk} \, \varepsilon_{rst} \, T_{ir} T_{js} T_{kt} \, . \qquad \text{(P2.15a)}$$

Prove that the determinant of **T** may also be expressed in the form

$$\det \mathbf{T} = \frac{1}{6} \ (\mathbf{T} \times \mathbf{T}) \bullet \mathbf{T} \ . \qquad \text{(P2.15b)}$$

Problem 2.16: Let **L** be a second order tensor with components L_{ij} and let **s** be a vector with components s_i.

(a) Show that

$$\mathbf{L} \bullet (\mathbf{s} \otimes \mathbf{s}) = \mathbf{s} \bullet \mathbf{L}\, \mathbf{s} = s_i\, L_{ij}\, s_j \quad . \qquad \text{(P2.16a)}$$

(b) Let **W** be a skew-symmetric second order tensor with components W_{ij}. Show that

$$\mathbf{W} \bullet (\mathbf{s} \otimes \mathbf{s}) = 0 \quad . \qquad \text{(P2.16b)}$$

PROBLEM SET 3

Problem 3.1: Let **T**' be the deviatoric part of a <u>symmetric</u> second order tensor **T** and define the scalar σ_e by the formula

$$\sigma_e^2 = \frac{3}{2}\ \mathbf{T}' \bullet \mathbf{T}'\ \ ,\ \ \mathbf{T}' = \mathbf{T} - \frac{1}{3}(\mathbf{T} \bullet \mathbf{I})\ \mathbf{I}\ . \qquad \text{(P3.1a,b)}$$

(a) Write an expression for σ_e^2 in terms of **T**.

(b) Expand this expression in terms of the rectangular Cartesian components T_{ij} of **T** to deduce that

$$\sigma_e^2 = \frac{1}{2}\ [(T_{11} - T_{22})^2 + (T_{11} - T_{33})^2 + (T_{22} - T_{33})^2] + 3\ [T_{12}^2\ + T_{13}^2\ + T_{23}^2\ ]\ . \qquad \text{(P3.1b)}$$

Problem 3.2: Using (5.4) the transformation tensor **A** may be expressed in the form

$$\mathbf{A} = A_{ij}\ \mathbf{e}_i \otimes \mathbf{e}_j\ \ . \qquad \text{(P3.2a)}$$

Show that the component forms of the orthogonality conditions

$$\mathbf{A}\mathbf{A}^T = \mathbf{I}\ \ ,\ \ \mathbf{A}^T\mathbf{A} = \mathbf{I}\ \ , \qquad \text{(P3.2b,c)}$$

may be written in the forms, respectively,

$$A_{im}A_{jm} = \delta_{ij}\ \ ,\ \ A_{mi}A_{mj} = \delta_{ij}\ \ . \qquad \text{(P3.2d,e)}$$

Problem 3.3: Define the base vectors $\mathbf{e}_i'$ by

$$\mathbf{e}_1' = -\frac{1}{\sqrt{2}}\ \mathbf{e}_1 + \frac{1}{\sqrt{2}}\mathbf{e}_3\ \ , \qquad \text{(P3.3a)}$$

$$\mathbf{e}_2' = \frac{1}{2}\ \mathbf{e}_1 + \frac{1}{\sqrt{2}}\ \mathbf{e}_2 + \frac{1}{2}\ \mathbf{e}_3\ \ , \qquad \text{(P3.3b)}$$

$$\mathbf{e}_3' = -\frac{1}{2}\ \mathbf{e}_1 + \frac{1}{\sqrt{2}}\ \mathbf{e}_2 - \frac{1}{2}\ \mathbf{e}_3\ \ , \qquad \text{(P3.3c)}$$

Calculate the components of A_{ij} and show that **A** satisfies the orthogonality condition (P3.2b).

Problem 3.4: Let $v_i = \mathbf{e}_i \bullet \mathbf{v}$ and $v_i' = \mathbf{e}_i' \bullet \mathbf{v}$ be the components of the vector **v** such that

$$v_i = A_{mi}\ v_m'\ \ ,\ \ v_i' = A_{im}\ v_m\ \ . \qquad \text{(P3.4a,b)}$$

Prove that

$$v_i \, v_i = v'_i \;\; v'_i \quad , \qquad \text{(P3.4c)}$$

which verifies that **v • v** is a scalar invariant.

Problem 3.5: Let T_{ij} be the components of a second order tensor **T** referred to the Cartesian base vectors $\mathbf{e}_i$ and let T'_{ij} be the components of the same tensor referred to another set of Cartesian base vectors $\mathbf{e}'_i$. Prove that

$$T_{ii} = T'_{ii} \quad , \quad T_{ij}T_{ij} = T'_{ij} \, T'_{ij} \quad , \qquad \text{(P3.5a,b)}$$

which verifies that the trace of **T** (denoted by tr **T** = **T • I**) and the magnitude squared **T • T** are scalar invariants.

Problem 3.6: Let φ(**x**) be a scalar function and **u**(**x**) be a vector function of position **x**. Obtain the indicial counterparts of

$$\text{curl (grad } \phi) = 0 \;\; , \;\; \text{div (curl } \mathbf{u}) = 0 \;\; . \qquad \text{(P3.6a,b)}$$

Problem 3.7: Recalling the divergence theorem in the form (3.46) show that

$$\int_{\partial P} \mathbf{x} \bullet \mathbf{n} \; da = 3 \; v \quad , \qquad \text{(P3.7)}$$

where v is the volume of the region P and **x** is the position vector of a point in P.

Problem 3.8: Let **v** be a vector and **T** be a second order tensor and use the divergence theorem (3.46) to show that

$$\int_{\partial P} \mathbf{v} \bullet \mathbf{Tn} \; da = \int_P \left(\partial \mathbf{v}/\partial \mathbf{x} \bullet \mathbf{T} + \mathbf{v} \bullet \text{div } \mathbf{T} \right) \; dv \quad . \qquad \text{(P3.8a)}$$

As a special case take **v** to be the position vector and show that

$$\int_{\partial P} \mathbf{x} \bullet \mathbf{Tn} \; da = \int_P \left(\mathbf{I} \bullet \mathbf{T} + \mathbf{x} \bullet \text{div } \mathbf{T} \right) \; dv \quad . \qquad \text{(P3.8a)}$$

Problem 3.9: Let X_A and x_i be the Cartesian components of **X** and **x**, respectively. Consider the motion defined by

$$x_1 = X_1 \cos\theta + X_2 \sin\theta \ , \ x_2 = -X_1 \sin\theta + X_2 \cos\theta \ , \ x_3 = X_3 \ , \quad \text{(P3.9a,b,c)}$$

where $\theta(t)$ is a function of time only.

(a) Calculate the inverse mapping (i.e. express X_A in terms of x_i and θ).

(b) Calculate the deformation gradient **F** and show that

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{I} \ . \quad \text{(P3.9d)}$$

(c) Explain the physical meaning of the result (P3.9d).

(d) Calculate the Lagrangian representation $\hat{v}_i(X_A,t)$ of the velocity.

(e) Calculate the Eulerian representation $\tilde{v}_i(x_j,t)$ of the velocity.

(f) Show that the velocity $\mathbf{v} = v_i \, \mathbf{e}_i$ may also be expressed in the form

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x} \ , \ \boldsymbol{\omega} = -\dot{\theta} \ \mathbf{e}_3 \ , \ \mathbf{x} = x_i \, \mathbf{e}_i \ . \quad \text{(P3.9e,f,g)}$$

(g) Calculate the components D_{ij} of the deformation tensor **D** and the components W_{ij} of the spin tensor **W**.

(h) Using the Eulerian representation of the velocity developed in part (e) calculate the components $a_i = \dot{v}_i$ of the acceleration **a**.

(i) Show that the results obtained in part (h) are consistent with the expression

$$\mathbf{a} = \dot{\mathbf{v}} = \dot{\boldsymbol{\omega}} \times \mathbf{x} + \boldsymbol{\omega} \times \mathbf{v} \ . \quad \text{(P3.9h)}$$

PROBLEM SET 4

Problem 4.1: Taking the material derivative of (7.17) and using the properties of the scalar triple product between three vectors, show that

$$\dot{J} = (\mathbf{Fe}_2\times\mathbf{Fe}_3)\bullet\dot{\mathbf{F}}\,\mathbf{e}_1 + (\mathbf{Fe}_3\times\mathbf{Fe}_1)\bullet\dot{\mathbf{F}}\,\mathbf{e}_2 + (\mathbf{Fe}_1\times\mathbf{Fe}_2)\bullet\dot{\mathbf{F}}\,\mathbf{e}_3 \ . \qquad \text{(P4.1)}$$

Problem 4.2: Thinking of J=det **F** as a function of **F**, the chain rule of differentiation yields

$$\dot{J} = (\partial J/\partial \mathbf{F})\bullet\dot{\mathbf{F}} \ . \qquad \text{(P4.2a)}$$

Next, using (7.13) and the result (P4.1), show that

$$\partial J/\partial \mathbf{F} = J\,\mathbf{F}^{-T}. \qquad \text{(P4.2b)}$$

Problem 4.3: With the help of the results (P4.2a,b) and equation (9.2), show that

$$\dot{J} = J\,\mathrm{div}\,\mathbf{v} = J\,\mathbf{L}\bullet\mathbf{I} = J\,\mathbf{D}\bullet\mathbf{I} \ . \qquad \text{(P4.3)}$$

Problem 4.4: Using the chain rule of differentiation we have

$$(\partial\mathbf{x}/\partial\mathbf{X})\,(\partial\mathbf{X}/\partial\mathbf{x}) = \partial\mathbf{x}/\partial\mathbf{x} = \mathbf{I} \ , \qquad \text{(P4.4a)}$$

It follows from (P4.4a) and the definition of the deformation gradient **F** that

$$\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X} \ , \ \mathbf{F}^{-1} = \partial\mathbf{X}/\partial\mathbf{x} \ , \ \mathbf{F}\mathbf{F}^{-1} = \mathbf{I} \ . \qquad \text{(P4.4b,c,d)}$$

Using (9.3a) and taking the material derivative of (P4.4d) show that

$$\dot{\overline{\partial\mathbf{X}/\partial\mathbf{x}}} = \dot{\overline{\mathbf{F}^{-1}}} = -\,\mathbf{F}^{-1}\mathbf{L} \ , \ \dot{\overline{\mathbf{F}^{-T}}} = -\,\mathbf{L}^{T}\mathbf{F}^{-T} \ , \qquad \text{(P4.4e,f)}$$

Problem 4.5: Let ϕ be a scalar function of position and let Grad $\phi = \partial\phi/\partial\mathbf{X}$ be the gradient of ϕ with respect to the reference position **X** and let grad $\phi = \partial\phi/\partial\mathbf{x}$ be the gradient of ϕ with respect to the present position **x**. Using the chain rule of differentiation and the result (3.28b) show that

$$\partial\phi/\partial\mathbf{X} = \partial\hat{\phi}(\mathbf{X})\,/\partial\mathbf{X} = \mathbf{F}^{T}\partial\tilde{\phi}(\mathbf{x})\,/\partial\mathbf{x} = \mathbf{F}^{T}(\partial\phi/\partial\mathbf{x}) \ . \qquad \text{(P4.5a)}$$

Multiplying (P4.5a) by $\mathbf{F}^{-T}$ on the left it follows that

$$\partial\phi/\partial\mathbf{x} = \mathbf{F}^{-T}\,(\partial\phi/\partial\mathbf{X}) \ . \qquad \text{(P4.5b)}$$

Problem 4.6: Derive the following formula

$$\dot{\overline{\partial\phi/\partial\mathbf{X}}} = \partial\dot{\phi}/\partial\mathbf{X} \ , \ \dot{\overline{(\phi_{,A})}} = \dot{\overline{\partial\phi/\partial X_A}} = \partial\dot{\phi}/\partial X_A = (\dot{\phi})_{,A} \ . \qquad \text{(P4.6a,b)}$$

Problem 4.7: Using (P4.5b) and (P4.6) derive the following formula

$$\dot{\overline{\partial\phi/\partial\mathbf{x}}} = \partial\dot{\phi}/\partial\mathbf{x} - \mathbf{L}^T(\partial\phi/\partial\mathbf{x}) \ , \ \dot{\overline{(\phi_{,i})}} = (\dot{\phi})_{,i} - v_{j,i}\,\phi_{,j} \ . \qquad \text{(P4.7a,b)}$$

Problem 4.8: Recalling that the material derivative $\dot{f}$ of the Eulerian representation $\tilde{f}(x_i,t)$ of a function may be expressed in the form

$$\dot{f} = \frac{\partial\tilde{f}(x_i,t)}{\partial t} + \tilde{f}(x_i,t)_{,m}\, v_m \ , \qquad \text{(P4.8)}$$

derive the result (P4.7b) directly by setting $f=\phi_{,i}$.

Problem 4.9: Consider the deformation characterized by

$$x_1 = X_1 \ , \ x_2 = X_2 \ , \ x_3 = X_3 + k\, X_2^2 \ . \qquad \text{(P4.9a,b,c)}$$

(a) Find the inverse mapping.

(b) Calculate the components of C_{AB} and $c_{ij} = (\mathbf{B}^{-1})_{ij}$.

(c) Derive the expression for determining the stretch λ of a line element $d\mathbf{x}$ in the present configuration in the direction $\mathbf{s}$

$$\frac{1}{\lambda^2} = \mathbf{B}^{-1} \bullet (\mathbf{s}\otimes\mathbf{s}) \ . \qquad \text{(P4.9d)}$$

(d) Determine the stretch λ of a material line element $d\mathbf{x}$ in the present configuration at the position $\mathbf{x}$ and in the direction $\mathbf{s}$ where $\mathbf{x}$ and $\mathbf{s}$ are given by

$$x_i = (0,1,1) \ , \ s_i = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \ . \qquad \text{(P4.9e,f)}$$

(e) Determine the direction $\mathbf{S}$ of the line element $d\mathbf{X}$ in the reference configuration which is associated with the $d\mathbf{x}$ at point $\mathbf{x}$ in part (d).

(f) Determine the stretch of the line element d**X** in the direction **S** of part (e) using the formula

$$\lambda^2 = \mathbf{C} \bullet (\mathbf{S} \otimes \mathbf{S}) \ , \qquad \text{(P4.9g)}$$

and compare your result with that derived in part (d).

Problem 4.10: Recalling from (7.9),(7.34), and (7.20b) that

$$ds = \lambda \, dS \ , \ \mathbf{n} \, da = J \, \mathbf{F}^{-T} \, \mathbf{N} \, dA \ , \ dv = J \, dV \ , \qquad \text{(P4.10a,b,c)}$$

calculate expressions for

$$\overset{\bullet}{\overline{ds}} / ds \ , \ \overset{\bullet}{\overline{da}} / da \ , \ \overset{\bullet}{\overline{dv}} / dv \ , \qquad \text{(P4.10d,e,f)}$$

in terms of the rate of deformation tensor **D** and the direction **s**. It is important to emphasize that the direction **s** of the material line element in the formula for (P4.10d) is different from the normal **n** to the material surface in the formula for (P4.10e).

Problem 4.11: Consider the velocity field defined by

$$v_1 = a \, x_2 \, (x_1^2 + x_2^2) \ , \ v_2 = - b \, x_1 \, (x_1^2 + x_2^2) \ , v_3 = d \, (x_3 - ct) \ , \qquad \text{(P4.11a,b,c)}$$

where a,b,c,d are constants.

(a) Calculate the components of the acceleration $\dot{v}_i$.

(b) Calculate the components of the velocity gradient L_{ij}.

(c) Calculate the components of the rate of deformation tensor D_{ij}.

(d) Calculate the components of the spin tensor W_{ij}.

(e) Does the deformation preserve volume?

Problem 4.12: The velocity field associated with rigid body motion is given by

$$\mathbf{v} = \dot{\mathbf{c}} + \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{c}) \ , \qquad \text{(P4.12a)}$$

where **c** and **ω** are vector functions of time only.

(a) Write the component form of equation (P4.12a).

(b) Calculate the components of the velocity gradient L_{ij}.

(c) Calculate the components of the rate of deformation tensor D_{ij}.

(d) Calculate the components of the spin tensor W_{ij}.

(e) What is the physical meaning of the result in part (c)?

PROBLEM SET 5

Problem 5.1: Consider the motion described by

$$x_1 = X_1(1 + a \sin \omega t) \ , \ x_2 = X_2 + b \sin k(x_1 - ct) \ , \ x_3 = X_3 \ , \quad \text{(P5.1a,b,c)}$$

where a,ω,b,k,c are constants. Let X_2=X_3=0 define a string (material fiber) in space.

(a) Calculate the velocity of the material point on the string which was initially (t=0) located at X_1=L.

(b) Calculate the velocity of the material point on the string which at time t is located at x_1=L.

(c) Calculate the vertical ($\mathbf{e}_2$ direction) velocity of the string as measured by an observer fixed at x_1=L (i.e. the velocity of the intersection of the string with the fixed plane x_1=L).

Problem 5.2:

Consider a line element d**X**=**S**dS in the reference configuration which is mapped to d**x**=**s**ds in the present configuration, where **S** and **s** are unit vectors and recall that

$$\lambda \, \mathbf{s} = \mathbf{F} \, \mathbf{S} \ , \ \lambda = \frac{ds}{dS} \ . \quad \text{(P5.2a,b)}$$

(a) Show that

$$\frac{\dot{\lambda}}{\lambda} \, \mathbf{s} + \dot{\mathbf{s}} = \mathbf{L} \, \mathbf{s} \ , \quad \text{(P5.2c)}$$

where **L** is the velocity gradient.

(b) Also show that

$$\frac{\dot{\lambda}}{\lambda} = \mathbf{D} \bullet (\mathbf{s} \otimes \mathbf{s}) \ . \quad \text{(P5.2d)}$$

(c) Use equations (P4.4e) and the chain rule of differentiation to show that

$$\mathbf{L} = \partial \mathbf{v}/\partial \mathbf{x} = \partial \mathbf{v}/\partial \mathbf{X} \, \mathbf{F}^{-1} \ , \ \dot{\mathbf{L}} = \partial \dot{\mathbf{v}}/\partial \mathbf{x} - \mathbf{L}^2 \ . \quad \text{(P5.2e,f)}$$

(d) Differentiating (P5.2c) show that

$$\ddot{\lambda} + \lambda \, \mathbf{s} \bullet \ddot{\mathbf{s}} = \lambda \, \partial \dot{\mathbf{v}}/\partial \mathbf{x} \bullet (\mathbf{s} \otimes \mathbf{s}) \ , \quad \text{(P5.2g)}$$

and that

$$\frac{\ddot{\lambda}}{\lambda} = \dot{\mathbf{s}} \bullet \dot{\mathbf{s}} + \partial\dot{\mathbf{v}}/\partial\mathbf{x} \bullet (\mathbf{s}\otimes\mathbf{s}) \quad . \tag{P5.2h}$$

Problem 5.3:

Given the velocity field

$$v_1 = e^{x_3 - ct} \cos \omega t \ , \ v_2 = e^{x_3 - ct} \sin \omega t \ , \ v_3 = c = \text{constant} \ , \tag{P5.3a,b,c}$$

(a) Show that the speed of every particle is constant.

(b) Calculate the acceleration components $\dot{v}_i$.

(c) Find the logarithmic stretching $\dot{\lambda}/\lambda$ for a line element which in the present configuration has the direction $\mathbf{s} = \frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_3)$ at $\mathbf{x} = 0$.

Problem 5.4:

Let $\tilde{\phi}(\mathbf{x},t)$ be a scalar function of position $\mathbf{x}$ and time t. Prove the transport theorem

$$\frac{d}{dt}\int_P \tilde{\phi}(\mathbf{x},t)\, dv = \int_P (\dot{\phi} + \phi \, \text{div}\, \mathbf{v})\, dv \ , \tag{P5.4}$$

indicating all steps of the proof in detail.

Problem 5.5:

Put $\phi=1$ in (P5.4) and use the divergence theorem to show that the rate of change of the volume of the part P is given by

$$\dot{v} = \frac{d}{dt}\int_P dv = \int_{\partial P} \mathbf{v} \bullet \mathbf{n}\, da \quad . \tag{P5.5}$$

Discuss the physical meaning of the formula (P5.5).

Problem 5.6:

Let $\sigma_1 \geq \sigma_2 \geq \sigma_3$ be the principal values and $\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3$ be the unit principal directions of the Cauchy stress $\mathbf{T}$, so that $\mathbf{T}$ admits the representation

$$\mathbf{T} = \sigma_1 \, \mathbf{p}_1\otimes\mathbf{p}_1 + \sigma_2 \, \mathbf{p}_2\otimes\mathbf{p}_2 + \sigma_3 \, \mathbf{p}_3\otimes\mathbf{p}_3 \quad . \tag{P5.6a}$$

Recall the **T** may be separated into its spherical part $-p\,\mathbf{I}$ and its deviatoric part $\mathbf{T}'$ such that

$$\mathbf{T} = -p\mathbf{I} + \mathbf{T}' \ , \ \mathbf{T}' \bullet \mathbf{I} = 0 \ . \qquad \text{(P5.6b,c)}$$

(a) Show that the pressure p in (P5.6a) is given by

$$p = -\frac{1}{3}\ \mathbf{T} \bullet \mathbf{I} = -\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \ . \qquad \text{(P5.6d)}$$

(b) Show that the deviatoric stress is given by

$$\mathbf{T}' = \sigma_1'\ \mathbf{p}_1\otimes\mathbf{p}_1 + \sigma_2'\ \mathbf{p}_2\otimes\mathbf{p}_2 + \sigma_3'\ \mathbf{p}_3\otimes\mathbf{p}_3 \ . \qquad \text{(P5.6e)}$$

$$\sigma_1' = \sigma_1 + p = \frac{1}{3}(2\sigma_1 - \sigma_2 - \sigma_3) \ , \qquad \text{(P5.6f)}$$

$$\sigma_2' = \sigma_2 + p = \frac{1}{3}(-\sigma_1 + 2\sigma_2 - \sigma_3) \ , \qquad \text{(P5.6g)}$$

$$\sigma_3' = \sigma_3 + p = \frac{1}{3}(-\sigma_1 - \sigma_2 + 2\sigma_3) \ . \qquad \text{(P5.6h)}$$

(c) The unit normal **n** to the octahedral plane is defined by

$$\mathbf{n} = \frac{1}{\sqrt{3}}\ (\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \ . \qquad \text{(P5.6i)}$$

Show that the stress vector **t** acting on the octahedral plane is given by

$$\mathbf{t} = \frac{1}{\sqrt{3}}\ (\sigma_1\ \mathbf{p}_1 + \sigma_2\ \mathbf{p}_2 + \sigma_3\ \mathbf{p}_3) \ . \qquad \text{(P5.6j)}$$

(d) In general the stress vector **t** acting on a plane **n** admits the representation

$$\mathbf{t} = \sigma\,\mathbf{n} + \tau\,\mathbf{s} \ , \ \sigma = \mathbf{t} \bullet \mathbf{n} \ , \qquad \text{(P5.6k,l)}$$

$$\tau = |\mathbf{t} - \sigma\,\mathbf{n}| \ , \ \mathbf{s} = \frac{\mathbf{t} - \sigma\,\mathbf{n}}{|\mathbf{t} - \sigma\,\mathbf{n}|} \ , \qquad \text{(P5.6m,n)}$$

where σ is the normal component of **t**, τ is the shearing component of **t** and **s** is the direction of shearing on the plane defined by the normal **n**. Show that for the octahedral plane

$$\sigma = -p \ , \ \tau\,\mathbf{s} = \mathbf{T}'\mathbf{n} = \frac{1}{\sqrt{3}}(\sigma_1'\ \mathbf{p}_1 + \sigma_2'\ \mathbf{p}_2 + \sigma_3'\ \mathbf{p}_3) \ , \qquad \text{(P5.6o,p)}$$

$$\tau = \frac{1}{\sqrt{3}}\ \left[(\sigma_1')^2 + (\sigma_2')^2 + (\sigma_3')^2\right]^{1/2} \ . \qquad \text{(P5.6q)}$$

Note that in this sense the octahedral plane is special because the normal stress σ equals minus the pressure p.

(e) Use (P5.6e) to show that the octahedral shear stress τ in (P5.6q) can also be written in the invariant form

$$\tau = \frac{1}{\sqrt{3}} \left[\mathbf{T}' \bullet \mathbf{T}' \right]^{1/2} \quad . \tag{P5.6r}$$

(f) Use the results in Appendix A to show that the Von Mises stress σ_e is related to the octahedral stress τ by

$$\sigma_e = \frac{3}{\sqrt{2}} \; \tau \quad , \tag{P5.6s}$$

and that a deviatoric state of torsion is characterized by

$$\sigma'_2 = 0 \quad . \tag{P5.6t}$$

Problem 5.7:

Consider two surfaces S and S' through the same point $\mathbf{x}$ in the present configuration and let $\mathbf{n}$ and $\mathbf{n}'$ be the normals to these surfaces, respectively. Recall that the stress vector $\mathbf{t}(\mathbf{n})$ acting on the surface shows outward normal is $\mathbf{n}$ is related to the symmetric Cauchy stress $\mathbf{T}$. Show that the component of $\mathbf{t}(\mathbf{n}')$ along the direction $\mathbf{n}$ is equal to the component of $\mathbf{t}(\mathbf{n})$ along the direction $\mathbf{n}'$

$$\mathbf{t}(\mathbf{n}) \bullet \mathbf{n}' = \mathbf{t}(\mathbf{n}') \bullet \mathbf{n} \quad . \tag{P5.7}$$

PROBLEM SET 6

Problem 6.1: Let the Cauchy stress **T** at a point be given by

$$\mathbf{T} = -\,p\,\mathbf{I} \quad . \tag{P6.1}$$

(a) Calculate the stress vector **t** acting on the surface defined by the unit normal **n**.

(b) Calculate normal component $\mathbf{t} \bullet \mathbf{n}$ of this stress vector.

(c) Calculate the shearing stress $\mathbf{t}_s = \mathbf{t} - (\mathbf{t} \bullet \mathbf{n})\,\mathbf{n}$ acting on this plane.

Problem 6.2: Let the Cartesian components T_{ij} of the Cauchy stress **T** referred to the base vectors $\mathbf{e}_i$ be given by

$$T_{ij} = -\,p\,\delta_{ij} + T'_{ij} \quad , \; T'_{ij} = \begin{pmatrix} \sigma'_1 & 0 & 0 \\ 0 & -(\sigma'_1+\sigma'_3) & 0 \\ 0 & 0 & \sigma'_3 \end{pmatrix} , \tag{P6.2a,b}$$

$$\sigma'_1 \geq \sigma'_3 \quad . \tag{P6.2c}$$

(a) Calculate the stress vector **t** acting on the octahedral plane defined by

$$\mathbf{n} = \frac{1}{\sqrt{3}}\,(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \quad . \tag{P6.2d}$$

(b) Show that the normal component $\mathbf{t} \bullet \mathbf{n}$ of this stress vector is given by

$$\mathbf{t} \bullet \mathbf{n} = -\,p \quad . \tag{P6.2e}$$

(c) Calculate the shearing stress $\mathbf{t}_s$ acting on this plane and show that

$$\mathbf{t}_s = \mathbf{T}'\,\mathbf{n} \quad . \tag{P6.2f}$$

(d) Calculate an expression for the Von Mises Stress σ_e in terms of σ'_1 and σ'_3.

(e) Show that the magnitude of $\mathbf{t}_s$ is given by

$$|\,\mathbf{t}_s\,| = \frac{\sqrt{2}}{3}\;\sigma_e \quad . \tag{P6.2g}$$

(f) Define the base vectors

$$\bar{\mathbf{e}}_1 = \frac{1}{\sqrt{6}}\,(-\,\mathbf{e}_1 + 2\,\mathbf{e}_2 - \mathbf{e}_3) \; , \; \bar{\mathbf{e}}_2 = \frac{1}{\sqrt{3}}\,(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \quad , \tag{P6.2h,i}$$

$$\bar{\mathbf{e}}_3 = \frac{1}{\sqrt{2}}\,(\mathbf{e}_1 - \mathbf{e}_3) \quad . \tag{P6.2j}$$

Show that the shearing stress vector $\mathbf{t}_s$ may be expressed in the form

$$\mathbf{t}_s = \frac{\sqrt{2}}{3} \; \sigma_e \, (\cos\beta \, \bar{\mathbf{e}}_3 + \sin\beta \, \bar{\mathbf{e}}_1) \, , \tag{P6.2k}$$

and derive expressions for cosβ and sinβ in terms of σ'_1 and σ'_3. Note that β defines the angle that the shear stress $\mathbf{t}_s$ makes with the direction $\bar{\mathbf{e}}_3$

(g) By substituting (A12b,d) into your results of part (f) show that β in (P6.2k) is the same as β in (A12a).

Problem 6.3: Consider uniaxial strain in the $\mathbf{e}_1$ direction which is given by

$$x_1 = x_1(X_1,t) \;\; , \;\; x_2 = X_2 \;\; , \;\; x_3 = X_3 \;\; . \tag{P6.3a,b,c}$$

Letting $a=\partial x_1/\partial X_1$, show that for uniaxial strain the components T_{11} of the Cauchy stress, Π_{11} of the nonsymmetric Piola-Kirchhoff stress, and S_{11} of the symmetric Piola-Kirchhoff stress are related by

$$T_{11} = \Pi_{11} = a \, S_{11} \;\; . \tag{P6.3d}$$

Problem 6.4: Consider the deformation given by

$$x_1 = a \, X_1 \;\; , \;\; x_2 = b \, X_2 \;\; , \;\; x_3 = c \, X_3 \;\; , \tag{P6.4a,b,c}$$

where a,b, and c are constants. Assuming that the components of the Cauchy stress T_{ij} are restricted such that

$$T_{12} = T_{13} = T_{23} = 0 \;\; , \tag{P6.4d}$$

determine expressions for the components Π_{iA} and S_{AB} of the Piola-Kirchhoff stresses $\mathbf{\Pi}$ and $\mathbf{S}$.

Problem 6.5: Show that under superposed rigid body motions the shearing component $\mathbf{t}_s$ of the stress vector $\mathbf{t}$ transforms by

$$\mathbf{t}_s^+ = \mathbf{Q} \, \mathbf{t}_s \;\; . \tag{P6.5b}$$

Problem 6.6: Starting with the assumption (20.16) prove the result (20.17). Be sure to carefully state the important points in your proof.

PROBLEM SET 7

Problem 7.1: Using the relationship (18.18a) between the Cauchy stress **T** and the symmetric Piola-Kirchhoff stress **S**, and the invariance relations (19.16b) prove that **S** is trivially invariant (19.17c).

Problem 7.2: Recall that the material derivative of a scalar function $\phi(\mathbf{x},t)$ may be expressed in the form

$$\dot{\phi} = \frac{\partial\phi}{\partial t} + \partial\phi/\partial\mathbf{x} \bullet \mathbf{v} \ . \tag{P7.2a}$$

Also recall that under superposed rigid body motions the material point that is located at position $\mathbf{x}$ at time t moves to the position $\mathbf{x}^+$ at time t^+, such that

$$\mathbf{x}^+ = \mathbf{x}^+(\mathbf{x},t) = \mathbf{c}(t) + \mathbf{Q}(t)\,\mathbf{x} \ , \ t^+ = t^+(t) = t + a \ , \tag{P7.2b,c}$$

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{I} \ , \ \det\mathbf{Q} = +1 \ , \ \dot{\mathbf{Q}} = \boldsymbol{\Omega}\,\mathbf{Q} \ , \tag{P7.2d,e,f}$$

$$\mathbf{v}^+ = \dot{\mathbf{x}}^+ = \dot{\mathbf{c}} + \boldsymbol{\Omega}\,\mathbf{Q}\,\mathbf{x} + \mathbf{Q}\,\mathbf{v} \ , \tag{P7.2g}$$

where a is a constant and **c** and **Q** are functions of time only. Consider the function $\tilde{\phi}^+(\mathbf{x}^+,t^+)$ and think of it as a function of $\mathbf{x}$,t such that

$$\phi^+ = \tilde{\phi}^+(\mathbf{x}^+,t^+) = \tilde{\phi}^+\big(\mathbf{x}^+(\mathbf{x},t)\ ,\ t^+(t)\big) = \hat{\phi}^+(\mathbf{x},t) \ . \tag{P7.2h}$$

Calculate the partial derivatives $\partial\hat{\phi}^+/\partial t$ and $\partial\hat{\phi}^+/\partial\mathbf{x}$ in terms of the function $\tilde{\phi}^+$ and show that the material derivative of ϕ^+ may be expressed in the form

$$\dot{\tilde{\phi}}^+ = \frac{\partial\tilde{\phi}^+}{\partial t^+} + \partial\tilde{\phi}^+/\partial\mathbf{x}^+ \bullet \mathbf{v}^+ \ . \tag{P7.2i}$$

Problem 7.3: Using the linearized form of (23.8e) for **M** and the equation $\mathbf{C}=\mathbf{M}^2$ derive the linerized form (23.8c) for **C**.

Problem 7.4: Prove that for the linearized theory **R** given by (23.8g) is an orthogonal tensor.

Problem 7.5: Recalling that $\mathbf{C}'=\mathrm{I}_3^{-1/3}\mathbf{C}$ is a pure measure of distortion (det $\mathbf{C}'$=1) we may define a pure measure of distortional strain $\mathbf{G}'$ by

$$\mathbf{G}' = \frac{1}{2}(\mathbf{C}' - \mathbf{I}) \quad . \qquad \text{(P7.5a)}$$

Using (23.8c) and (23.13b) show that the linearized form of $\mathbf{G}'$ is given by

$$\mathbf{G}' = \boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon} - \frac{1}{3}(\boldsymbol{\varepsilon} \bullet \mathbf{I})\ \mathbf{I} \ , \qquad \text{(P7.5b)}$$

where here $\boldsymbol{\varepsilon}'$ is the defined as the deviatoric part of the linear strain $\boldsymbol{\varepsilon}$.

Problem 7.6: Taking the inner product of (18.5a) with the velocity $\mathbf{v}$, follow the derivation in section 24 and show that for the referential description the strain energy Σ is related to the mechanical power by

$$\rho_0 \dot{\Sigma} = \boldsymbol{\Pi} \bullet \dot{\mathbf{F}} \quad . \qquad \text{(P7.6)}$$

Problem 7.7: Assuming that for an elastic material the strain energy Σ and the stress $\boldsymbol{\Pi}$ depend on the deformation gradient $\mathbf{F}$ and are independent of the rate $\dot{\mathbf{F}}$, use (P7.6) to derive the result that

$$\boldsymbol{\Pi} = \rho_0 \frac{\partial \Sigma}{\partial \mathbf{F}} \quad . \qquad \text{(P7.7)}$$

Is this form of $\boldsymbol{\Pi}$ automatically properly invariant under superposed rigid body motions?

Problem 7.8: Given a nonlinear elastic material characterized by the strain energy function

$$2\rho_0\Sigma = 2\,k_0\,[\,(\mathrm{J} - 1) - \ln \mathrm{J}\,] + \mu_0\,(\alpha_1 - 3) \ , \qquad \text{(P7.8a)}$$

$$\mathrm{J} = \det \mathbf{F} = \mathrm{I}_3^{1/2} \ , \ \ \alpha_1 = \mathrm{J}^{-2/3}\,\mathbf{C} \bullet \mathbf{I} \ , \qquad \text{(P7.8b,c)}$$

where k_0 and μ_0 are constants. Show that pressure p and the deviatoric Cauchy stress $\mathbf{T}'$ are given by

$$\mathrm{p} = k_0\,(\frac{1}{\mathrm{J}} - 1) \ , \ \ \mathbf{T}' = \mu_0 \mathrm{J}^{-1}\,[\,\mathbf{B}' - \frac{1}{3}(\mathbf{B}' \bullet \mathbf{I})\ \mathbf{I}\,] \ , \qquad \text{(P7.8d,e)}$$

$$\mathbf{B}' = J^{-2/3}\,\mathbf{B} = J^{-2/3}\,\mathbf{F}\mathbf{F}^T \quad . \tag{P7.8f}$$

Problem 7.9: Consider simple shear

$$x_1 = X_1 + \gamma X_2 \ , \ x_2 = X_2 \ , \ x_3 = X_3 \ , \tag{P7.9a,b,c}$$

where γ is a function of time only, and show that the stresses associated with the nonlinear elastic material of problem 7.8 become

$$p = 0 \ , \ T_{12} = \mu_0 \gamma \ , \tag{P7.9d,e}$$

$$T_{11} = \frac{2}{3}\ \mu_0 \gamma^2 \ , \ T_{22} = T_{33} = -\frac{1}{3}\ \mu_0\, \gamma^2 \ . \tag{P7.9f,g}$$

Notice that the normal stresses are quadratic function of the shear γ whereas the shear stress is a linear function of γ.

Problem 7.10: Consider simple shear (P7.9a,b,c) of the Reiner-Rivlin fluid characterized by (29.15) and show that the stresses become

$$p = -\frac{1}{3}\ (3d_0 + \frac{1}{2}\, d_2\, \dot{\gamma}^2) \ , \ T'_{12} = \frac{1}{2}\ d_1\, \dot{\gamma} \ , \tag{P7.10a,b}$$

$$T'_{11} = \frac{1}{12}\ d_2\, \dot{\gamma}^2 \ , \ T'_{22} = \frac{1}{12}\ d_2\, \dot{\gamma}^2 \ , T'_{33} = -\frac{2}{12}\ d_2\, \dot{\gamma}^2 \ , \tag{P7.10c,d,e}$$

where p is the pressure and T'_{ij} are the components of the deviatoric stress $\mathbf{T}'$. Notice that the normal components of stress are quadratic function of the shearing rate $\dot{\gamma}$ whereas the shear stress is a linear function of $\dot{\gamma}$.

Problem 7.11: An attempt is made to develop a constitutive equation for an anisotropic viscous fluid by assuming that the Cauchy stress T_{ij} is related to J=det $\mathbf{F}$ and the rate of deformation D_{ij} by the constitutive equation

$$T_{ij} = \hat{T}_{ij}(J, D_{mn}) = \hat{A}_{ij}(J) + \hat{A}_{ijmn}(J)\ D_{mn} \ , \tag{P7.11a}$$

where A_{ij} and A_{ijkl} are tensor functions of J only which have the symmetries

$$A_{ij} = A_{ji} \ , \ A_{ijmn} = A_{jimn} = A_{ijnm} \ . \tag{P7.11b,c}$$

By requiring that the constitutive equation (P7.11a) be properly invariant under superposed rigid body motions prove that A_{ij} and A_{ijmn} must be isotropic tensors of the forms

$$A_{ij} = - p_1(J)\, \delta_{ij} \ , \quad A_{ijmn} = \lambda(J)\, \delta_{ij}\delta_{mn} + \mu(J)\, (\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) \ , \qquad \text{(P7.11d,e)}$$

so that the Cauchy stress **T** must reduce to the form

$$\mathbf{T} = - p_1\, \mathbf{I} + \lambda\, (\mathbf{D} \bullet \mathbf{I})\, \mathbf{I} + 2\mu\, \mathbf{D} \ . \qquad \text{(P7.11f)}$$

Note that since **T** in (P7.11f) is an isotropic function of its arguments the proposed form (P7.11a) did not work because the resulting fluid response cannot be anisotropic.