עיבוד אותות – רעידות ודיאגנוסטיקה (035039)

- Introduction
  - Areas of Applications –
    - General
    - Mechanical Engineering
- Types of general problems
- Types of signals

DSP in various disciplines

- Communication (Electronics, speech)
- EE – home appliances (TV, Music)
- Image processing
- Control
- Civil Engineering –Earthquakes, ground motions
- Aeronautical Engineering
- Medicine (EEG, ECG etc)
- Physics (Optical devices)
- Economics, Finance (Stock market)
- and many more

DSP in mechanical Engineering

- Control
- Measurements
- Manufacturing
- Diagnostics
- Design (dynamic aspects)
- Products (Mechatronics)
- Vibration (noise) control
- and many others

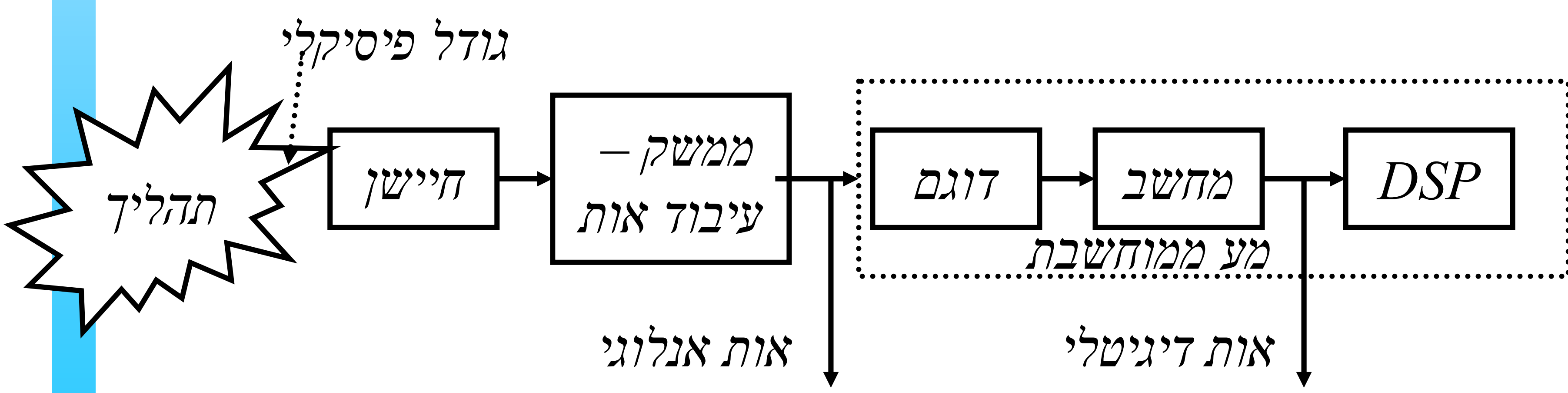
Signals of interest

Physical phenomena

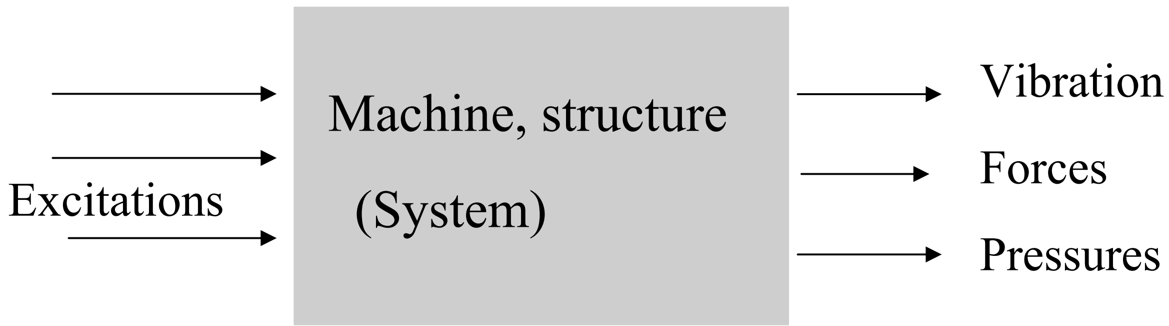
→ measurement system

→ amplitude versus time function

- Phenomena of interest:
  - Displacement, velocity, acceleration, force, angle, temperature
- “Time” axis
  - Relative time, spatial location, angular position



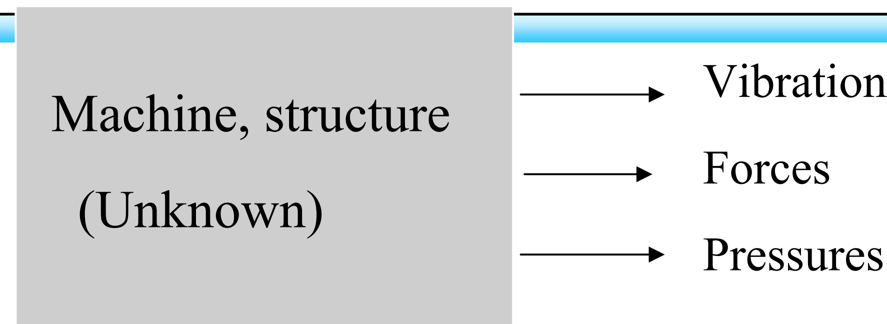
Black Box Approach



- *Three types of problems:*

1. *Monitoring*
2. *System Identification*
3. *Excitation estimation*

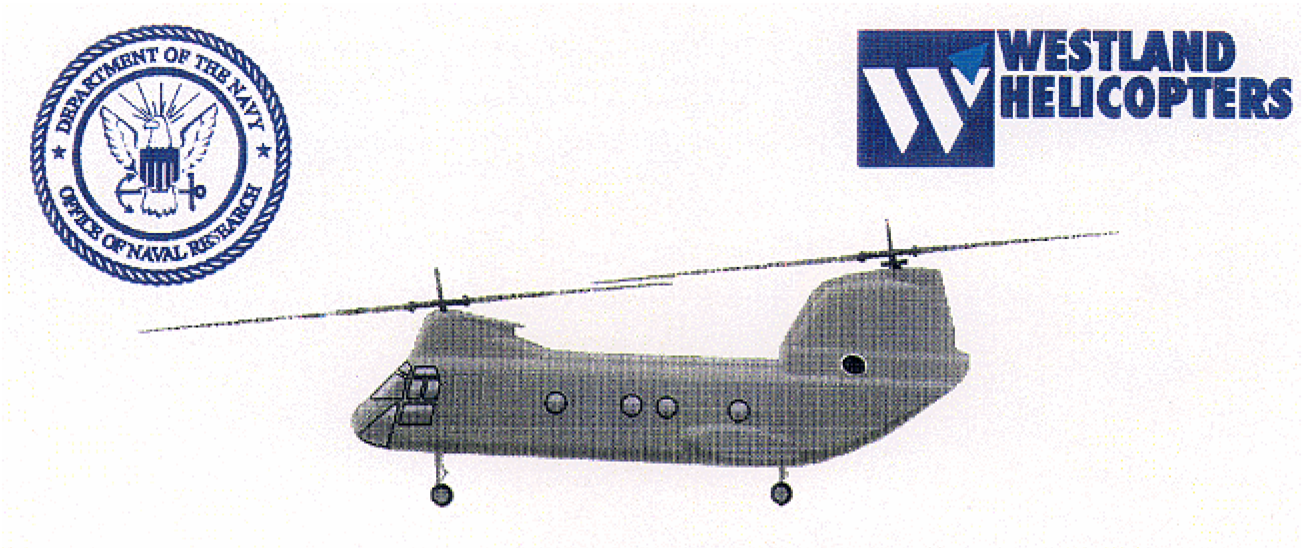
Types of problems –Monitoring



- *Monitoring:*
  - *Surveillance*
  - *Diagnostics*
  - *HUMS (Health Usage Monitoring Systems*
    - *Helicopter HUMS (Gear-Boxes, Jet Engines)*
  - *Tool Condition Monitoring (State of wear)*
  - *Automatic Monitoring of Manufacturing Processes*
  - *Structural Fault Detection and Localization*

The Westland Database

The US Navy funded Westland Helicopters LTD
to carry out a series of tests
on the CH-46E helicopter transmission
for generating a database
to evaluate Diagnostics tools in general
& Neural Networks in particular



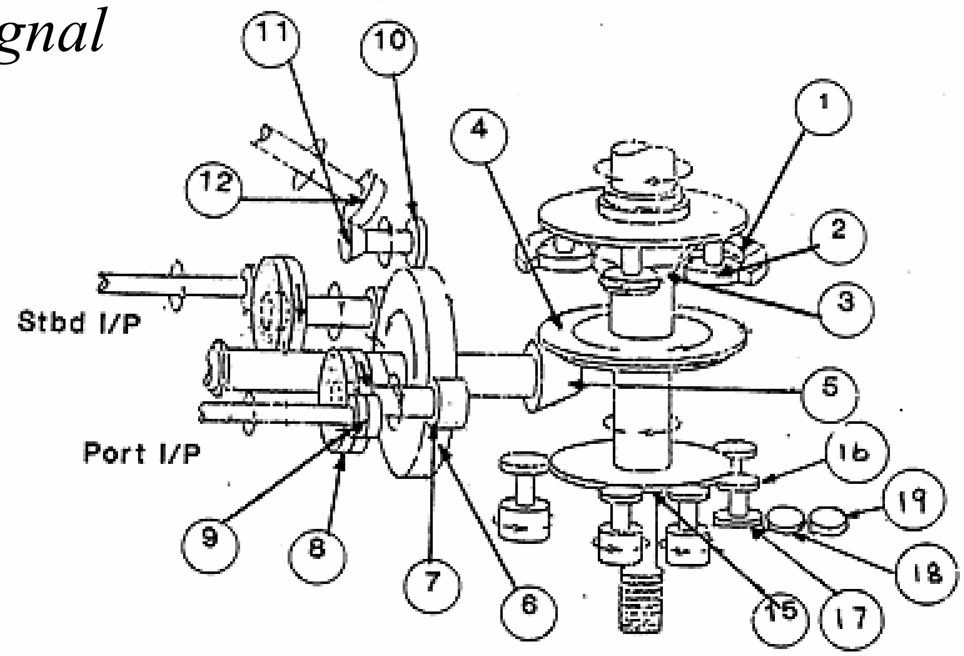
The Westland Fault Testing

System:

Main power Transmission
of a US Navy CH-46E helicopter

Measurements:

8 accelerometers recorded simultaneously
Tachometer Signal

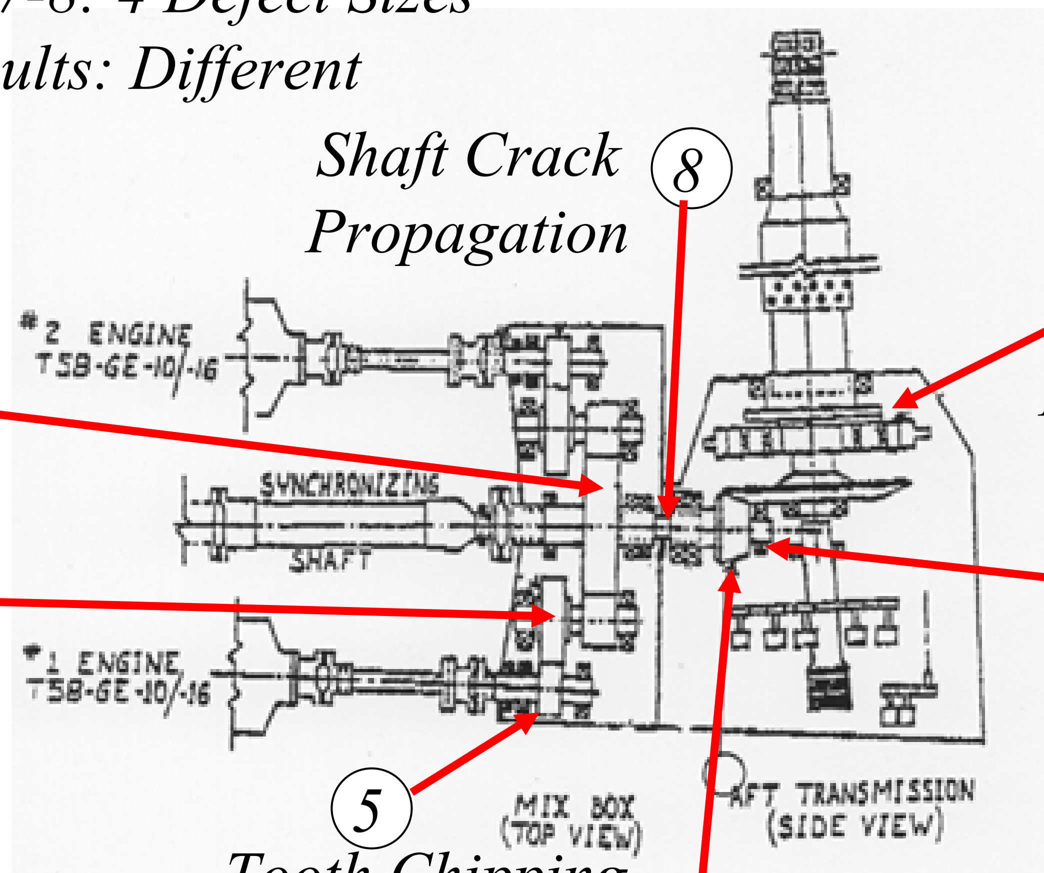


Fault Types & Test Conditions

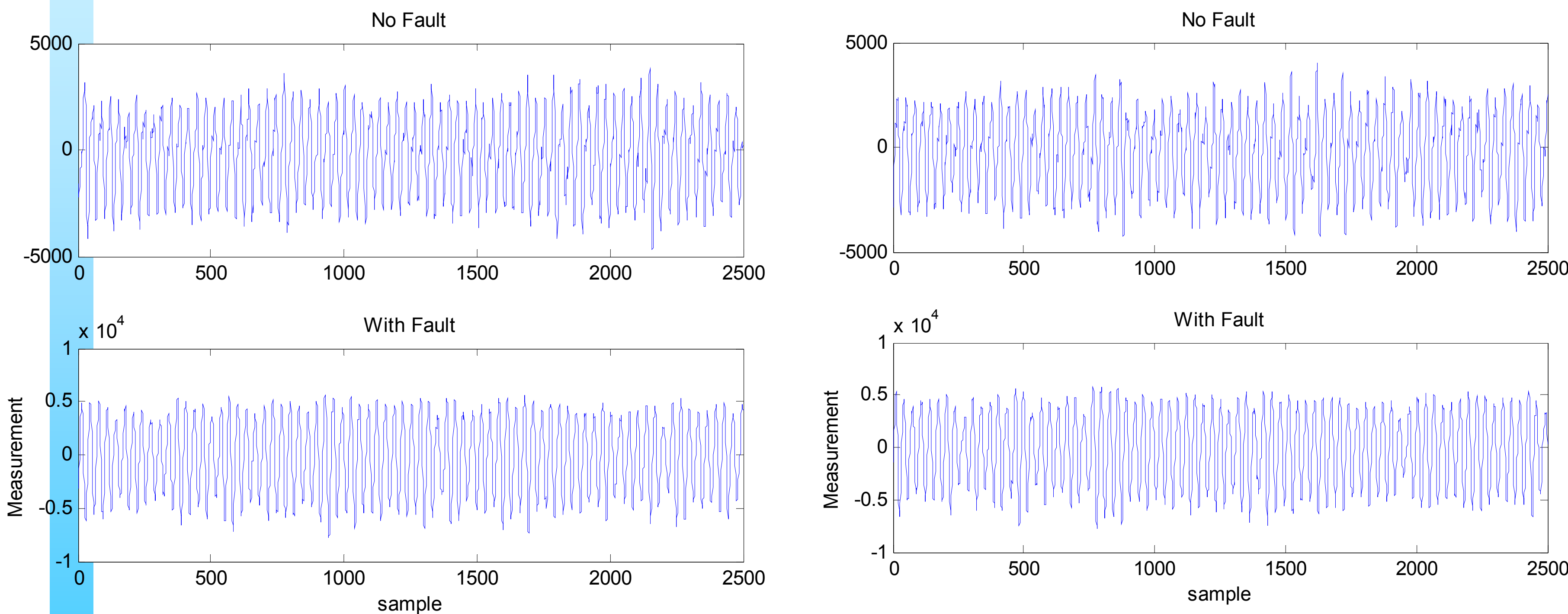
Faults 2-5: 2 Levels

Fault 7-8: 4 Defect Sizes

All Faults: Different Load levels



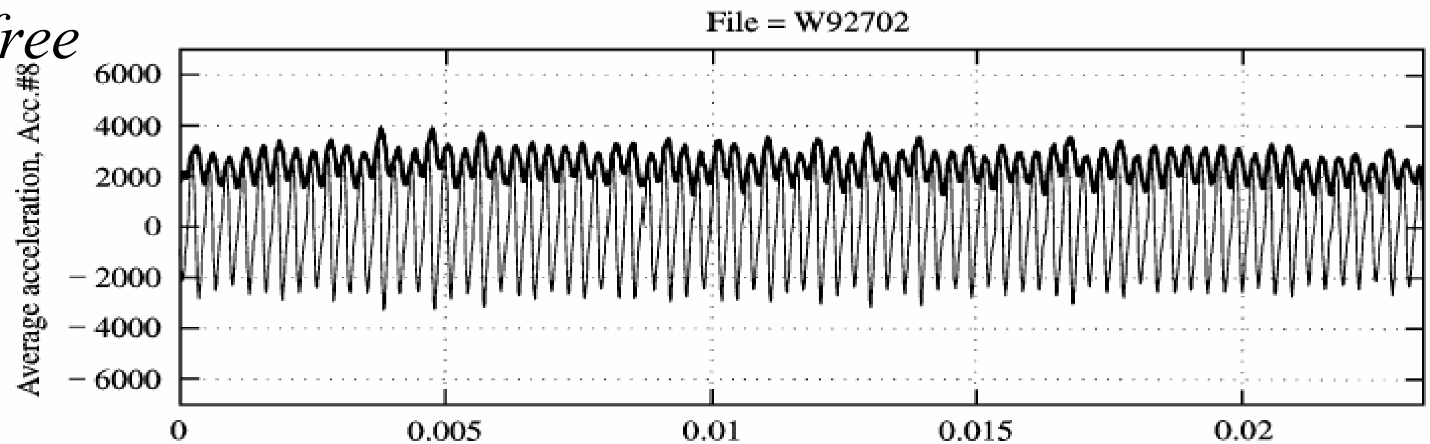
Example of accelerometer records under Normal and Faulty conditions



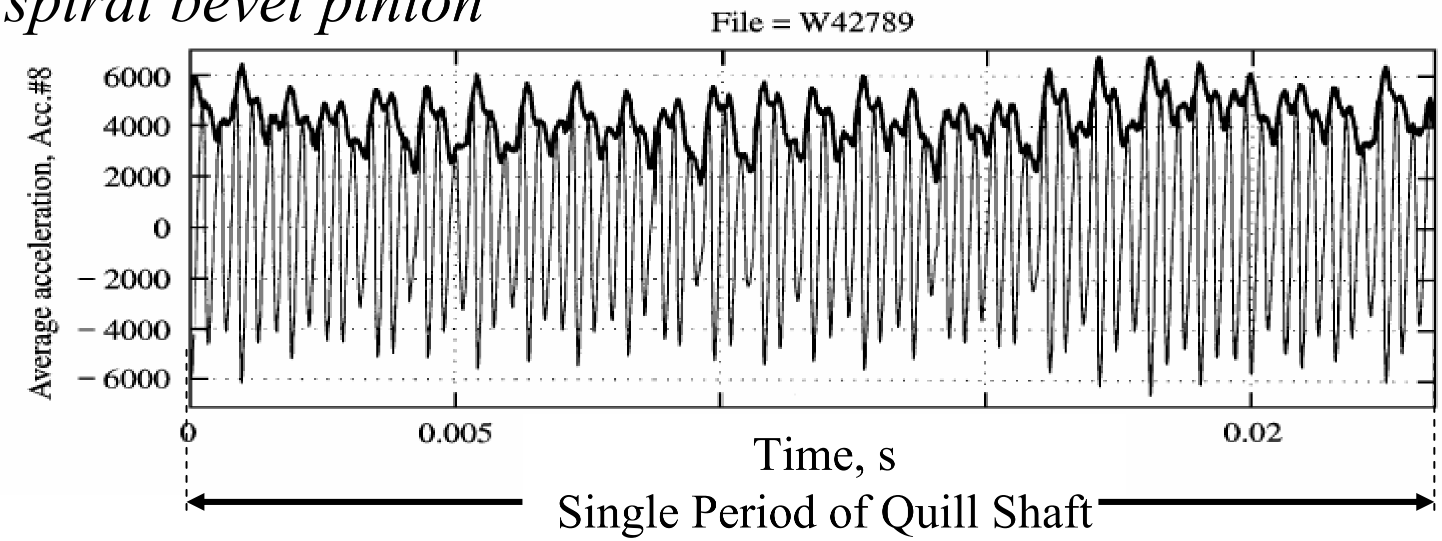
Cross-gear pair modulation

Time-Averaged Signals

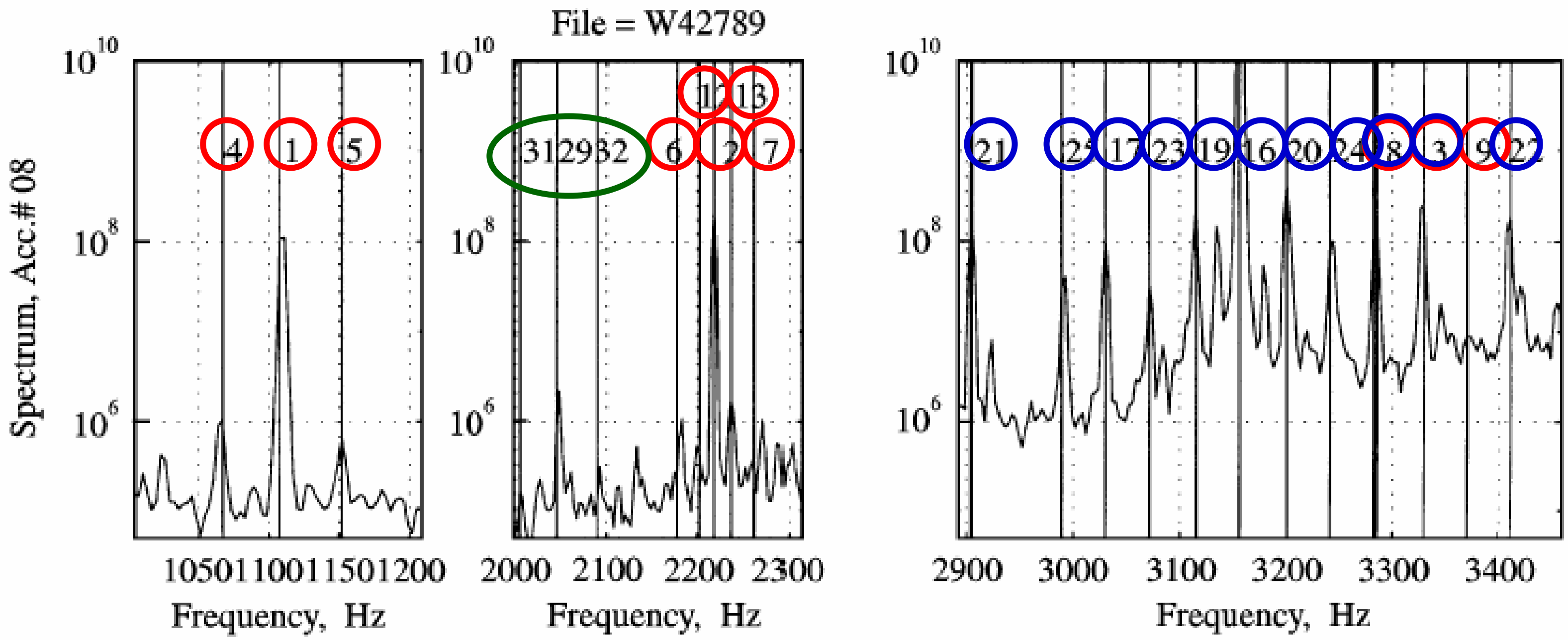
Fault free



Faulty spiral bevel pinion



Signal Generation Model
Predicted Spectral Lines



Vibrations induced by spiral bevel gear tooth meshing (1-15)

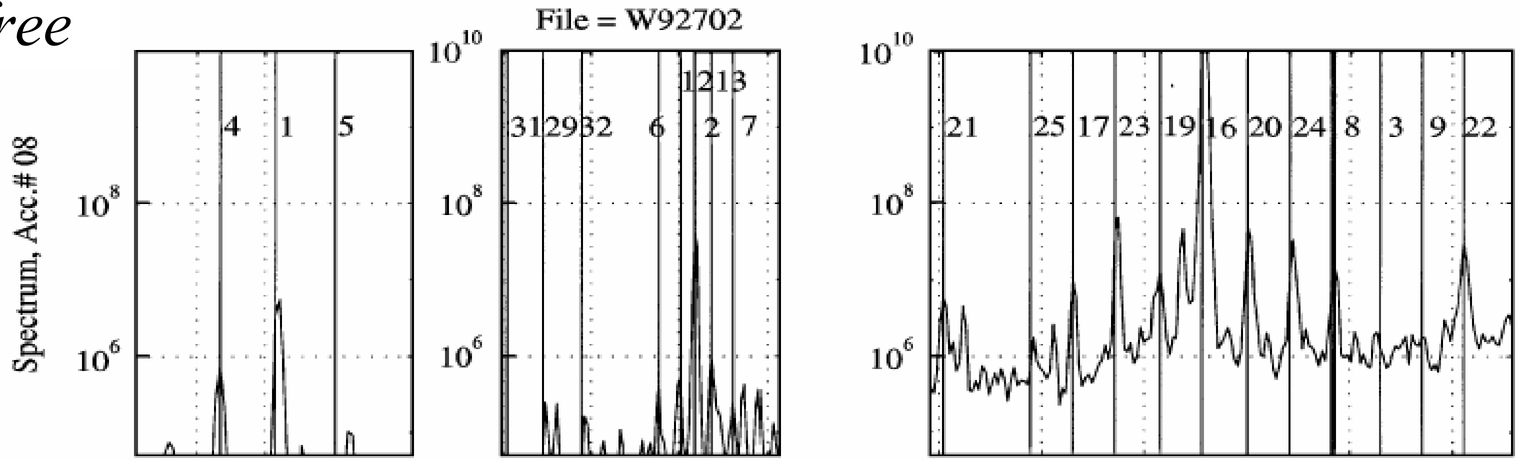
Vibrations induced by the collector gear tooth meshing (16-28)

Vibrations induced by cross gear pair interaction (29-32)

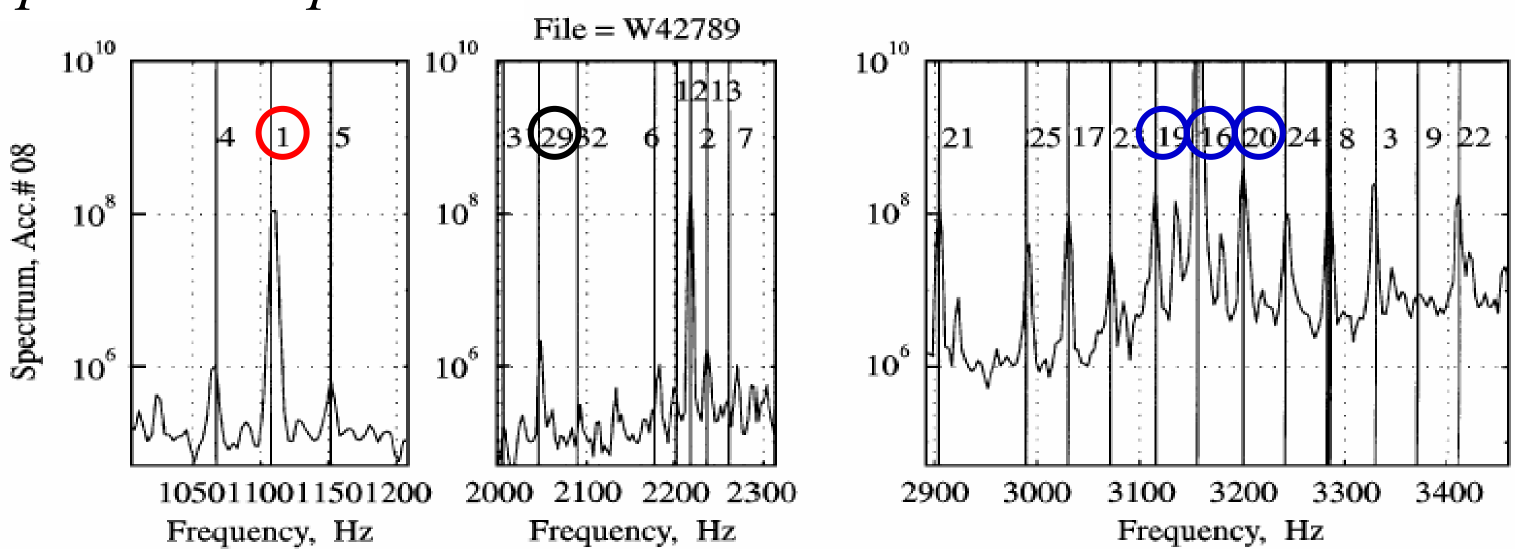
Effect of Fault on spectral lines

(Faulty spiral bevel pinion)

Fault free

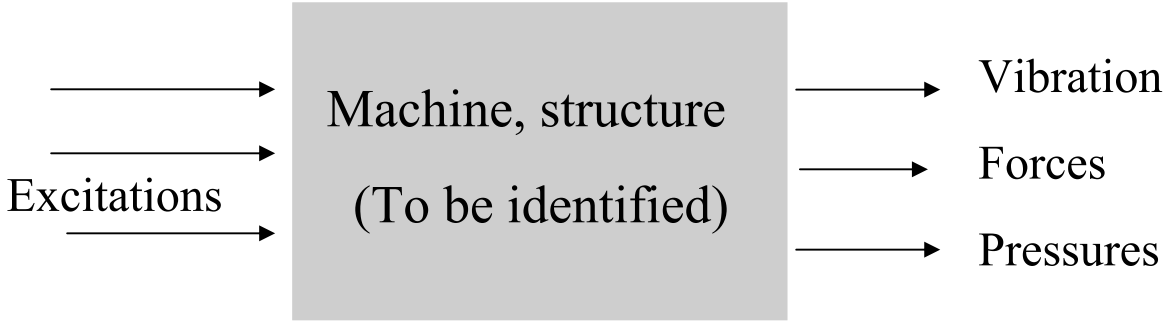


Faulty spiral bevel pinion

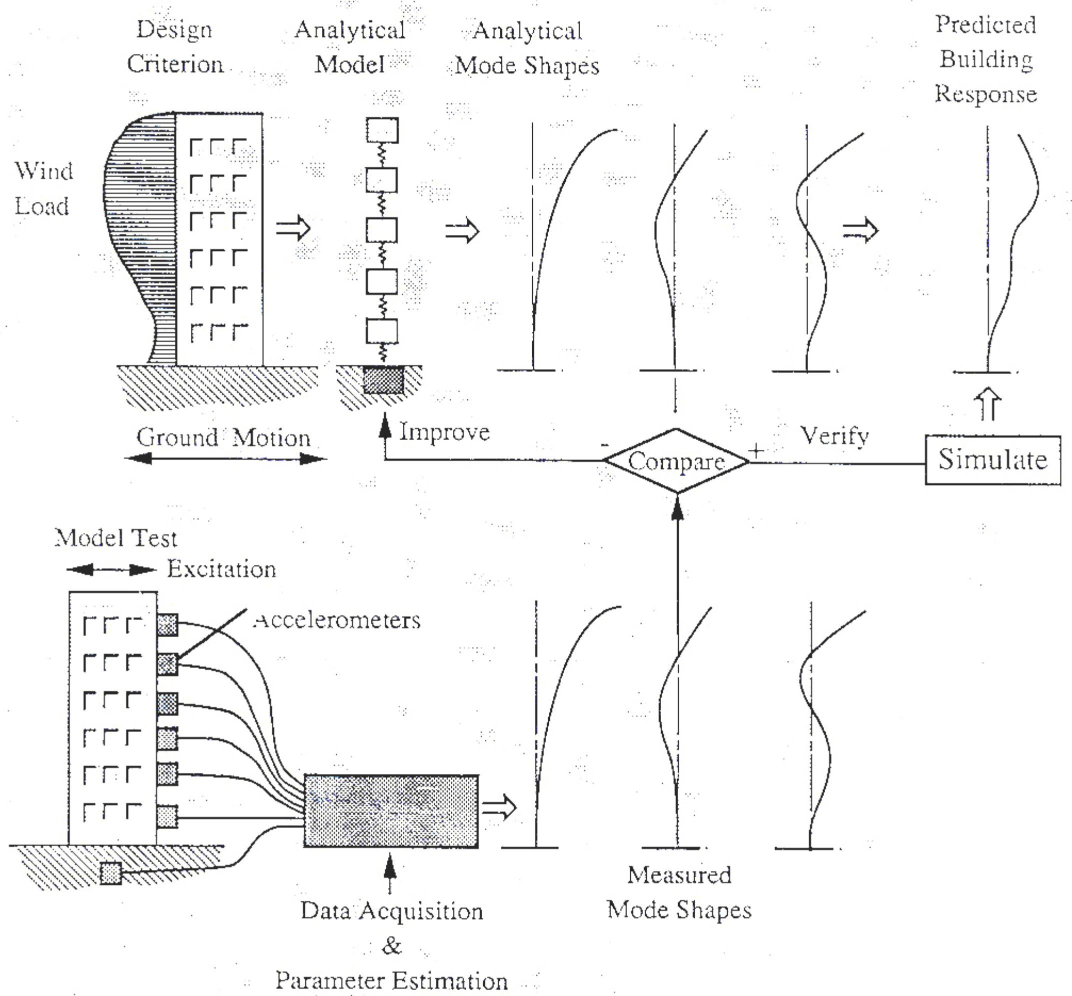


Types of problems – Identification

- *Identification*
  - Experimental Modal Analysis
  - Control of systems



Modal Analysis



Control of Systems

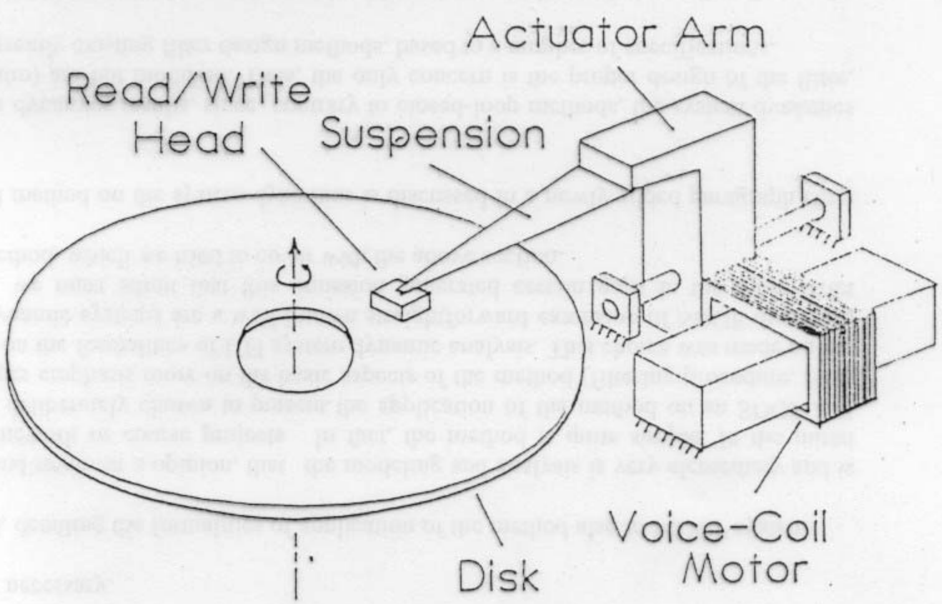
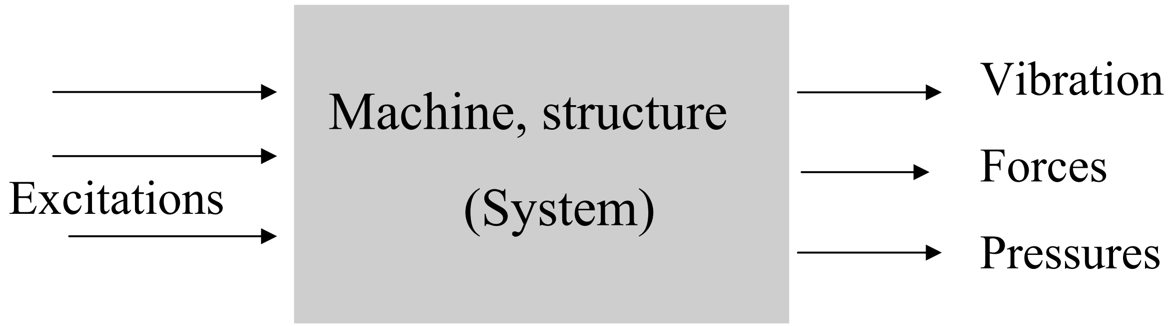


Fig. 1 Relevant mechanical components of small form factor rigid disk drives

JANUARY 1990, Vol. 112 / 33

Types of problems – Characterization of excitation

- *Estimating the exciting forces, accelerations*
- Identification of acoustical sources

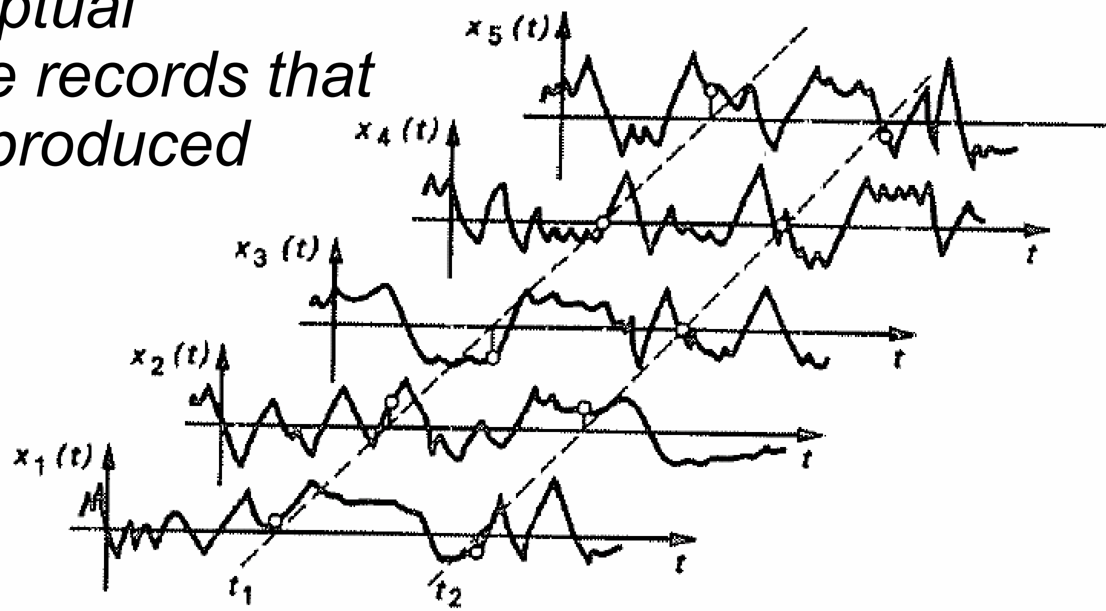


Deterministic versus Stochastic Signals

- *Deterministic signal: predictable (with reasonable accuracy) based on*
  - *The physics of the system*
  - *Previous measurements*
  - *Examples*
    - *Force of unbalanced wheel*
    - *Orbit of satellite*
- *Stochastic signal: a unique time history record which is not likely to be repeated and cannot be accurately predicted in detail.*

Stochastic data and Stochastic processes

- *Random signal: one possible realization of what might have occurred*
- *Stochastic process: Conceptual process that generated the different realizations*
- *Ensemble: Conceptual collection of all the records that might have been produced*



Random Signal Analysis

- *Random Signal Analysis: Characterize the random signals*
  - *Temporal domain - Statistical properties*
    - *E.g., probability that the acceleration at time t0 is within some limits*
  - *Frequency domain – Spectral properties*
    - *E.g., power in some frequency range*
- *Random Vibration Analysis: How the statistical characteristics of the motions of a randomly excited system depend on the statistics of the excitation and the dynamic properties of the vibrating system*

Ensemble Statistics

- Given an ensemble of records $\{x_i(t)\}$
- Ensemble statistics:
  - Ensemble average (mean) at time t1:

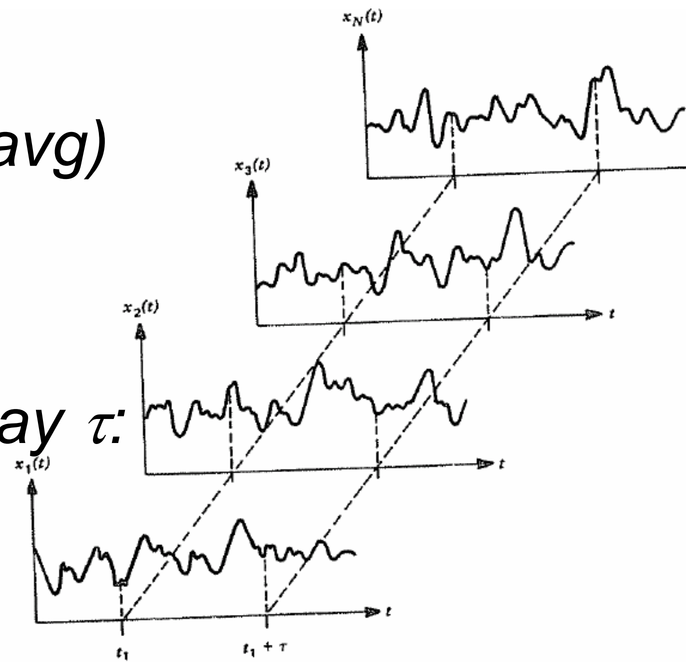
$$\mu_x(t_1) = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} x_i(t_1)$$

  - Mean square (and higher avg)

$$\mu_x(t_1) = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} x_i^2(t_1)$$

  - Autocorrelation at time delay τ:

$$R_{xx}(t_1,\tau) = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} x_i(t_1)x_i(t_1+\tau)$$



Stationary Data (Wide Sense)

- *Stationary (Wide sense, WSS):*
  - *Ensemble Mean is constant*
  - *Ensemble autocorrelation depends only on time delay τ*
- *Stationary: All averages and correlations of interest remain constant with changes in the time t1*
- *Stationary (Strict sense, SSS): The pdf at tany given time t1, evaluated across the ensemble, does not depend on the time t1.*

Stationary in practice

- *Stationary refers to the statistics not the records!!!*
- *Stationary >> the ensemble mean does not vary with time*
  - *Never holds strictly since actual data are limited in time!!!*
  - *In practice: a data is considered stationary if it is for the majority of its interval*
  - *Limited data implies that averages can only be estimated*
  - *Finite sampling results in computational errors !!*

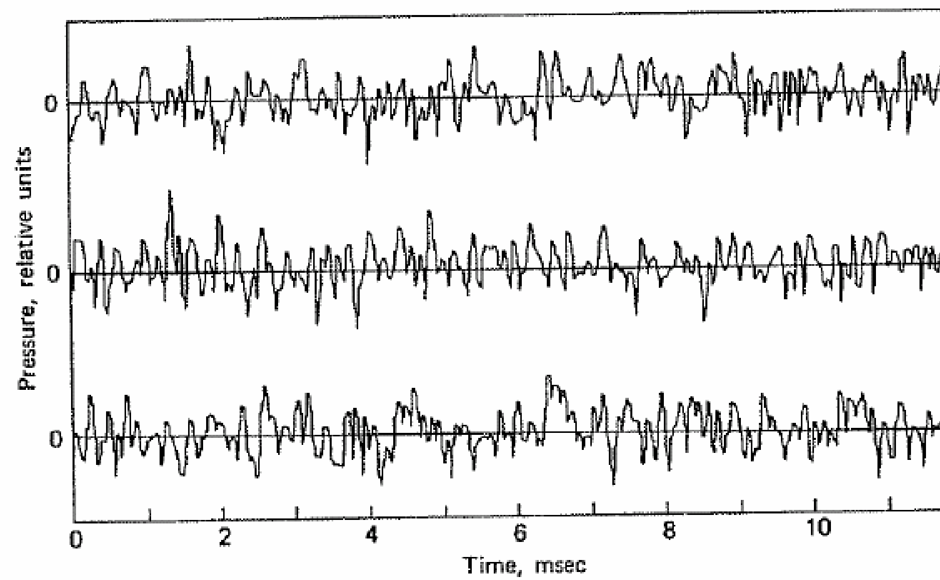
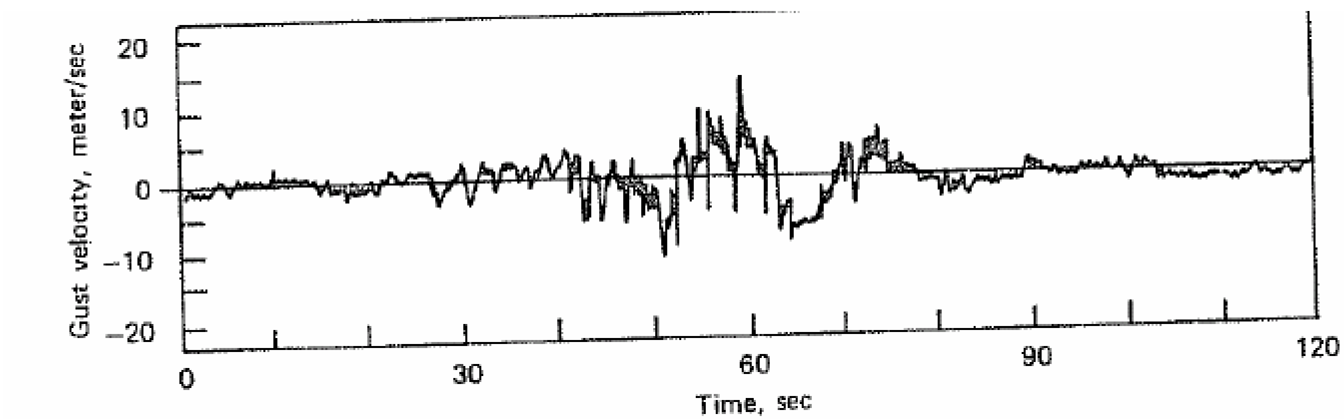


Figure 1.2 Ensemble of pipe-flow-pressure time history records

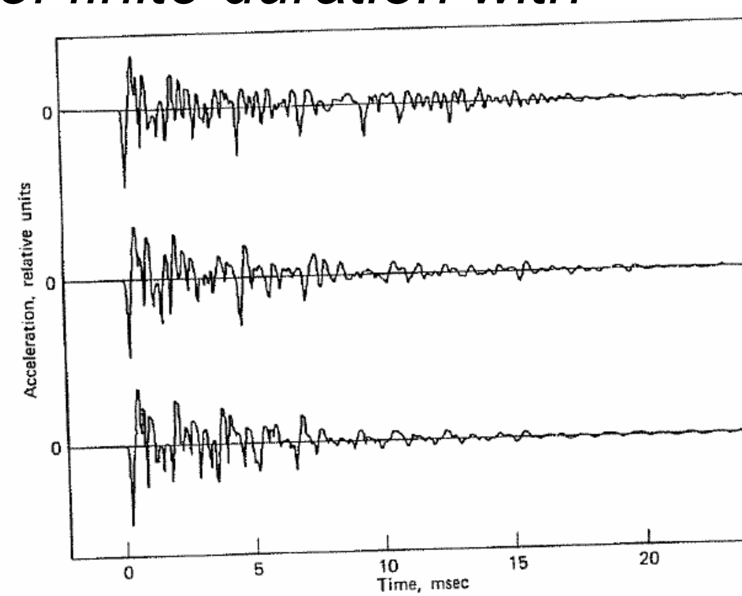
Non-Stationary Signals

- *Mechanisms producing the data depend on time (but can be repeated to get ensemble)*
  - *Shock, vibrations during take off or landing*
- *Parameters of mechanisms not under control*
  - *Acts of nature: Wind gust velocity, waves height;*
  - *Biological phenomenon: tremors*



Other practical considerations

- Stationary data: enhanced by maintaining constant experimental conditions
- Quasi-stationary: variations with time are slow, so the signal can be divided into long stationary intervals
  - Long compared to characteristic correlation delays
- Non-stationary data that can also be analyzed with similar tools: transient signals of finite duration with clear beginning and end



Ergodic Data

- *Ergodic data: ensembles averages equal (appropriate) time averages over individual records*
- *Temporal statistics:*
  - *Temporal average (mean):* $$\mu_x = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t) dt$$
  - *Higher order average values …*
  - *Autocorrelation at delay τ:* $$R_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t) x(t+\tau)$$
- *Ergodic condition: Stationary data is ergodic if (sufficient but not necessary)*

$$\frac{1}{T} \int_{-T}^{T} \left| R_{xx}(\tau) - \mu_x^2 \right| d\tau \underset{T \to \infty}{\longrightarrow} 0$$

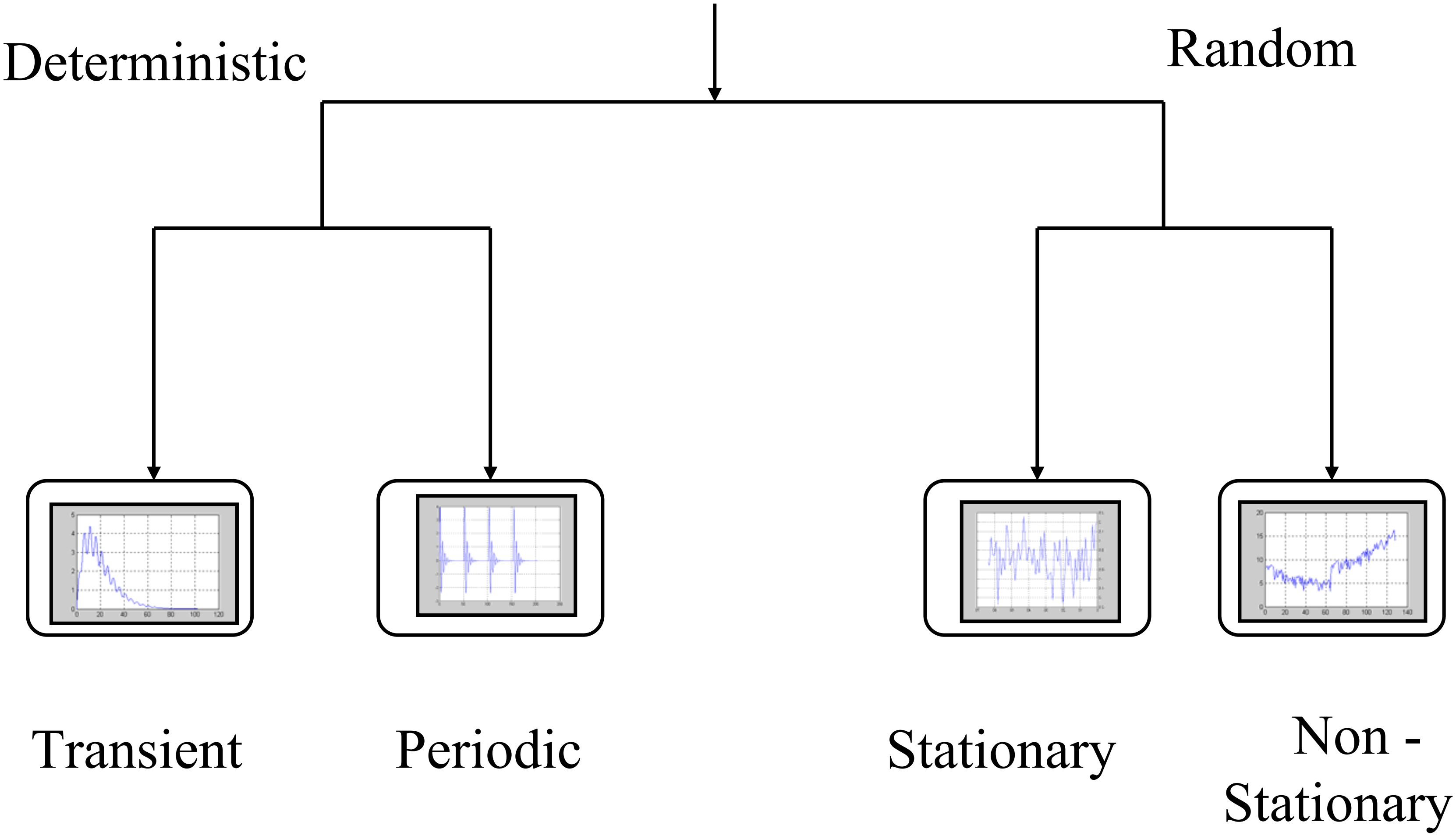
SIGNALS

The characterization as well as analysis methods depends on the signal structure.
The following are some classification possibilities.

Deterministic vs. random
Transient vs. continuous
Stationary vs. nonstationary

In practice we often encounter combinations of signal types
An example would be a harmonic signal contaminated by random noise.

Classification of Signals



Descriptions

Transient signals - *energy*

This is defined as

$$E = \int_{0}^{T} x^{2}(t)\,dt$$

where T is the signal duration.

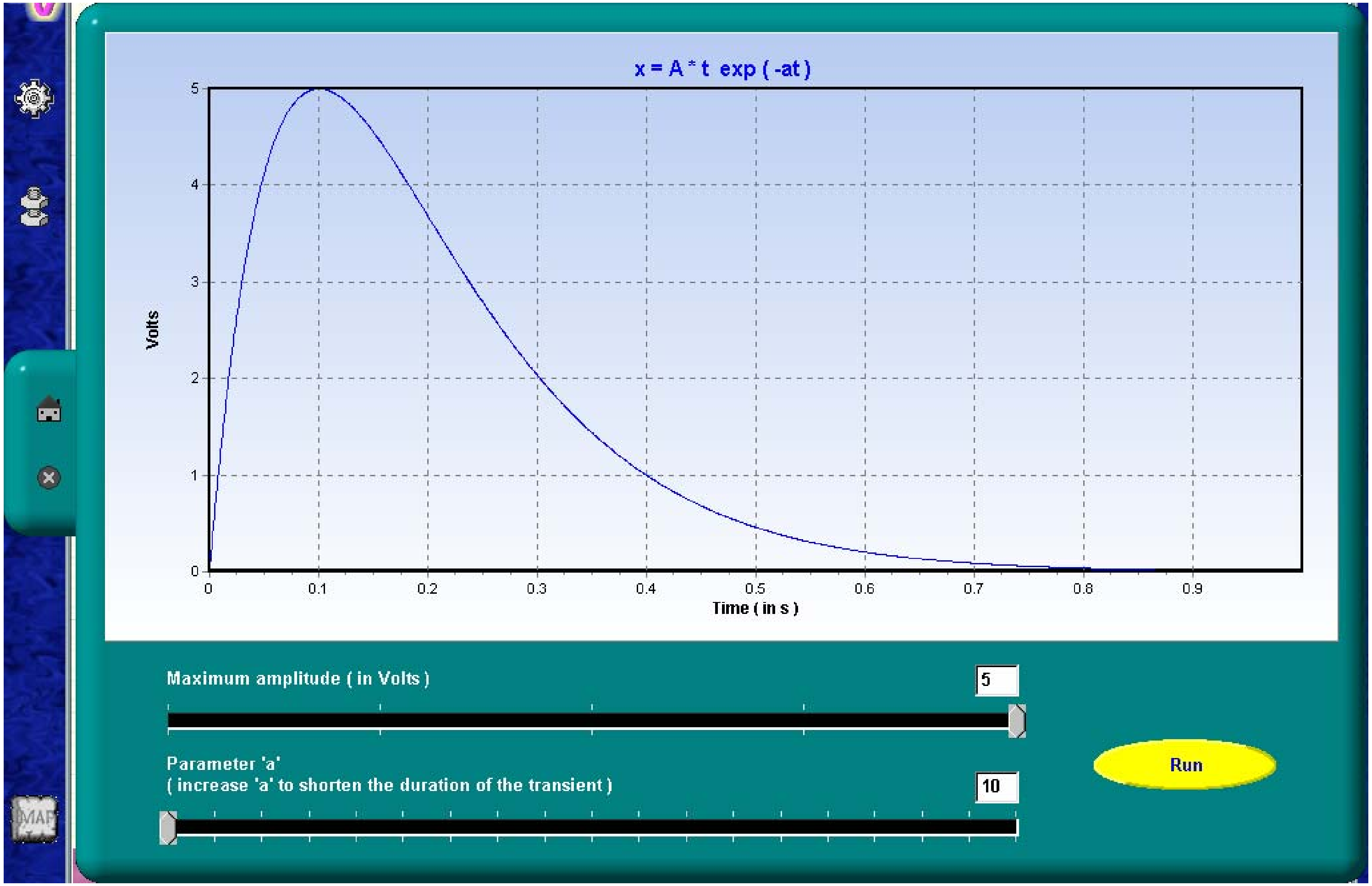
The energy is finite for a signal limited within an interval T.

The energy is finite for a signal limited within an interval T.

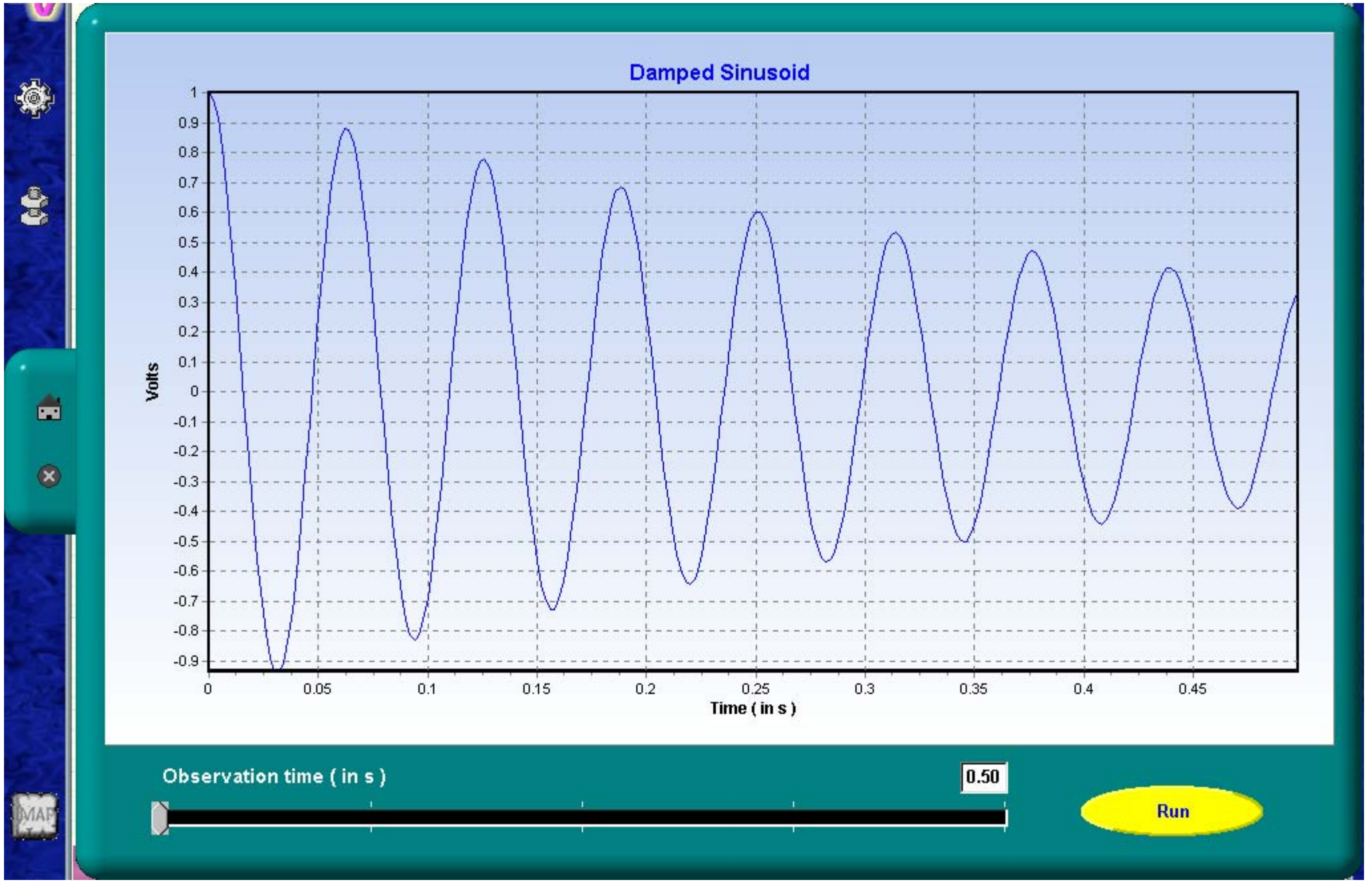
The units are

E [V^2 - sec, G^2 - sec...]

and such a signal is also called an energy signal.



x = A * t exp (-at)
Volts
Time (in s)
Maximum amplitude (in Volts)
5
Parameter 'a'
(increase 'a' to shorten the duration of the transient)
10
Run



Damped Sinusoid
Volts
Time (in s)
Observation time (in s)
0.50
Run
MAP

Continuous signals - *power*

For such a signal

$E \to \infty$

as

$T \to \infty$

and power can be used instead of energy.

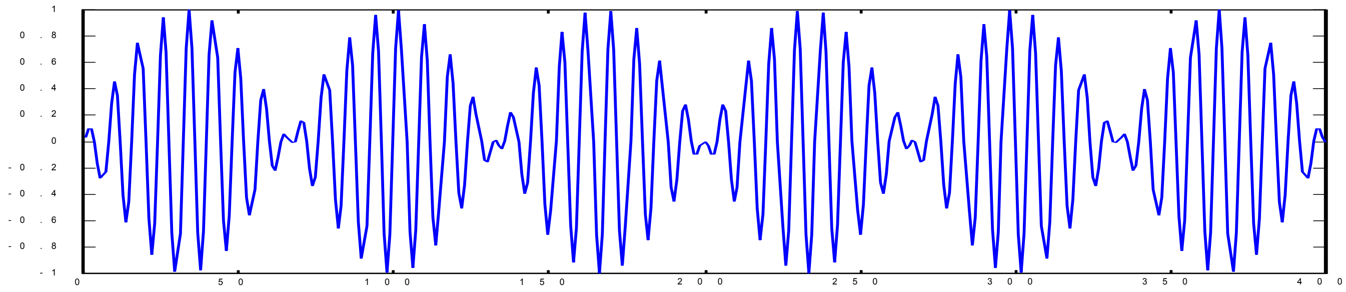
$$P = \frac{1}{T}\int_0^T x^2(t)\,dt$$

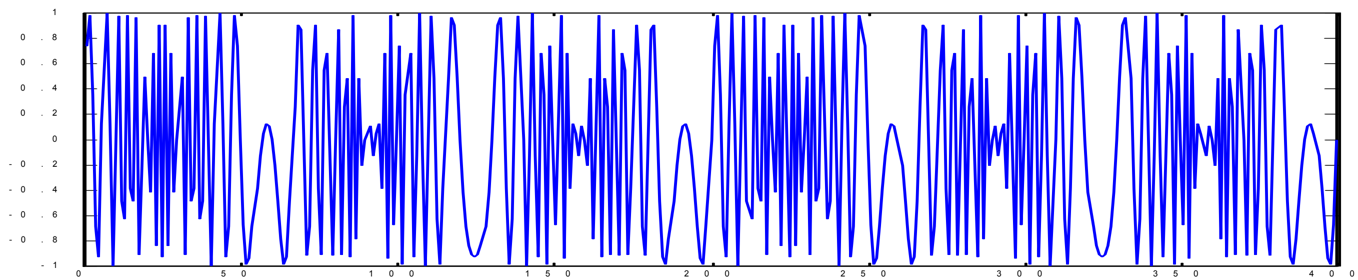
and P exists for $T \to \infty$ (but not E). The units are

P [V^2 , G^2 ...]

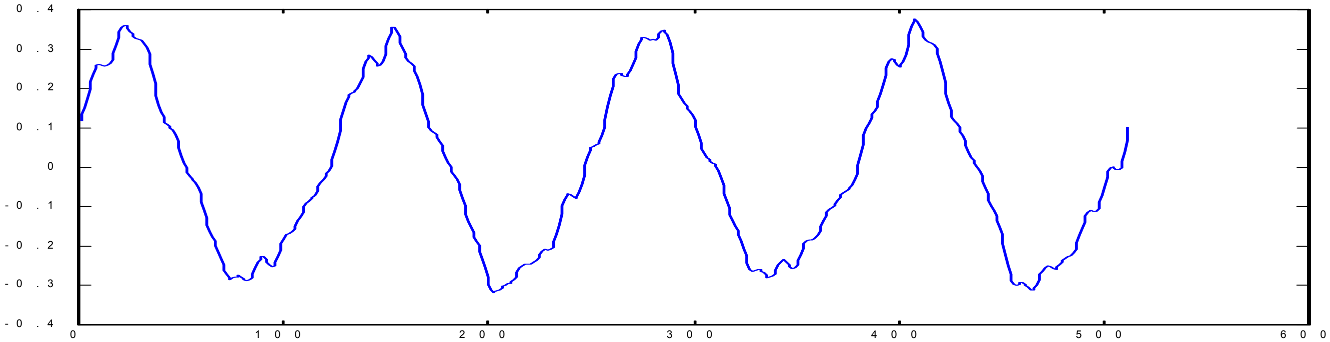
and such signals are called power signals.

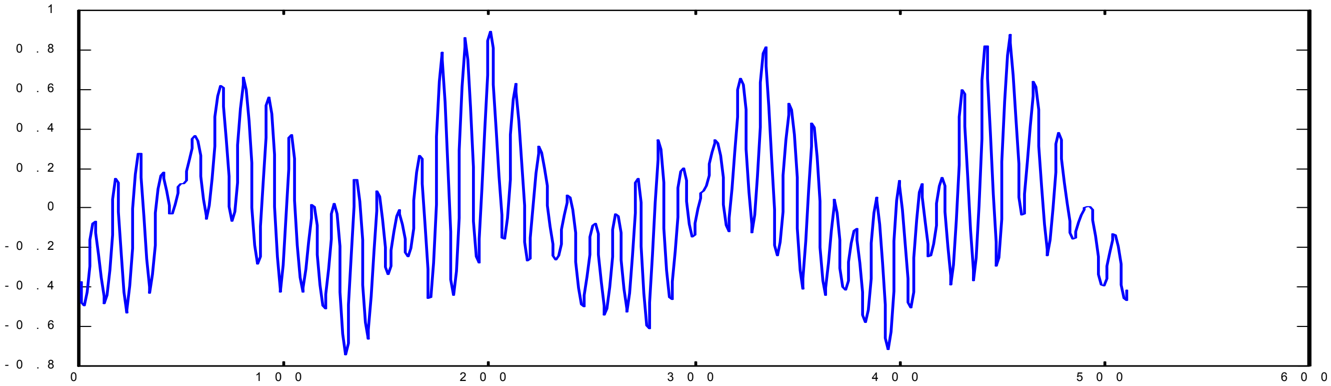
Modulations: Periodic



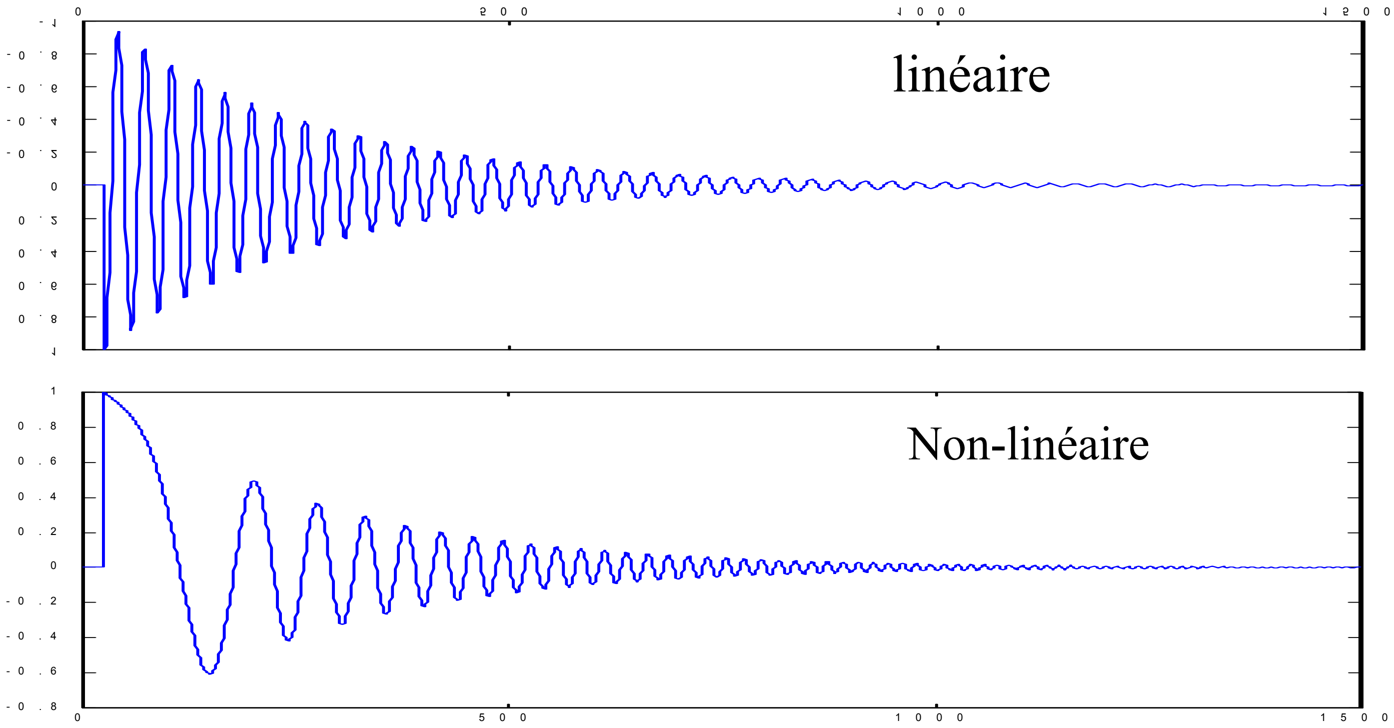


Vibrations of rotating machines

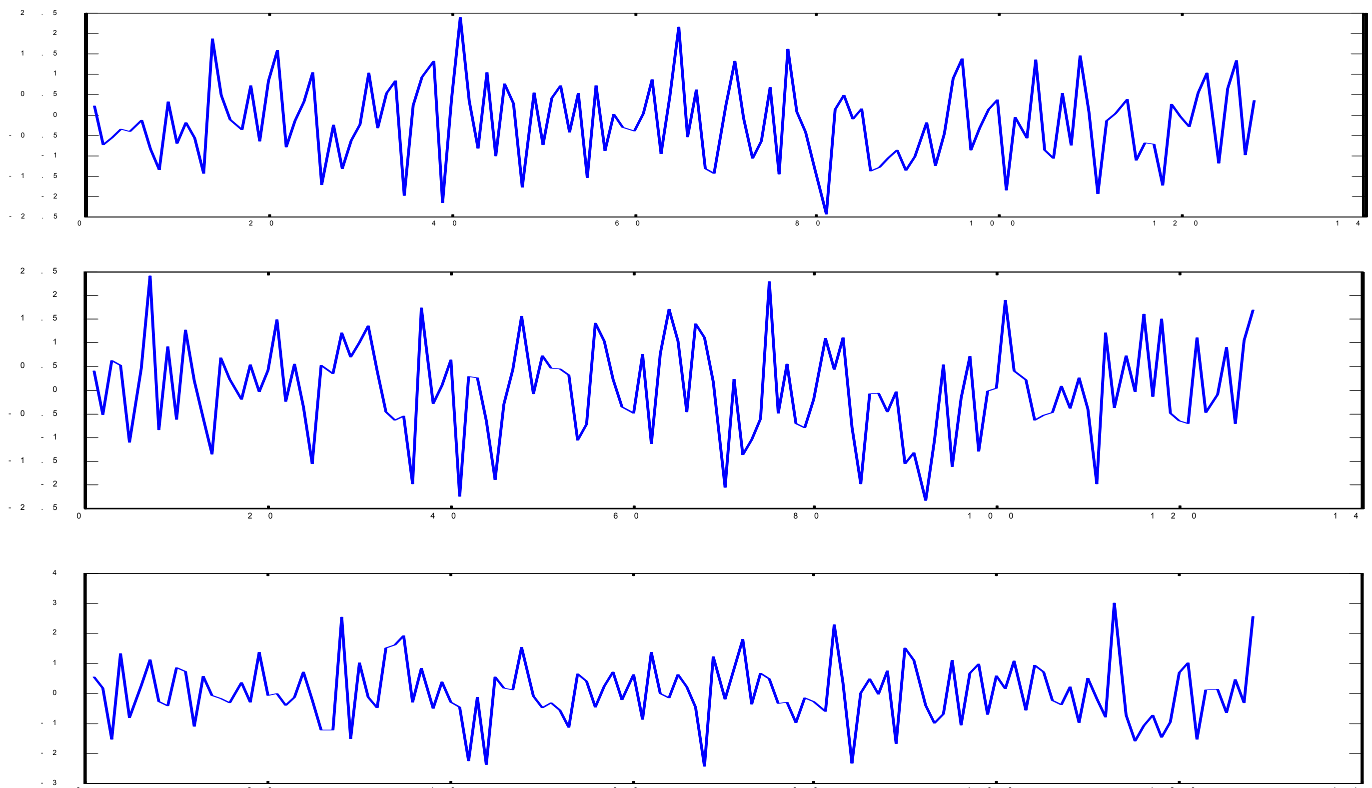




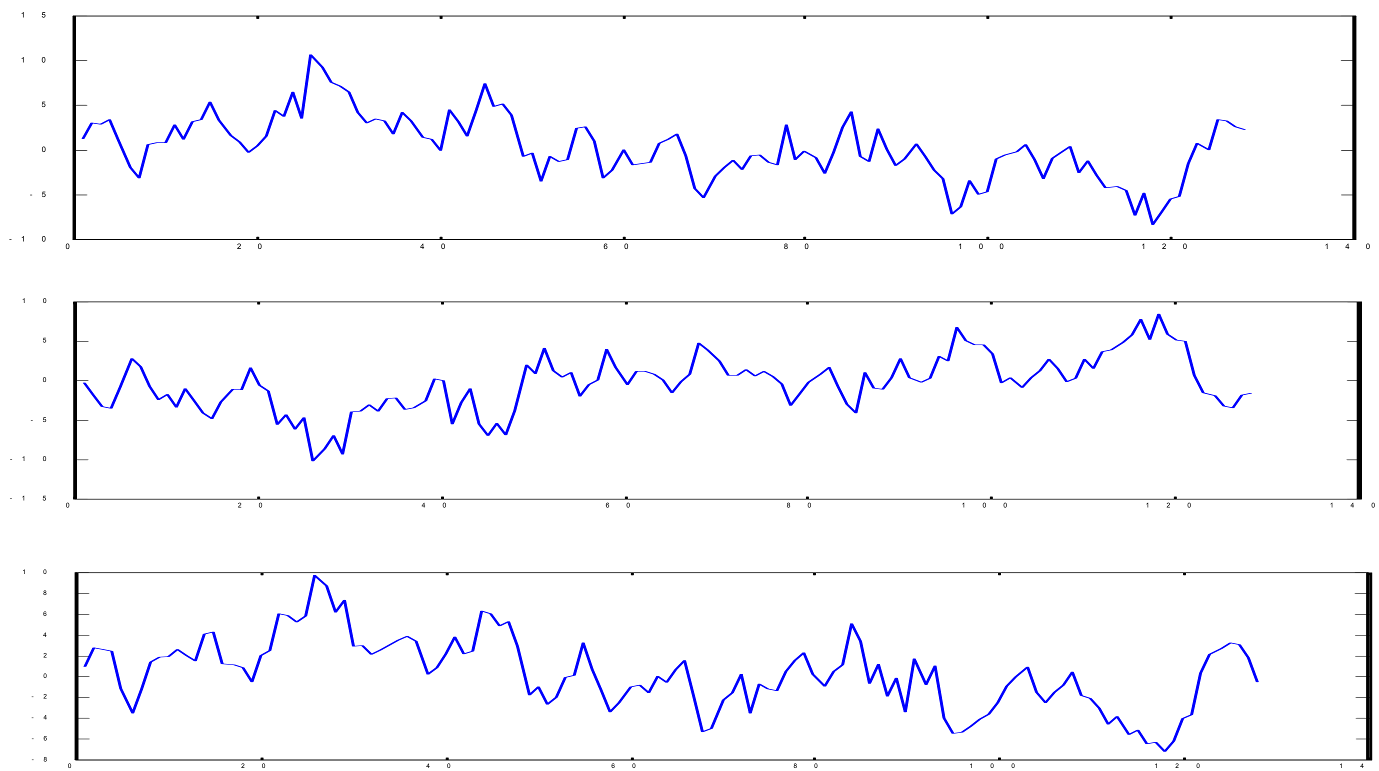
Impulse response of SDOF system



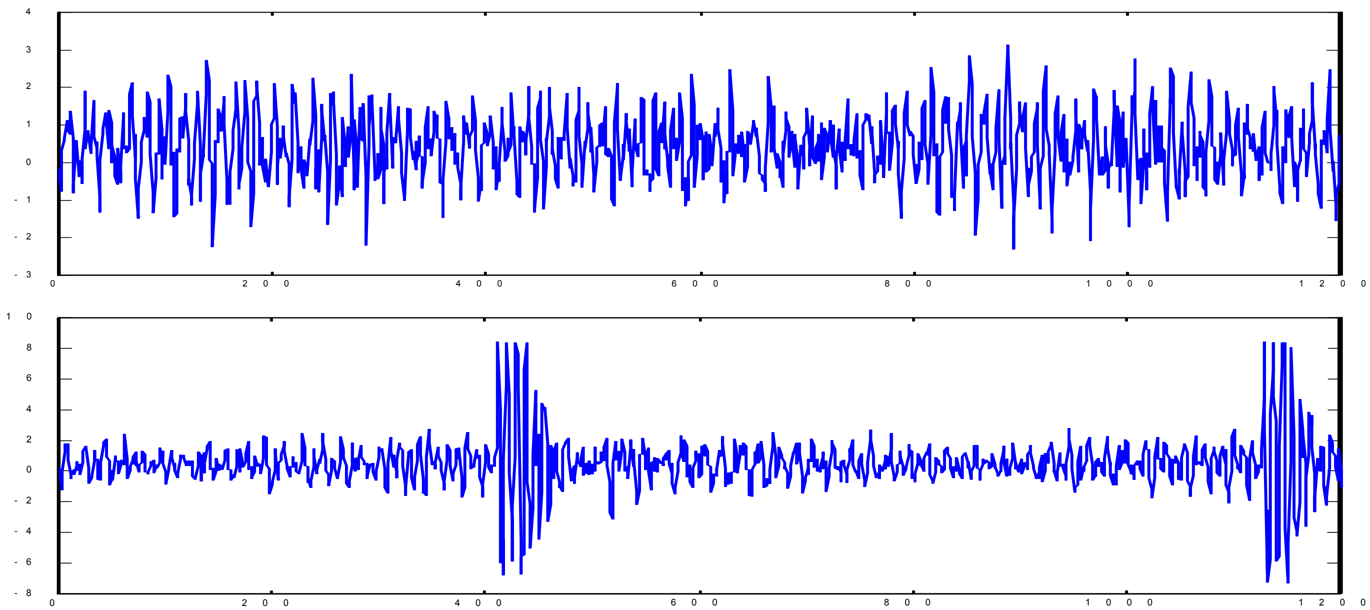
White noise:
3 realizations



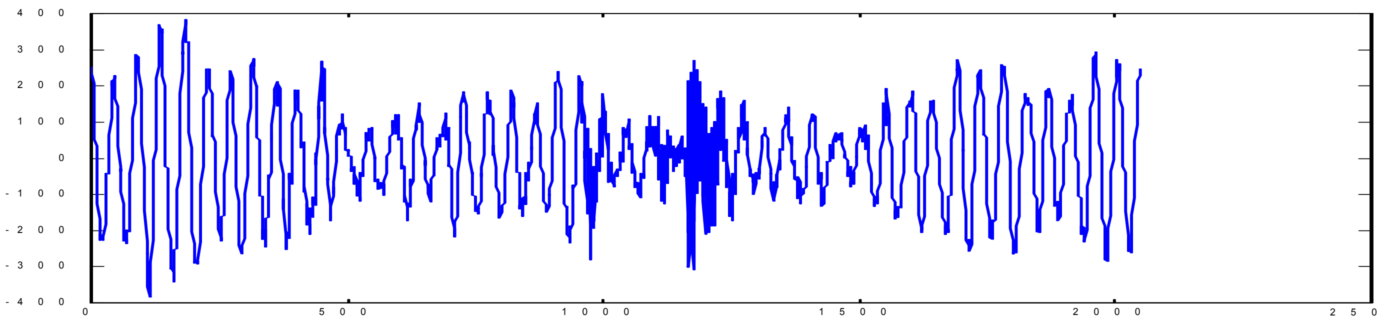
- *Response of system excited by white noise*
- *3 Realizations*

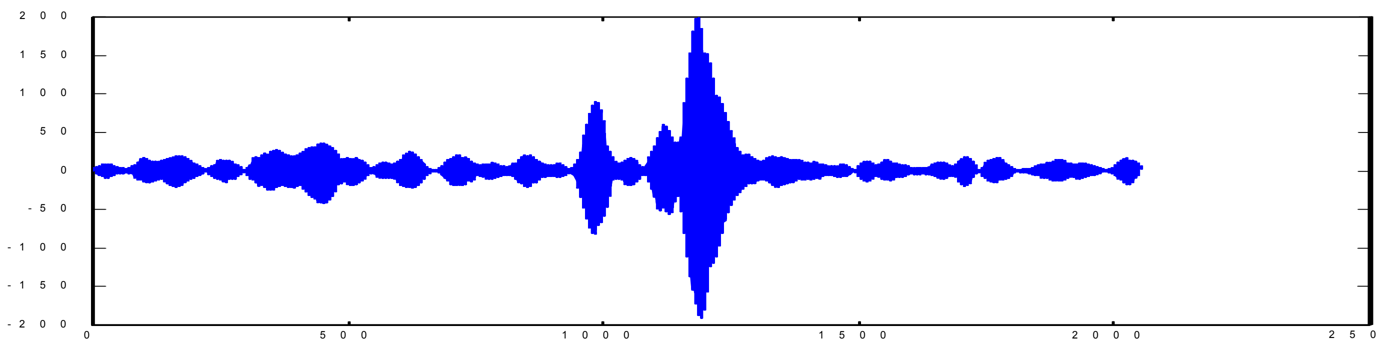


- *Gear system:*
- *Good and faulty gear*

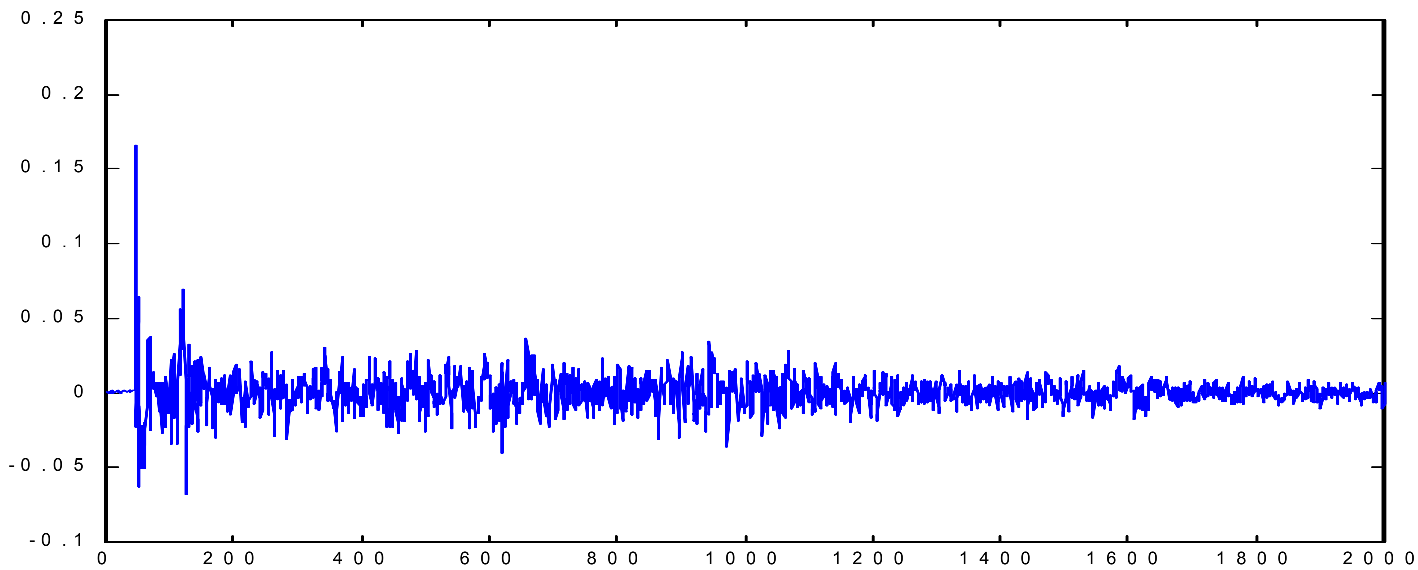


Monitoring of tool wear

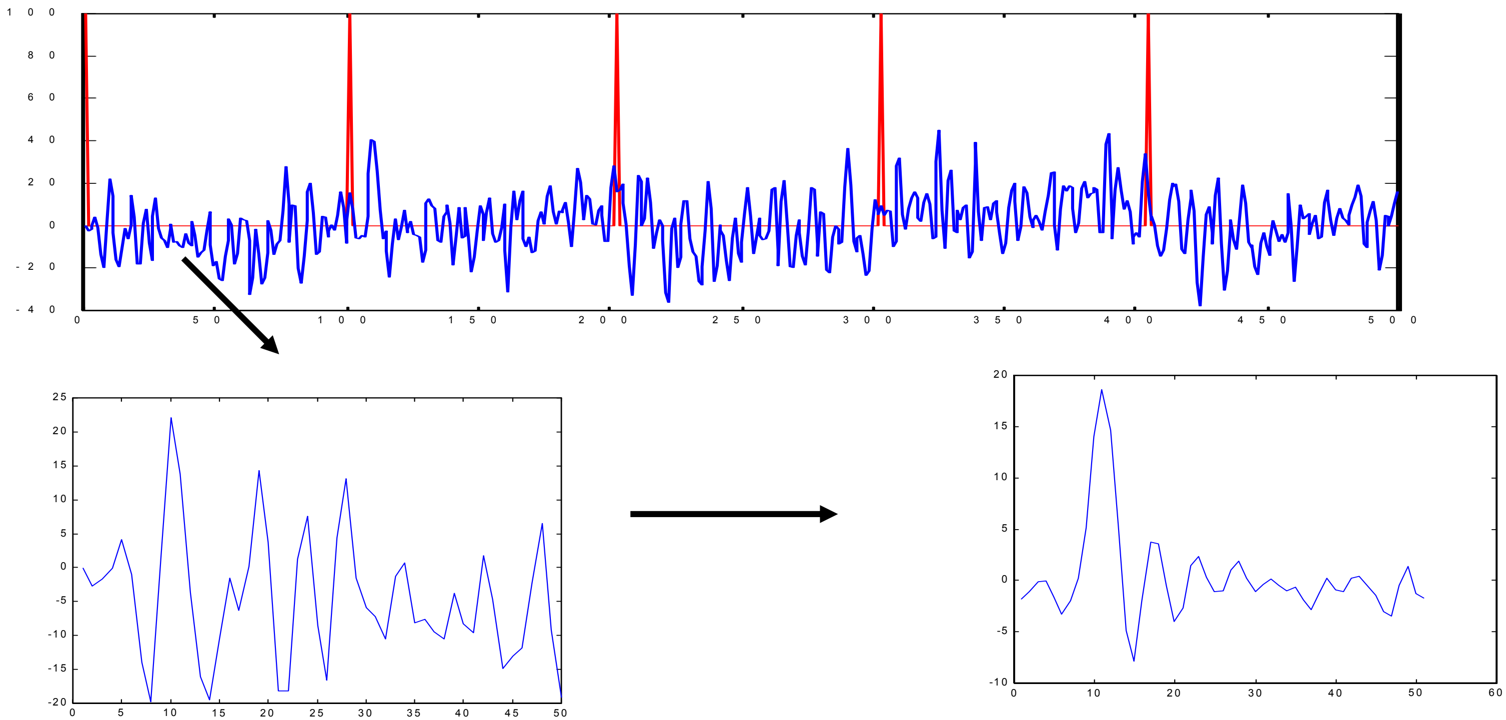




- *Acoustic impulse response of room*



Evoked potentials , response to flash (measured on scalp)



- ***Second class:***
  - *pdf*
  - *Sampling errors*
- ***Applications of pdf***
  - *Basic prob description of data*
  - *Evaluation of normality*
  - *Detection of data acquisition errors*
  - *Indication of non-linear effects*
  - *Analysis of extreme values*

Probability Distribution Function – General case

- *In general, PDF refer to the value of the process at a specific time (i.e., t1) and the probability that it is less then some threshold x*

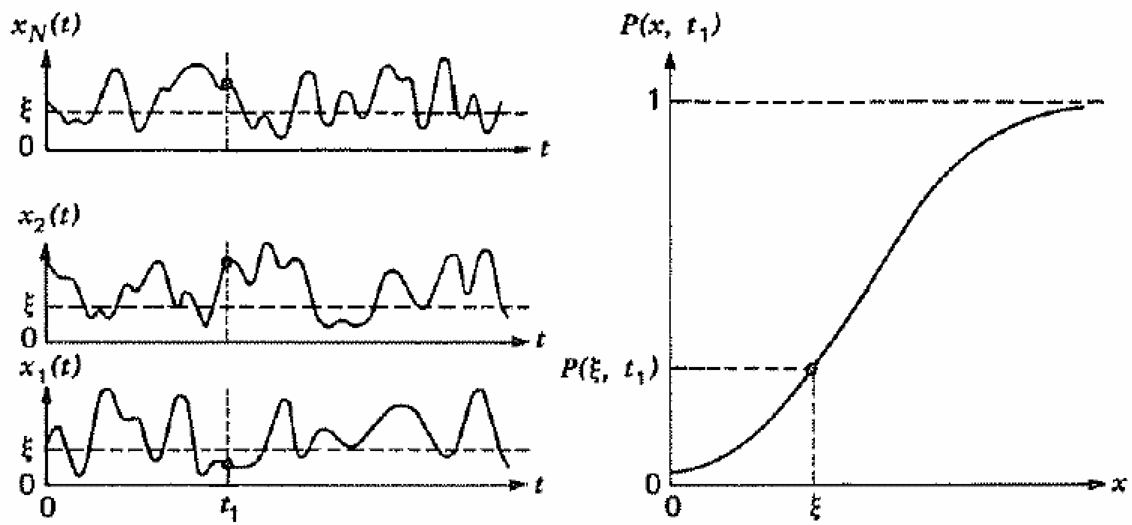


Figure 2.1 General probability distribution function

$$P_x(\xi,t_1)=\Pr[x(t_1)\le\xi]=\lim_{N\to\infty}\frac{\#[x(t_1)\le\xi]}{N}$$

Probability Distribution Function – Stationary case

- *Strict Sense Stationary Data: PDF is the same at all time and (assuming Ergodicity) can be determined from a single measurement*

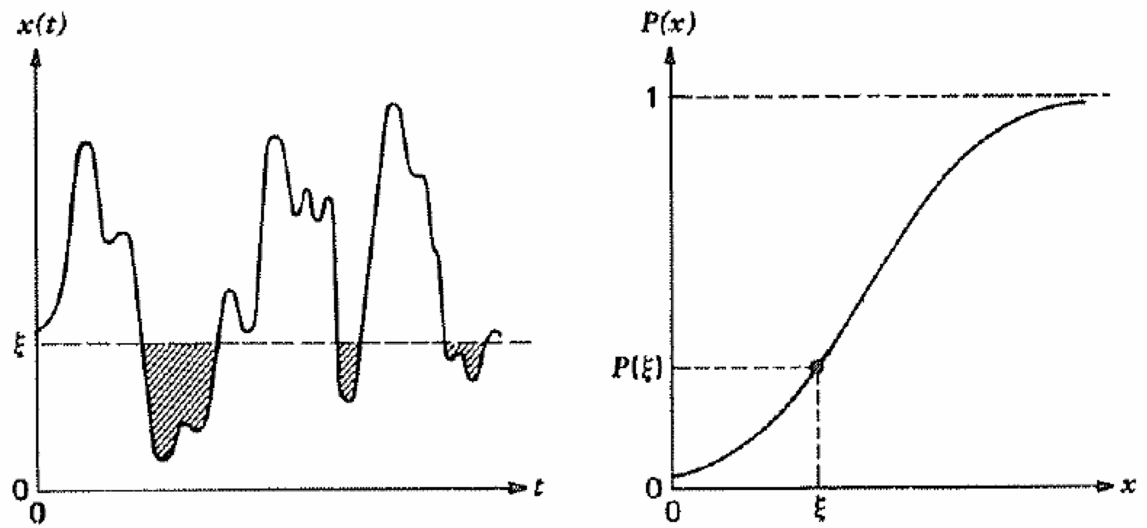


Figure 2.2 Probability distribution function of stationary data.

$$P_x(\xi) = \Pr[x(t) \le \xi] = \lim_{T \to \infty} \frac{T[x(t) \le \xi]}{T}$$

Probability Density Function – Stationary case

- *In general: derivative of PDF*
- *Computationally (for stationary data):*
  - *Prob in finite bins:* $$\Pr[x(t) \in \Delta x_\xi] = \lim_{T \to \infty} \frac{T[x(t) \in \Delta x_\xi]}{T}$$
  - *Pdf:* $$p(x) = \lim_{\Delta x \to 0} \frac{\Pr[x(t) \in \Delta x_\xi]}{\Delta x}$$

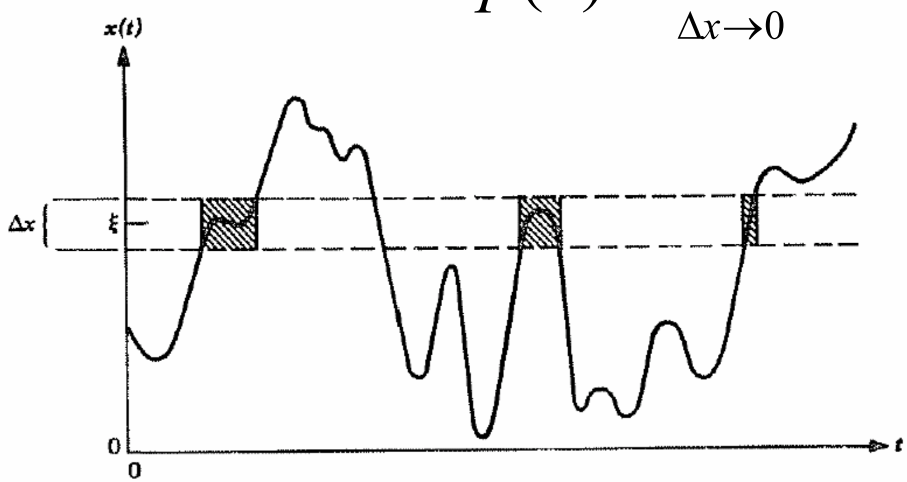


Figure 2.3 Probability density measurement.

Gaussian Random Process

- *The pdf of the value is Gaussian:*

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-m)^2}{2\sigma^2} \right)$$

- *Reasonable (but ***) model for the majority of vibratory random signals encountered in the real environment.*
- *Insight:*
  - *Central Limit Theorem: sum of infinitely many iid random events is roughly Gaussian*
  - *The response of a linear time-invariant system to a Gaussian process is a Gaussian process*
  - *Response of a linear system (convolution with impulse response) can be considered as the limit of a sum. Hence even if the excitation is not Gaussian the response of a linear system tend to be Gaussian (provided it is a narrow band response derived from a broad band excitation)*

Gaussian Limitation

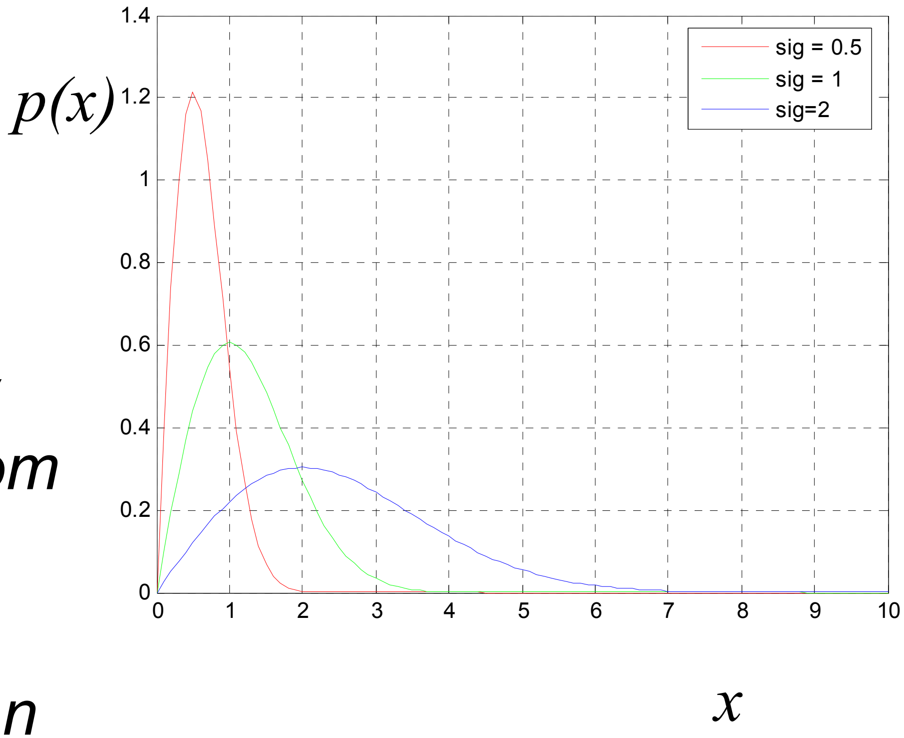
- *Gaussian pdf may give poor approximation at the tails of the distribution*
- *>>Predictions of maximum excursion of a random processes based on Gaussian assumption are suspicious*
- *The tails are crucial in many decision problems*
  - *Other distributions*
  - *Robust decisions*
  - *Info-gap*

Rayleigh Random Process

- *Rayleigh pdf:*

$$p(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2} \right) \quad x \geq 0$$

- *Representation of:*
  - *Envelope of a narrow band Gaussian random process*
  - *Peak distribution of a narrow band Gaussian process*



Sinusoidal Random Process

- *Deterministic sine wave* $x(t) = x_0 \sin(2\pi f t + \theta)$
- *Sinusoidal Random process: sine wave with uniformly distributed phase in* $\pm\pi$

$$x_k(t) = x_0 \sin(2\pi f t + \theta_k)$$

- *The resulting pdf is: (sd* $\sigma = X/\sqrt{2}$*)*

$$p(\xi) = \left(\pi\sqrt{x_0^2 - \xi^2}\right)^{-1} \quad for \quad |\xi| < x_0$$

  - *Depends only on the sd (for zero mean processes)*
  - *Minimum probability at the mean !!! (mean least likely)*
  - *Distinguish sinusoidal process from narrow-band Gaussian noise no matter how narrow the noise bandwidth may become*

Pdf of Sinusoidal Random Process

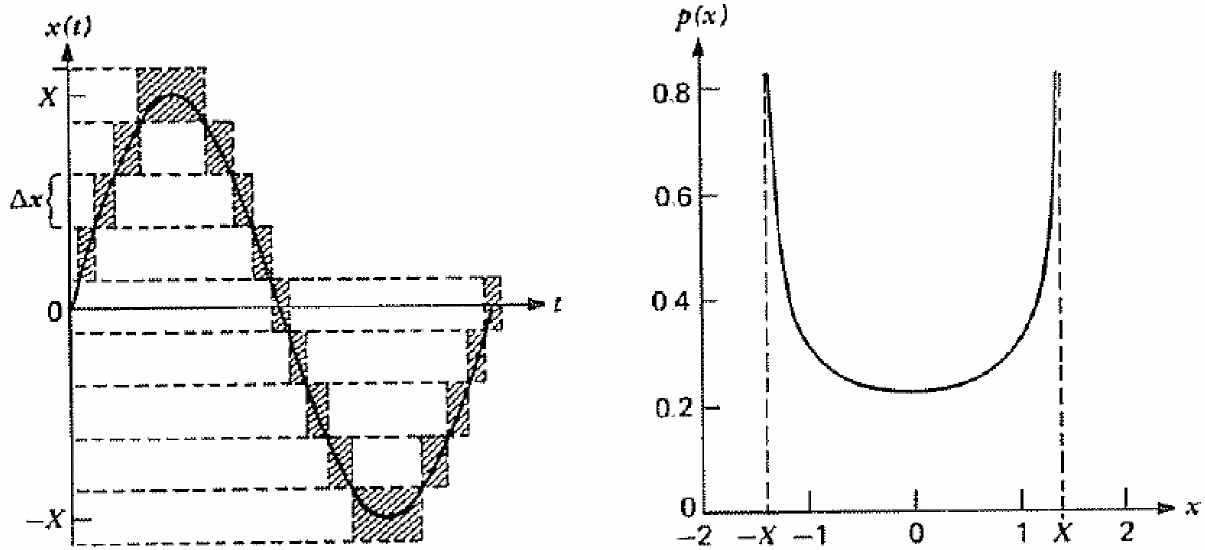


Figure 2.6 Standardized probability density function of a sine wave.

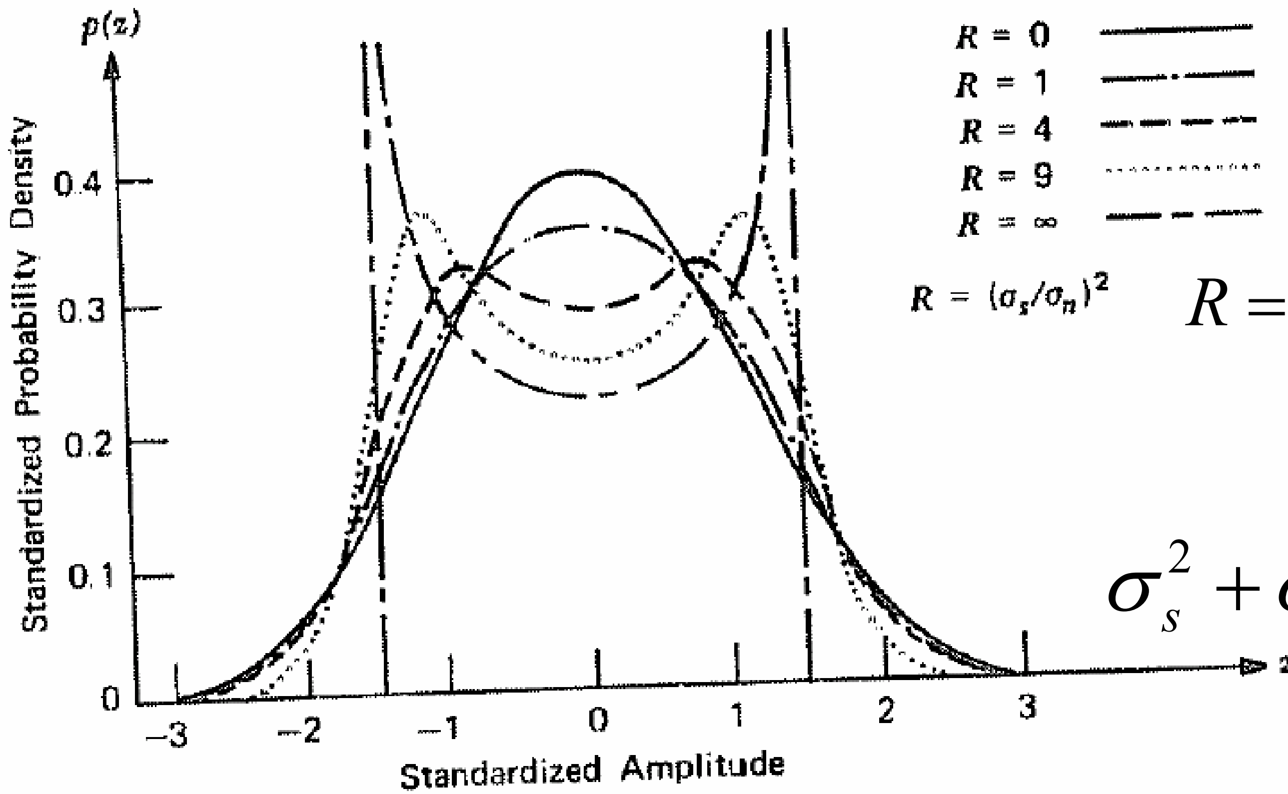
$$p(\xi) = \left(\pi\sqrt{x_0^2 - \xi^2}\right)^{-1} \quad for \quad |\xi| < X$$

$$= \left(\pi\sqrt{2\sigma^2 - \xi^2}\right)^{-1} \quad for \quad |\xi| < X$$

$$\sigma = x_0 / \sqrt{\ }$$

Sine Wave in Gaussian Noise

$$p(z) = \frac{1}{\pi\sqrt{2\pi\sigma_n}} \int_0^{\pi} \exp\left(-\frac{(z - x_0\cos\phi)}{2\sigma_n^2}\right) d\phi$$



$$R = \left(\sigma_s / \sigma_n\right)^2$$

$$\sigma_s^2 + \sigma_n^2 = 1$$

Moments (Stationary data)

- *Moments of a stationary random process with pdf p(x):* $\mu_k = E[x^k] = \int_{-\infty}^{\infty} x^k p(x)dx$
- *Central Moments:*

$$\mu_k^c = E[(x-\mu_1)^k] = \int_{-\infty}^{\infty} (x-\mu_1)^k p(x)dx$$

  - *1st moment = Mean (central tendency):* $\mu_1 = E[x] \equiv \mu$
  - *2nd moment = Mean sq. value* $\mu_2 = E[x^2] \equiv \psi^2$
    - ψ *is the rms (root mean square) – in analogy with ac*
  - *2nd central moment = variance (dispersion):*
    - σ *is the standard deviation* $\mu_2^c = E[(x-\mu)^2] \equiv \sigma^2$
- *Central tendency and dispersion:*

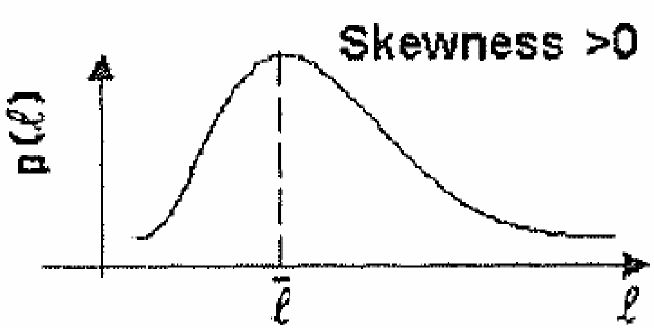
$$\psi^2 = \sigma^2 + \mu^2$$

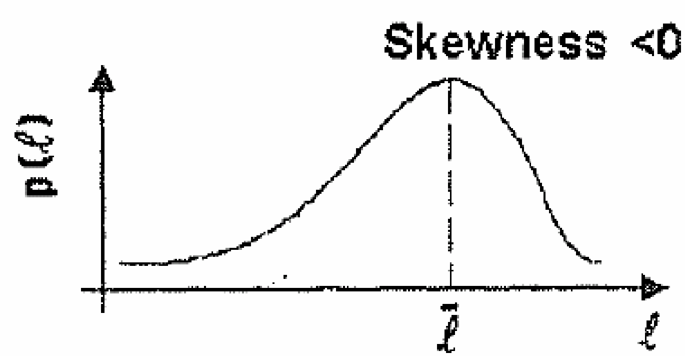
Higher Moments - Skewness

- *3rd central moment* $\mu_3^c = E[(x-\mu)^3]$
  - *> Skewness:*

$$skewness = \frac{\mu_3^c}{\sigma^3} = \frac{E[(x-\mu)^3]}{\sigma^3}$$

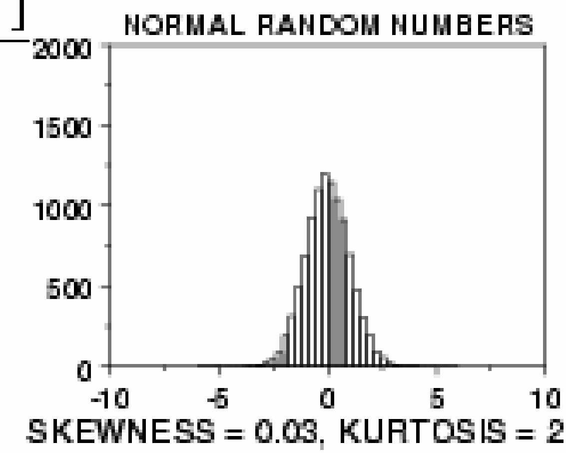
  - *Skewness (Gaussian) = 0*
  - *Skewness>0 →peak toward the left*
  - *Skewness<0 → peak toward the right*

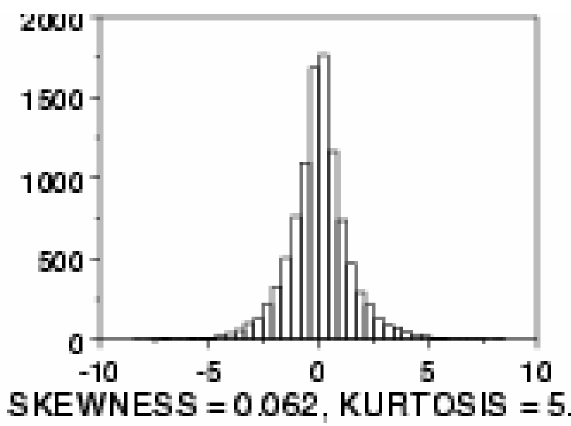




Higher Moments - Kurtosis

- *4th central moment* $\mu_4^c = E[(x-\mu)^4]$
  - *> Kurtosis:*
    $$\kappa = kurtosis = \frac{\mu_4^c}{\sigma^4} = \frac{E[(x-\mu)^4]}{\sigma^4}$$
  - *Kurtosis (Gaussian) = 3*
  - *Kurtosis>3 →*
    - *stronger peak, more rapid decay, and heavier tails*
    - *presence of peaks of high values more than in the Gaussian case*
  - *Kurtosis<3 →*
    - *Existence of a sinusoidal component (kurtosis =1.5 for sine)*
    - *Truncated signal (truncated extremes)*
  - *Excess kurtosis = κ-3*





Data Averaging

- *Moments computed directly on data records (instead of using the pdf)*
  - *Ensemble averaging (through the process)*
  - *Stationary data: Temporal averaging (along the process)*
- *Moment estimation based on*
  - *Ensemble averaging: Finite number of records N*
  - *Temporal averaging: Finite record of duration T*

Statistical Sampling Errors

- Estimation of a parameter ϕ from independent sample records >> collection of estimates $\left\{\hat{\phi}_i\right\}_{i=1,2,\ldots}$
  - Bias error: $b_{\hat{\phi}} = E[\hat{\phi}] - \phi = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N}\hat{\phi}_i - \phi$
  - Random error: $\sigma_{\hat{\phi}} = \left(\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\phi}_i - E[\hat{\phi}]\right)^2\right)^{1/2}$
  - Normalized errors: $\varepsilon_b = \frac{b_{\hat{\phi}}}{\phi}$ Coeff of Variation $\varepsilon_r = \frac{\sigma}{\phi}$
  -
  - Sampling distribution of the estimate
    - Complicated in general
    - Gaussian approximation

    Reasonable for ε_r<0.2)

$$\mu_{\hat{\phi}} = (1+\varepsilon_b)\phi \qquad \sigma_{\hat{\phi}} = \varepsilon_r\phi$$

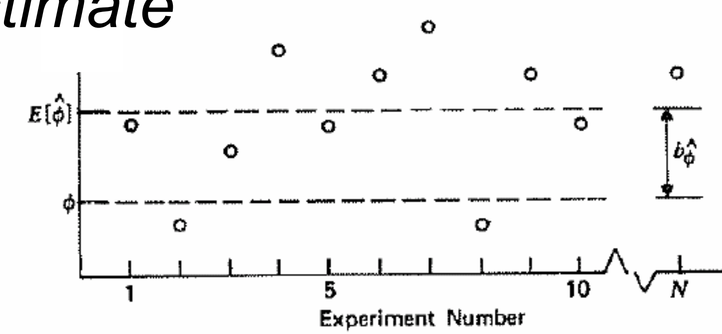


Figure 2.8 Random and bias errors in parameter estim

Errors in Mean Values

- No bias errors
- Random errors – see Table
  - Ensemble average with N independent records
  - Temporal averaging over a record of length T, when the energy is uniformly distributed over freq bandwidth B (N~2BT)
- Implications:
  - To cut ε_r by half: need to quadruple (x4) N or T
  - B is as important as T for the effective sample size
    - Wide bandwidth data (communications); short record might provide highly accurate estimates
    - Narrow bandwidth data (turbulence, vibrations) very long record may be required to obtain acceptably accurate estimates

Estimation and Random Error

Table 2.1

Random Errors in Mean and Mean Square Value Estimates

Estimation Procedure	Estimates and Corresponding Random Error, ε_r			
	Mean Value Estimates		Mean Square Value Estimates	
	Estimate, $\hat{\mu}_x$	Error, ε_r	Estimate, $\hat{\psi}_x^2$	Error, ε_r
Ensemble Averaging	$\frac{1}{N}\sum_{i=1}^{N} x_i$	$\frac{\sigma_x}{\mu_x\sqrt{N}}$	$\frac{1}{N}\sum_{i=1}^{N} x_i^2$	$\sqrt{\frac{2}{N}}$
Time Averaging	$\frac{1}{T}\int_0^T x(t)\,dt$	$\frac{\sigma_x}{\mu_x\sqrt{2BT}}$	$\frac{1}{T}\int_0^T x^2(t)\,dt$	$\frac{1}{\sqrt{BT}}$

Errors in pdf estimation) - bias error

- Bias error due to finite bin-width Δx:
  - Unless the first derivative of p is constant
  - First approx of bias error:

$$\varepsilon_b[\hat{p}(x)] \approx \frac{(\Delta x)^2}{24} \frac{p''(x)}{p(x)}$$

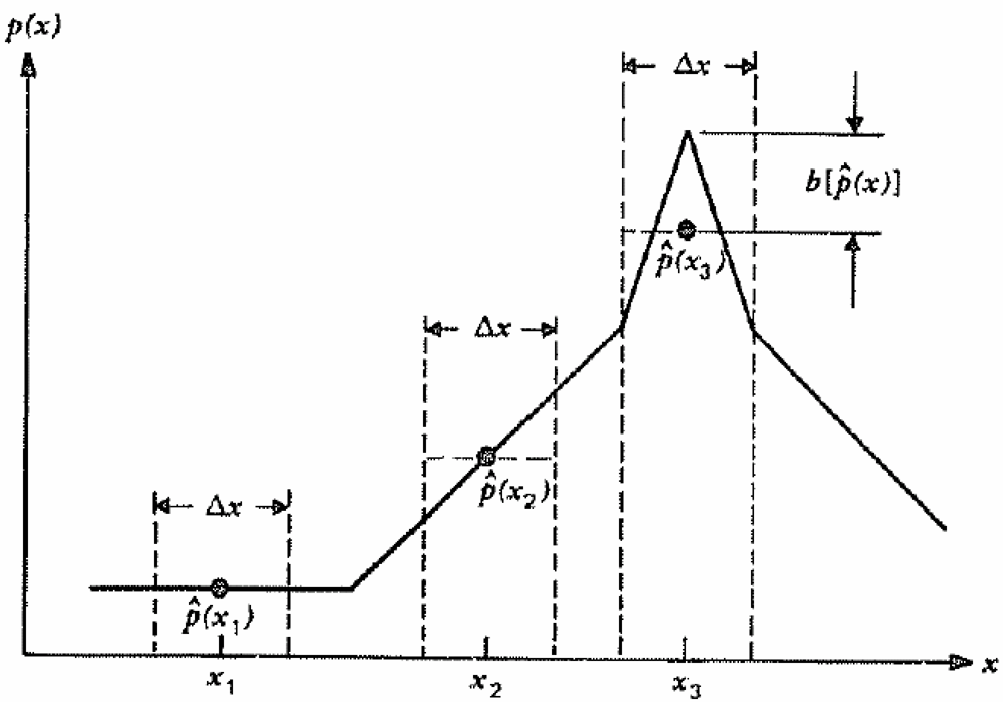


Figure 2.9 Bias error in probability density estimate.

Errors in pdf estimation) – random error

- *Random error due to*
  - *finite number of records in ensemble averaging:*

$$\varepsilon_r[\hat{p}(x)] \approx \frac{1}{\sqrt{N(\Delta x)\,p(x)}}$$

  - *finite length record in temporal averaging*

$$\varepsilon_r[\hat{p}(x)] \approx \frac{1}{\sqrt{2BT(\Delta x)\,p(x)}}$$

- *Implications:*
  - *Same dependence on N, T and B as in mean-estimation*
  - *Bin-size Δx: compromise between*
    - *bias error – increasing effect*
    - *random error – decreasing effect*

Fourier methods

- Continous signals

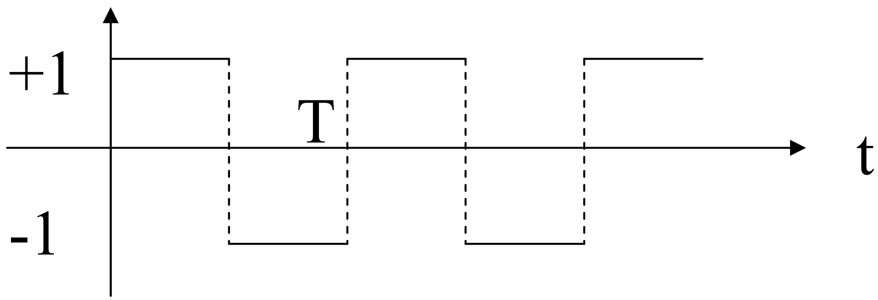
  FS – Fourier Series : Periodic

  FT – (Integral) Fourier Transform: Transients (aperiodic)

- Discrete (sampled) signals

  DFS –Discrete Fourier Series

  DFT – Discrete Fourier Transform

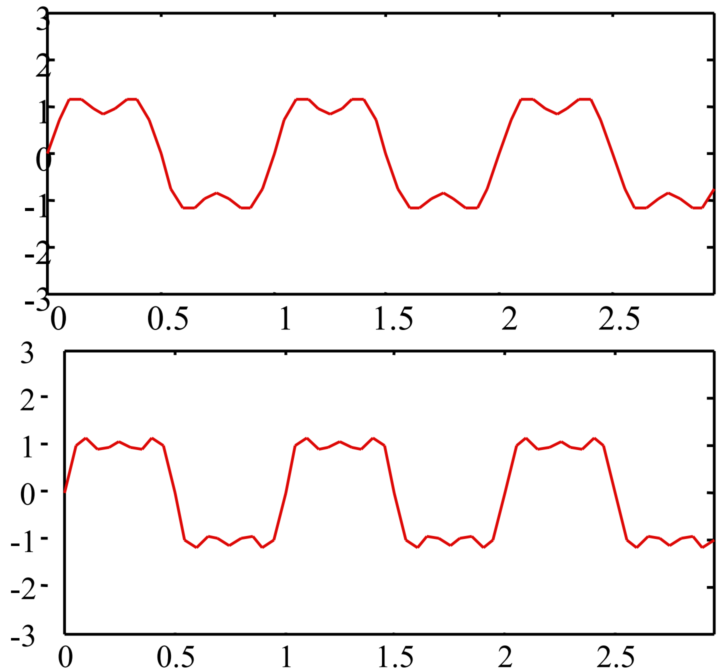


$$a_n = \frac{2}{T}\int_o^r x(t)\cos(2\pi\frac{nt}{T}) = 0$$

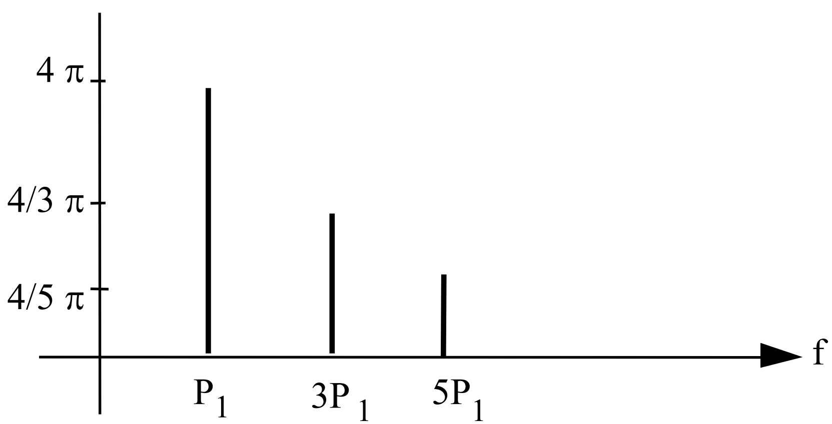
$$b_n = \frac{1}{T}\int_o^T x(t)\sin(2\pi\frac{nt}{T})dt = \begin{cases} 0 & n = 2,4,6... \\ \dfrac{2}{n\pi} & n = 1,3,5... \end{cases}$$

$$x(t) = \frac{4}{\pi}\left[Sin(\omega_o t) + \frac{1}{3}Sin(3\omega_o t) + \frac{1}{5}Sin(5\omega_o t).....\right], \omega_o = \frac{2\pi}{T}$$

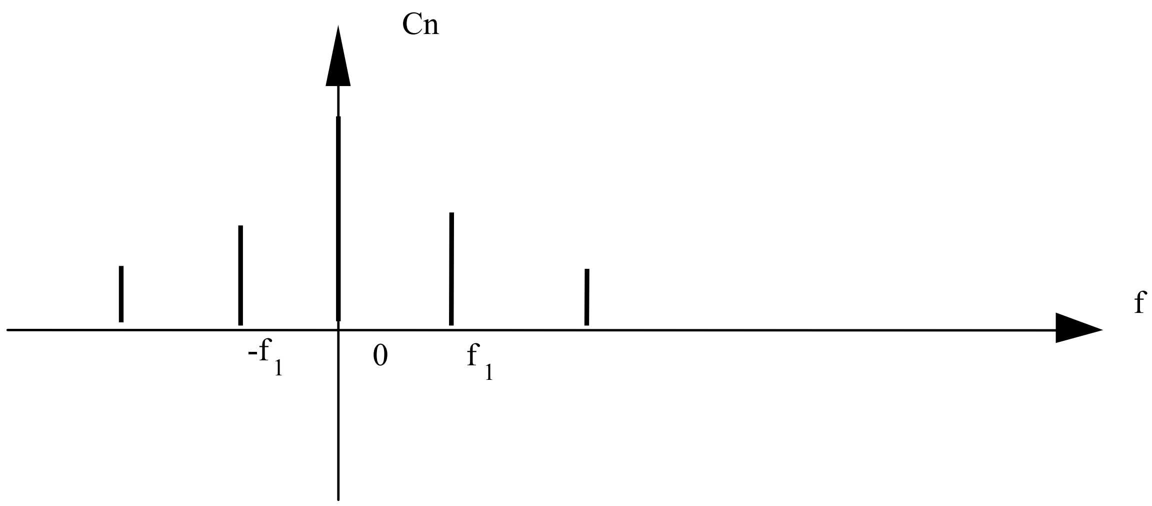
wing figure.



We may represent the decomposition by the spectrum. The 2-D representation is of magnitude at frequency location. The spectrum of the periodic signal is discrete: only elements with discrete frequencies nwo (n = 1,3,.....) exist.



The spectrum corresponding to the complex Fourier Series in 2 sided shows positive and negative frequencies.



טורי פורייה – קירוב של פונקציות מחזוריות (א)

- טור פורייה – סכום לינארי של אותות טריגונומטריים בהרמוניות של התדירות הבסיסית

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

- מרחב גדול של פונקציות מחזוריות ניתנות לקירוב בעזרת טור פורייה
  - תנאי: אנרגית מחזור סופית $\int_{T_0} |x(t)|^2 dt < \infty$
  - טיב הקירוב: האנרגיה של השגיאה שואפת לאפס

$$E_N = \int_{T_0} |e_N(t)|^2 dt \underset{N\to\infty}{\rightarrow} 0; \quad e_N(t) = x(t) - x_N(t)$$

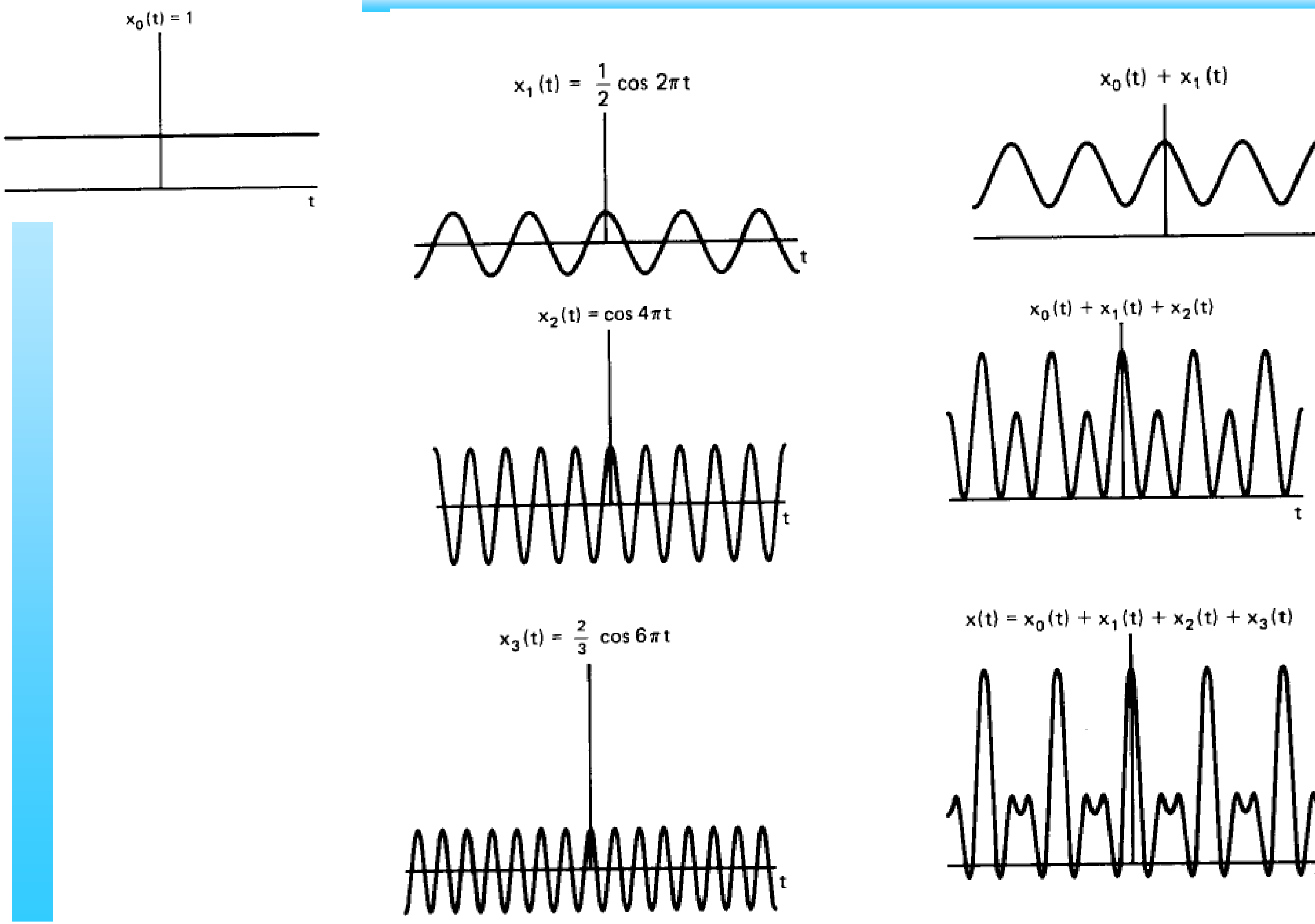
טורי פורייה – קירוב של פונקציות מחזוריות (ב)

- ייצוג של פונקציה מחזורית בעזרת טור פורייה
  - אקפוננציאלי:
$$x(t)=\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_0 t}$$
  - סינוסים:
$$x(t)=a_0+2\sum_{k=1}^{\infty}\left[B_k\cos(k\omega_0 t)-C_k\sin(k\omega_0 t)\right]$$
  - קשר בין המקדמים:
$$a_{\pm k}=B_k\pm jC_k$$

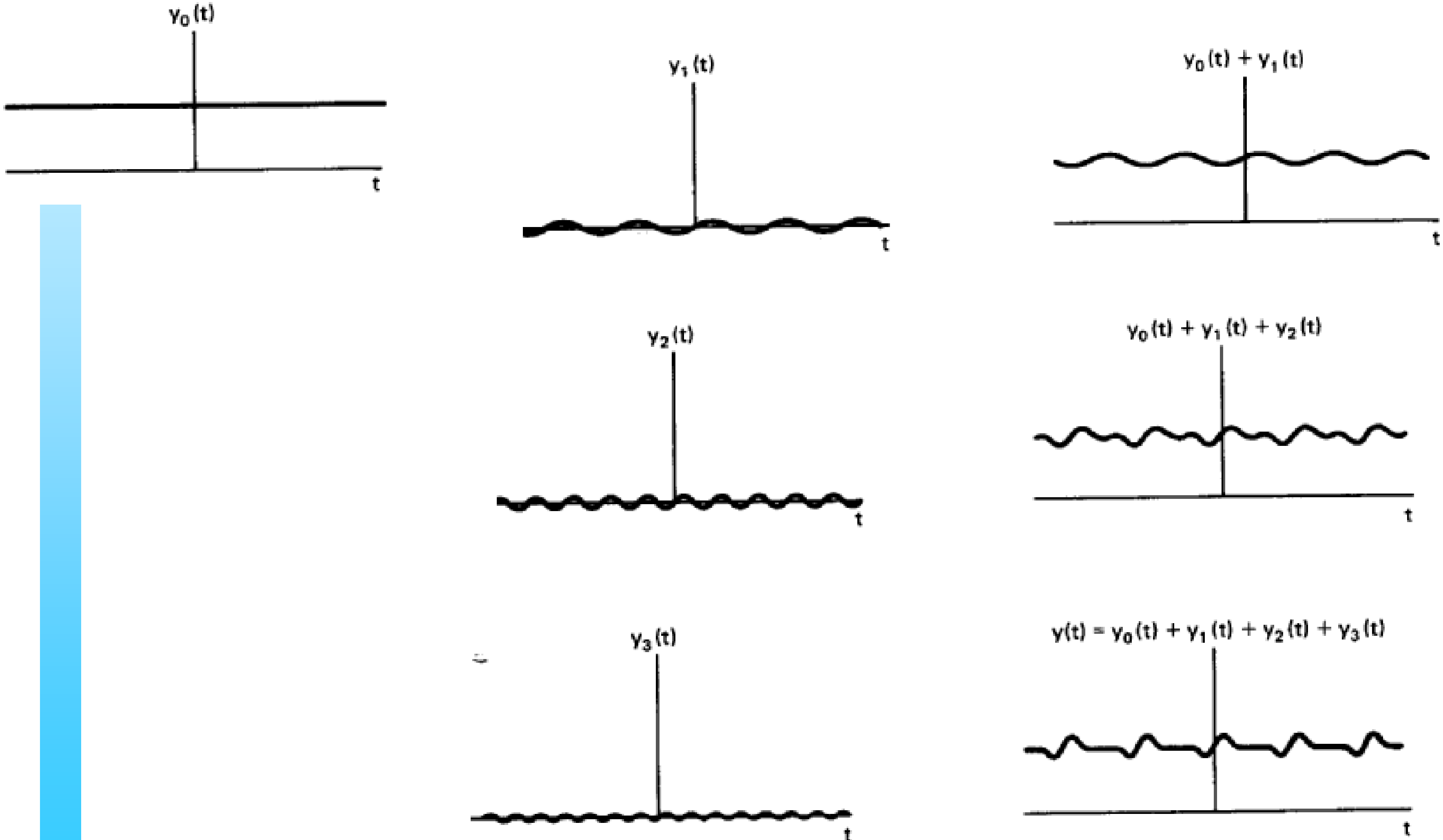
- קביעת המקדמים בטור פורייה

$$a_k=\frac{1}{T_0}\int_{T_0}x(t)e^{-jk\omega_0 t}dt$$

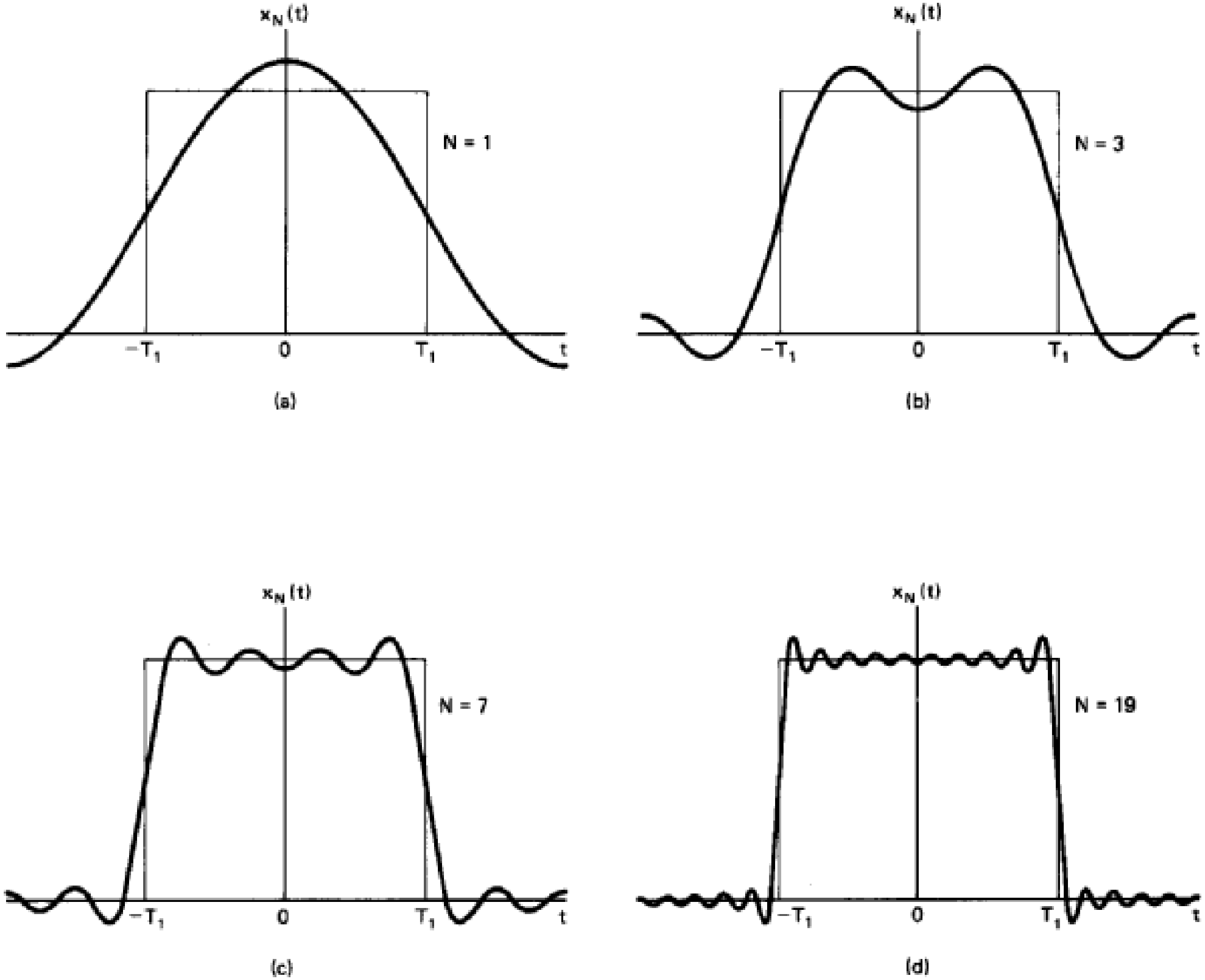
טורי פורייה – הדגמה: קירוב של פונקציה מחזורית



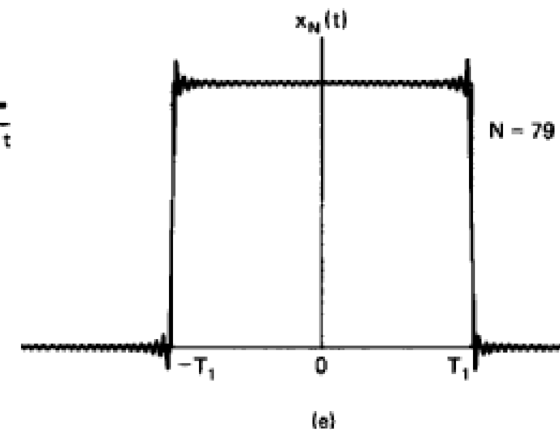
טורי פורייה – הדגמה: תגובה של מערכת לינארית לפונקציה מחזורית



טורי פורייה – הדגמה: קירוב של פונקציה מחזורית לא-רציפה (גל ריבועי מחזורי)



ריבועי מחזורי: $T_0=4T_1$



התכנסות בכל זמן –
עבור N מספיק גדול
התכנסות באי-הרציפות –
לממוצע
תופעת Gibbs

טורי פורייה – הדגמה: מקדמי פורייה של
גל ריבועי מחזורי

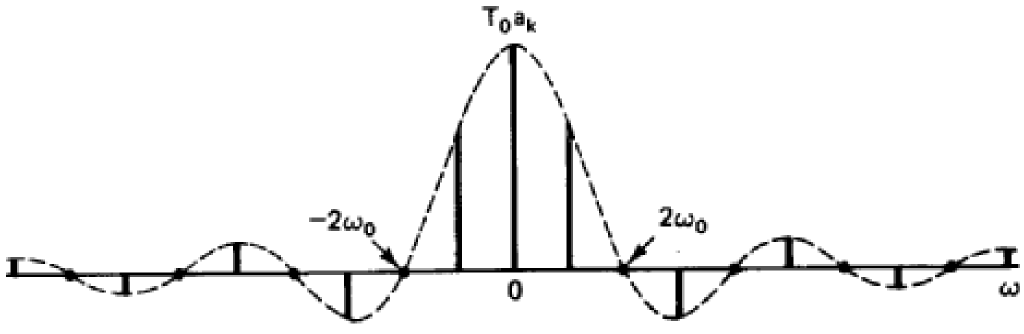
$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T_0} = \frac{\sin(k\omega_0 T_1)}{k\pi} \qquad k \neq$$

$$a_0 = \frac{2T_1}{T_0}$$

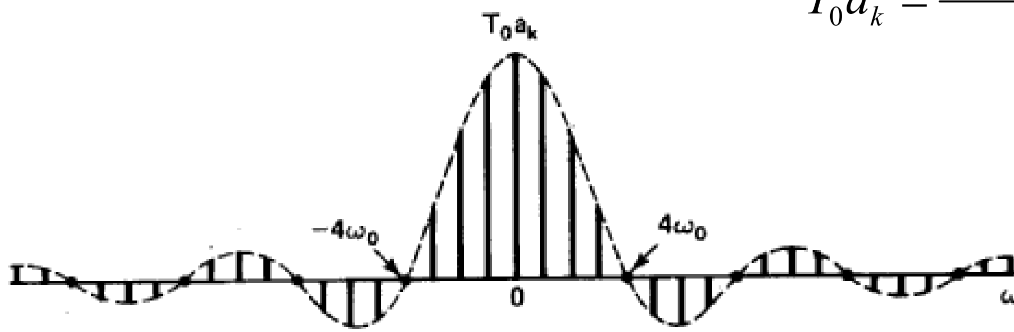
קדמים מהווים דגימות שוות-מרחק

ל המעטפת:

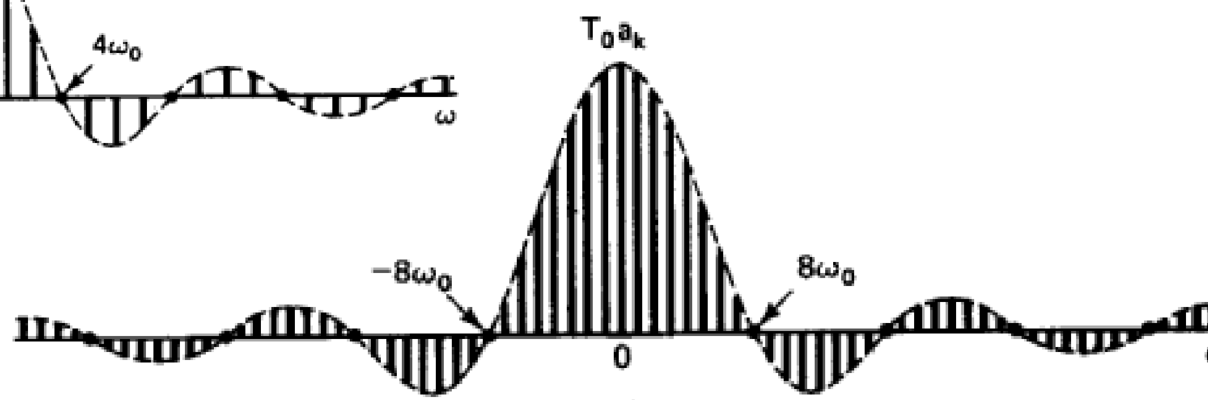
$$T_0 a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0} = \frac{\sin(\omega T_1)}{\omega}\bigg|_{\omega = k\omega_0}$$



(a)



(b)



(c)

Figure 4.11 Fourier coefficients and their envelope for the periodic squ wave: (a) $T_0 = 4T_1$; (b) $T_0 = 8T_1$; (c) $T_0 = 16T_1$.

טרנספורמצית פורייה – קירוב של פונקציות לא-מחזוריות (א)

- התמרת פורייה: $X(\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
- התמרת פורייה הפוכה (ספקטרום): $x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$
- עבור מרחב גדול של פונקציות התמרת פורייה סופית (מתכנסת) והתמרת פורייה ההפוכה משחזרת היטב את הפונקציה המקורית
  - התנאי: אנרגיה סופית $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$
  - טיב השחזור: אנרגית השגיאה שווה לאפס

$$\int_{-\infty}^{\infty} |e(t)|^2 dt = 0 \qquad e(t)=\hat{x}(t)-x(t)$$

טרנספורמצית פורייה – קירוב של פונקציות לא-מחזוריות (ב)

- התמרת פורייה של פונקציה לא-מחזורית (עם תחום סופי) כגבול של טור פורייה של פונקציה מחזורית עם מחזור ששואף לאינסוף
- הדגמה:

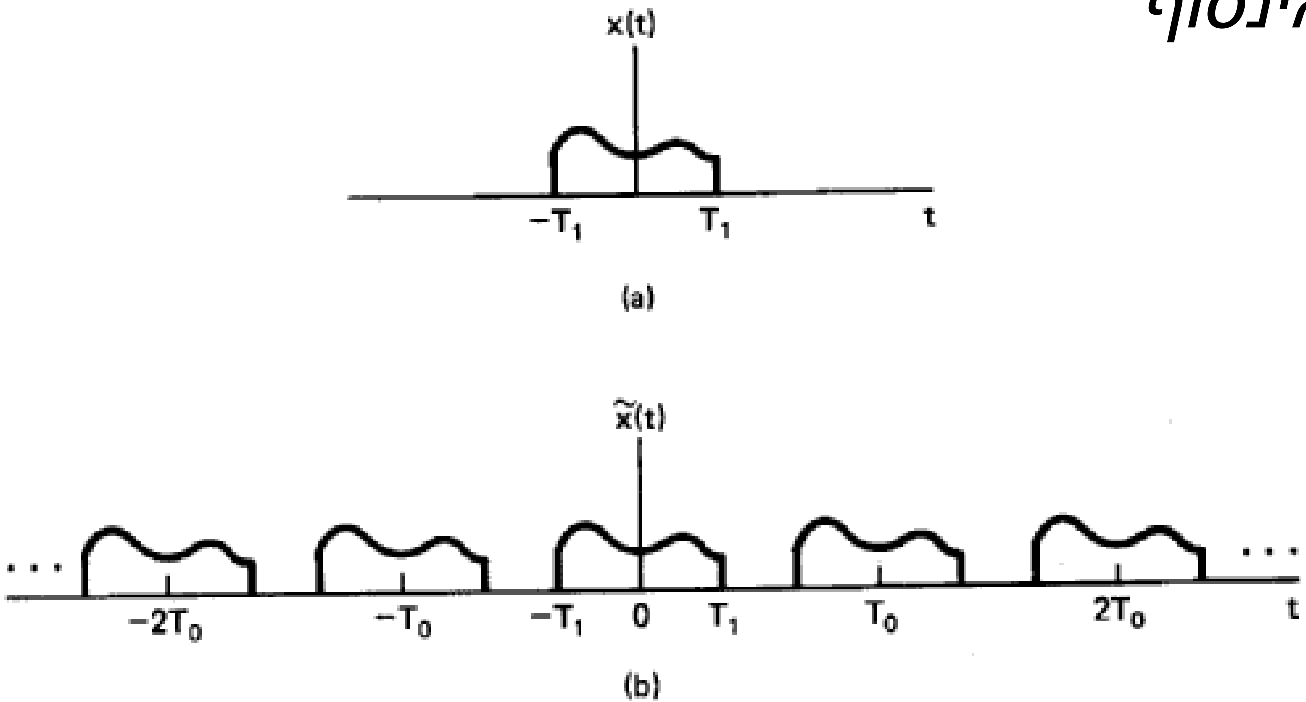
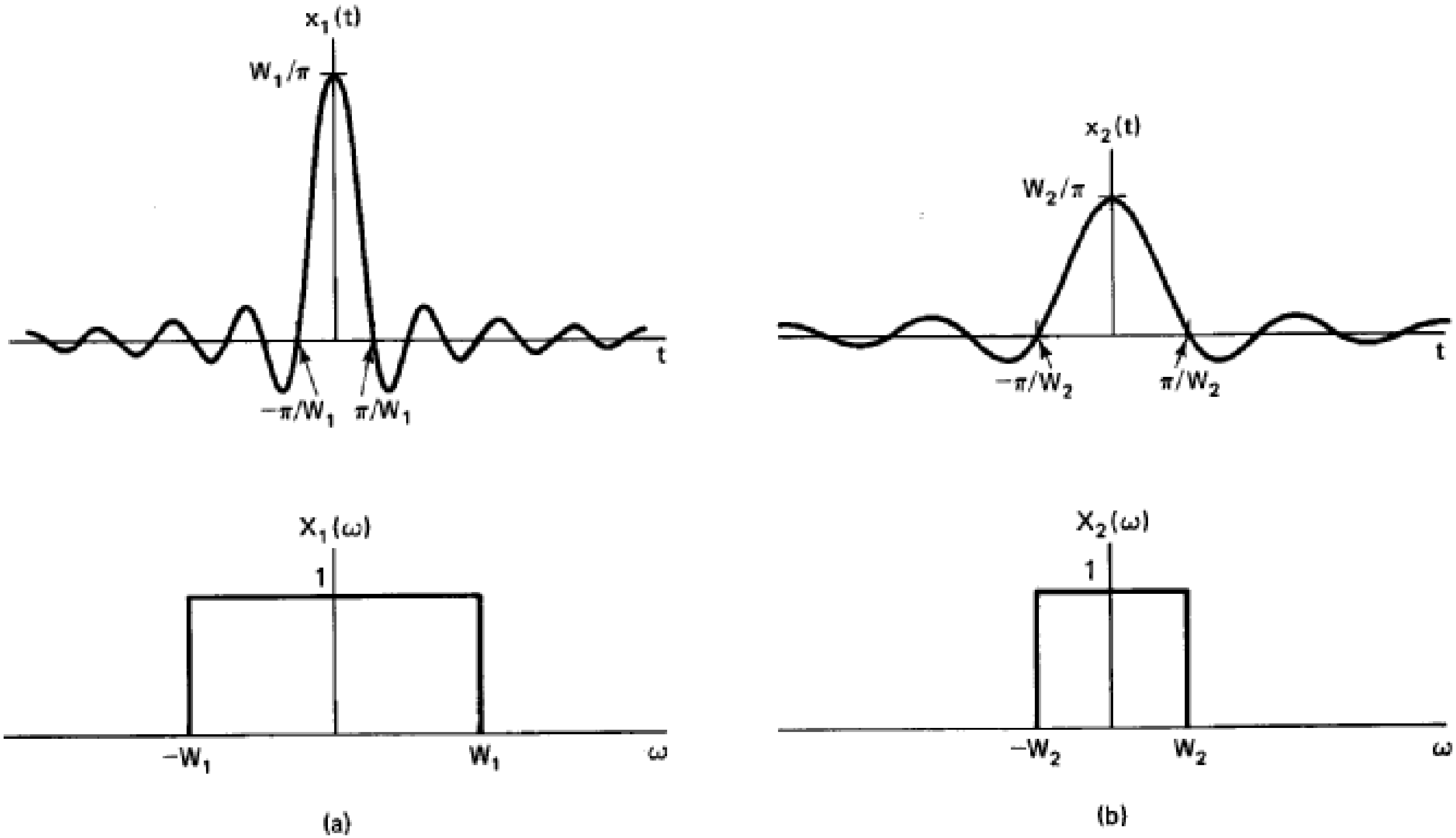


Figure 4.12 (a) Aperiodic signal $x(t)$; (b) periodic signal $\tilde{x}(t)$, constructed to be equal to $x(t)$ over one period.

התמרת פורייה: דוגמא – פולס ריבועי



התמרת פורייה של אותות מחזוריים

- הבעיה: לאותות אלו אין אינטגרל ריבועי סופי
- ההקשר: מקדמי פורייה של פונקציה מחזורית מהווים דגימות של טרנספורמצית פורייה של מחזור בודד
- התמרת פורייה של גל מחזורי: $X(\omega)=\sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega-k\omega_0)$
- התמרת פוריה הפוכה: $x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$
- דוגמא:

גל ריבועי מחזורי: $T_0=4T_1$

$$X(\omega)=\sum_{k=-\infty}^{\infty} 2\frac{\sin(k\omega_0 T_1)}{k}\delta(\omega-k\omega_0$$

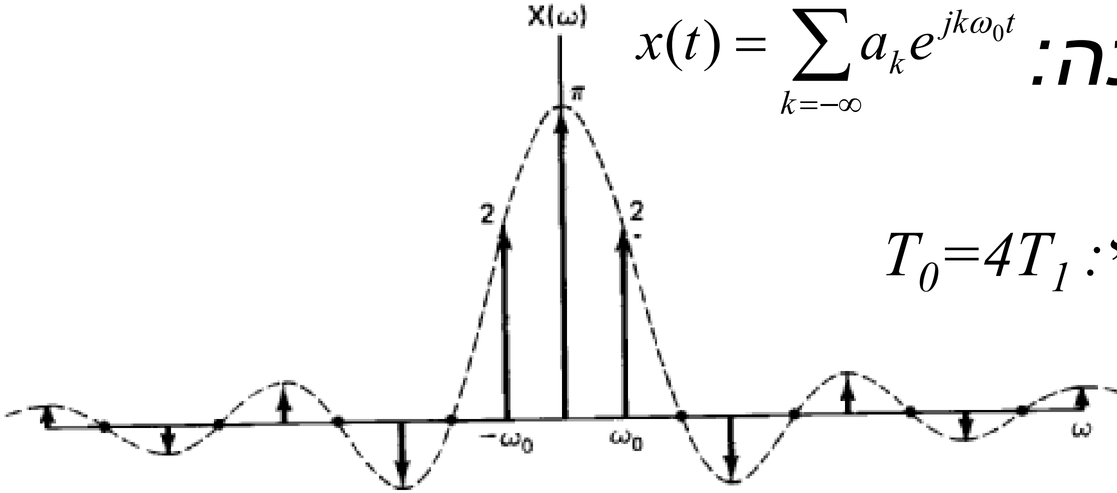


Figure 4.23 Fourier transform of a symmetric periodic square wave.

תכונות של התמרת פורייה (א)

- סימון: זוג פורייה $x(t) \xleftrightarrow{F} X(\omega)$ $\quad X(\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
- ליניאריות: $ax_1(t)+bx_2(t) \xleftrightarrow{F} aX_1(\omega)+bX_2(\omega)$
- סימטריות (עבור אותות ממשיים) $X(-\omega)=X^*(\omega)$
- הזזה בזמן: $x(t-t_0) \xleftrightarrow{F} e^{-j\omega t_0}X(\omega)$
- נגזרת: $\frac{dx(t)}{dt} \xleftrightarrow{F} j\omega X(\omega)$
- אינטגרציה: $\int_{-\infty}^{t} x(t)dt \xleftrightarrow{F} \frac{X(\omega)}{j\omega}+\pi X(0)\delta(\omega)$
- קנה מידה: $x(at) \xleftrightarrow{F} \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$
- קשר Parseval $\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$

תכונות של התמרת פורייה (ב)

- דואליות:

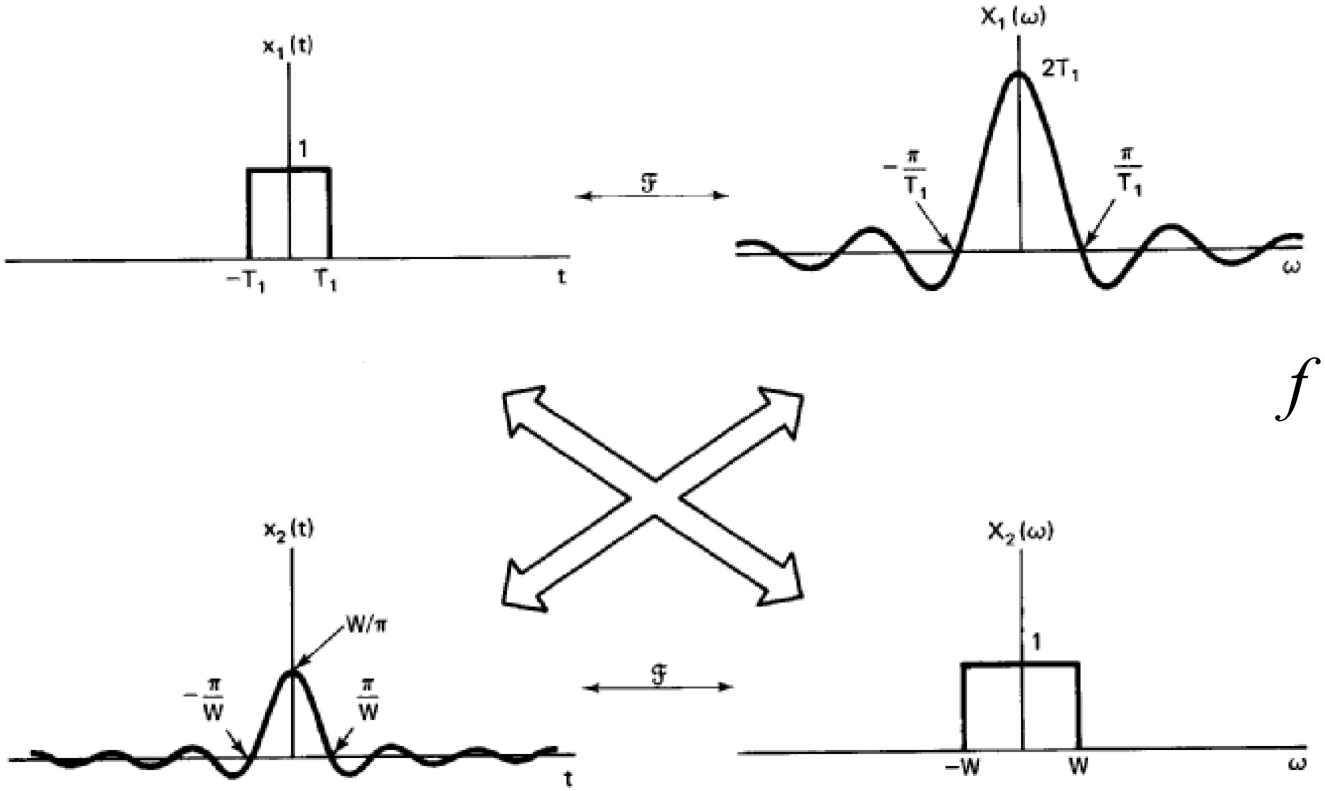


Figure 4.27 Relationship between the Fourier transform pair of eqs. (4.95) and (4.96).

$$g(t) \xleftrightarrow{F} f(\omega)$$

$$\Downarrow$$

$$f(t) \xleftrightarrow{F} 2\pi g(-\omega)$$

תכונות של התמרת פורייה (ג)
תכונת הקונבולוציה ומשמעותה

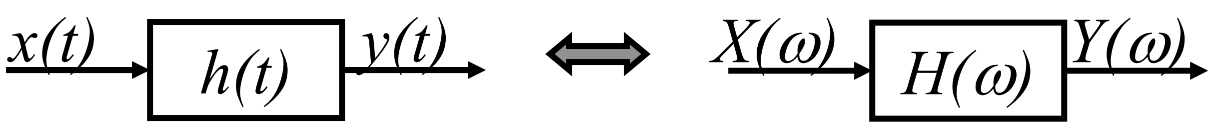
- תגובה של מערכת לינארית עם תגובת הלם $h(t)$ לכניסה $x(t)$ ניתנת ע"י קונבולוציה:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- תכונת הקונבולוציה:

$$y(t) = h(t) \otimes x(t) \xleftrightarrow{F} Y(\omega) = H(\omega)X(\omega)$$

- משמעותה: חישוב פשוט בתחום התדר



תכונות של התמרת פורייה (ג-המשך)
תכונת הקונבולוציה

- תכונת הקונבולוציה נובעת מ:
  - ייצוג $x(t)$ כגבול של טור פורייה

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)e^{j\omega t}d\omega=\lim_{\omega_0\to 0}\frac{1}{2\pi}\sum_{-\infty}^{\infty}X(k\omega_0)e^{jk\omega_0 t}\omega_0$$

  - תגובת התדירות (של פונקציות טריגונומטריות)

$$x(t)=e^{jk\omega_0 t}\longrightarrow\boxed{h(t)}\longrightarrow y(t)=H(k\omega_0)e^{jk\omega_0 t}$$

  - סופרפוזיציה של מערכות לינאריות

$$x(t)=\frac{1}{2\pi}\sum_{-\infty}^{\infty}X(k\omega_0)e^{jk\omega_0 t}\omega_0\longrightarrow\boxed{h(t)}\longrightarrow y(t)=\frac{1}{2\pi}\sum_{-\infty}^{\infty}X(k\omega_0)H(k\omega_0)e^{jk\omega_0 t}\omega_0$$

$$\Rightarrow\qquad X(\omega)\longrightarrow\boxed{H(\omega)}\longrightarrow Y(\omega)$$

$H(\omega)=$ תגובת התדירות

Discrete Fourier Transform (DFT)

- ***We start by discretizing the FT pair***
- ***The line interval*** $t \rightarrow \Delta T$
- ***The frequency interval*** $f \rightarrow \kappa \Delta T$
- ***The integral*** $\int \rightarrow \sum$

Let us analyze a signal of duration T_{total}

$$T_{total} = N\,\Delta T$$

The time interval ΔT ***is chosen according to the sampling theorem.***

The frequency interval Δf ***is chosen as the reciprocal of the analyzed signal length***

$$\Delta f = \frac{1}{N\,\Delta T}$$

Then

$$X(f) \rightarrow X(k\Delta f) = \Delta T \sum x(i\Delta T)\exp(-j\frac{2\pi}{N})^{ik}$$

$$x(t) \rightarrow x(i\Delta T) = \frac{1}{N\Delta T}\sum X(k\Delta f)\exp(j\frac{2\pi}{N})^{ik}$$

The expression is periodic for *ik* = *N*, hence the summation for $X(k\,\Delta f)$ and $x(i\,\Delta T)$ are periodic within N.

We thus limit the summation to N samples, resulting in the DFT pair

$$X(k\Delta f) = \Delta T \sum_{o}^{N-1} x(i\Delta T)\exp(-j\frac{2\pi}{N})^{ik} \quad E.U.\ X(kf)[V-\sec]$$

$$x(i\Delta T) = \frac{1}{N\Delta T} \sum_{o}^{N-1} X(d\Delta f)\exp(j\frac{2\pi}{N})^{ik} \quad E.U.\ X(k)[V]$$

A normalized DFT pair, for $\Delta T = 1$***, is usually computed by most procedures***

$$X(k) = \sum_{o}^{N-1} x(i) W_N^{ik}$$

$$x(i) = \frac{1}{N} \sum_{o}^{N-1} X(k) W_N^{-ik}$$

with the compact notation

$$W_N = \exp(-j \frac{2\pi}{N})$$

The DFT is a transform between two sequences of N samples

$$X(k) \leftrightarrow x(i)$$

In general it is a transform between two complex sequences

The frequency scale

a) $k = 0\,,\ \ f = k\,\Delta f = 0$

This is the zero (DC) frequency. Here

$$X(0) = \sum_{o}^{N-1} x(i)$$

and equals N times the average of x(i)

b) $k = \dfrac{N}{2} \quad , \quad f = \dfrac{N}{2}\Delta f = \dfrac{N}{2}\dfrac{1}{N\Delta T} = \dfrac{1}{2\Delta T} = f_{NYQUIST}$

c) From the periodicity of $\exp\left(j\,2\pi/N\right)$, and hence the periodicity of *x(i)* and *X(k)*, we may define negative indices, and

$$X(-k) = X(N-k)$$

The upper part of X(k), with *k>N/2*, may thus be interpreted as transforms for negative frequencies.

The DFT of real signals.

- ***For most engineering measurements, x(t) and hence x(i) is real. From***

$$x(i) = \frac{1}{N}\sum_{o}^{N-1} X(k)\exp(j\frac{2\pi}{N})^{ik}$$

it follows that $X(k)$ must satisfy specific conditions in order to have the summation result in real samples. These conditions are one of a specific symmetry, with

$$X(-k) = X(k)^*$$

thus

$$|X(-k)| = |X(k)|$$

and

$$\arg[X(-k)] = -\arg[X(k)]$$

b) The necessary

$$\Delta f = 2 \quad [Hz]$$

$$N = \frac{1}{\Delta f \times \Delta T} = 750$$

and we may choose

N = 512 resulting in

$$\Delta f = 2.44 \quad [Hz]$$

or

N = 1024 resulting in

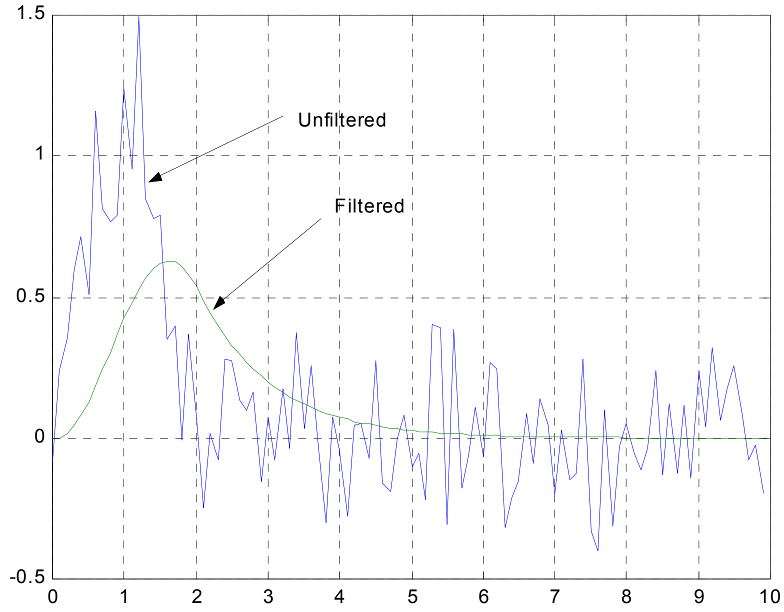
$$\Delta f = 1.22 \quad [Hz]$$

Filtering

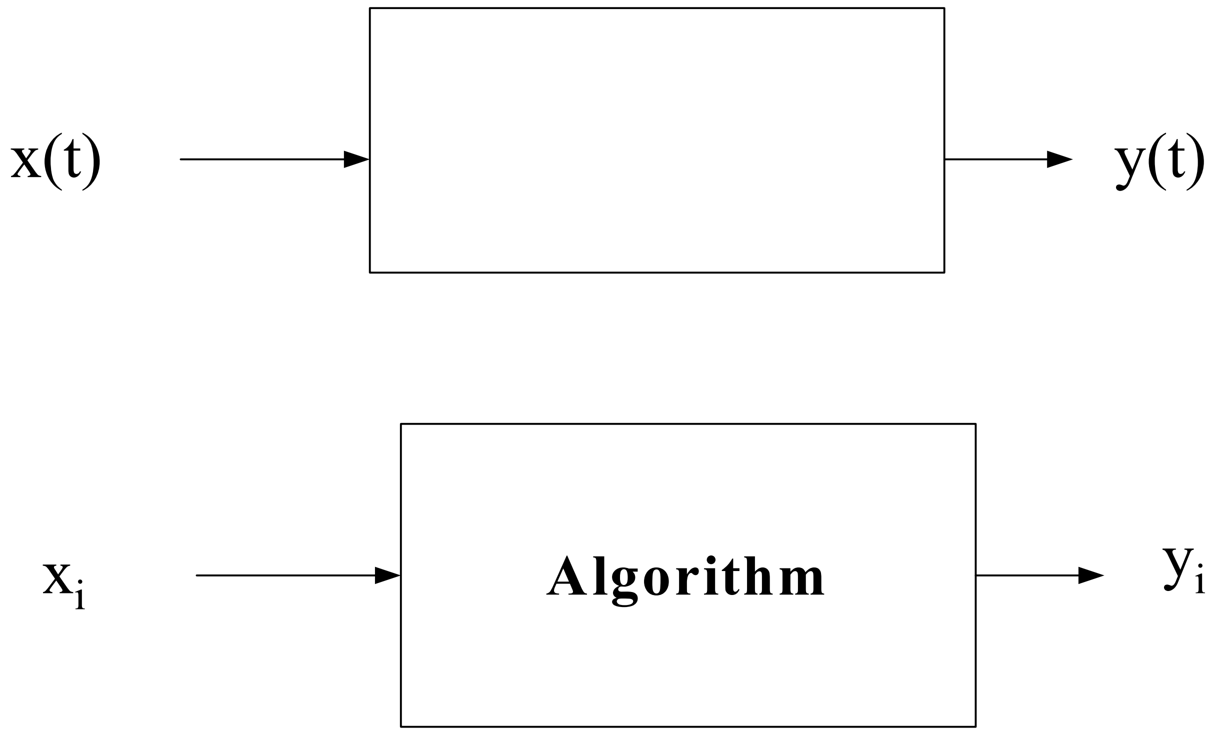
- *Manipulate the spectrum of the signal*
  - *Change relative amplitudes*
  - *Pass signals in one or a set of freq bands and attenuate or signals in remaining freq. bands*
- *Can be accomplished through the use of a linear time-invariant system (with an appropriately chosen FRF)*

Filters

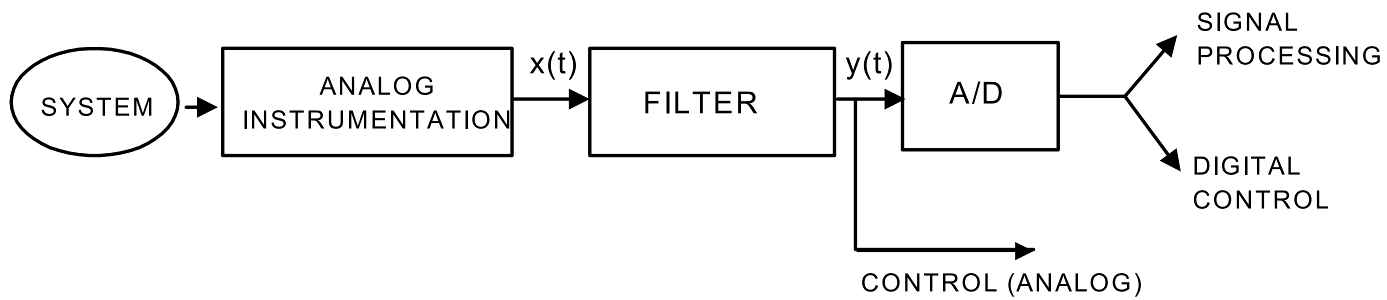
Reject/pass signal components according to frequency

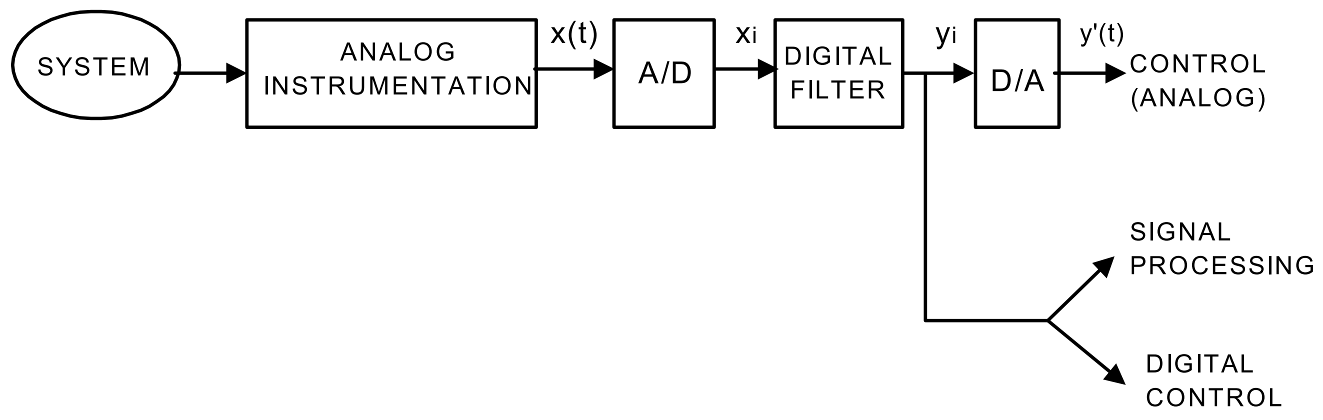


Algorithm as system



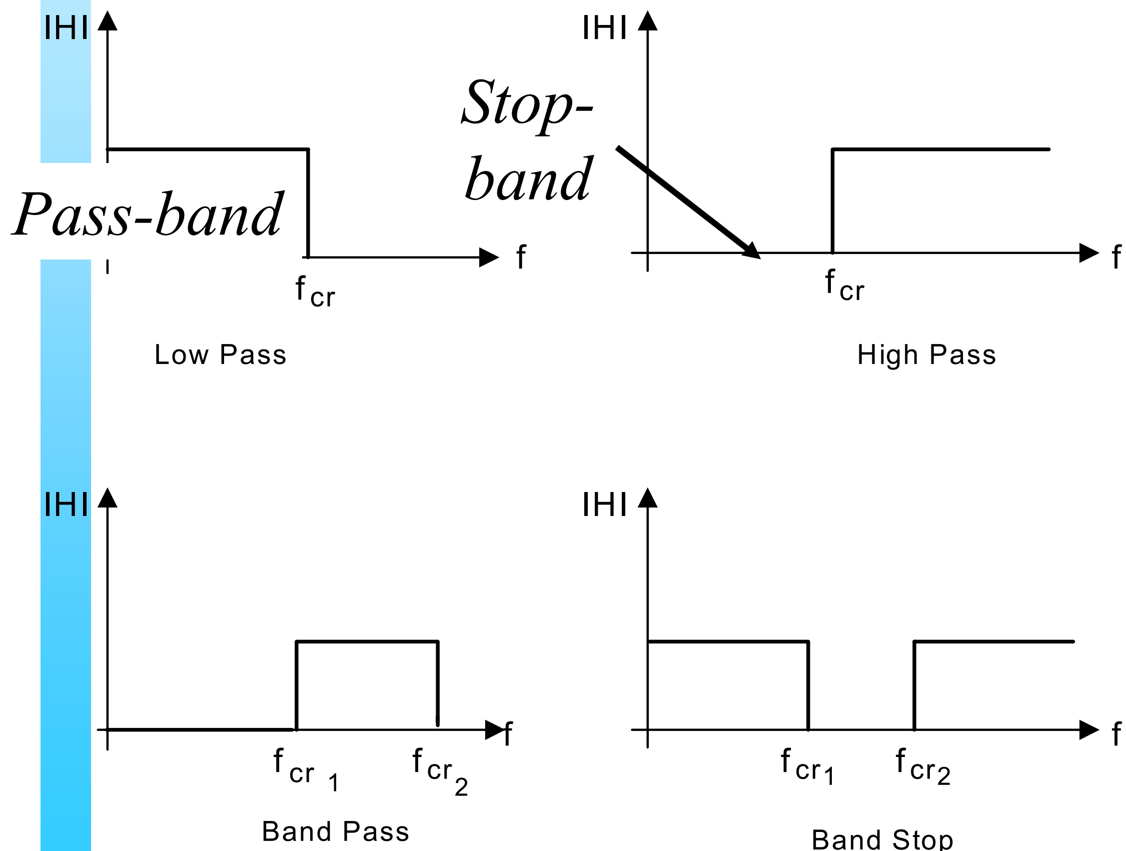
Anti-aliasing filters





Classification – ideal (continuous) filters

Note: The FRF is symmetric around f=0
Discrete time filters: periodic with period = 2π



Phase characteristics:

1. *IDEAL: Zero (i.e., FRF is real and non negative)*
2. *Acceptable "distortion": linear phase → time shift*

$$x(t-t_0) \leftrightarrow \exp(-i\omega t_0)X($$

Ideal Freq selective filters – time domain characteristics

- *Ideal low-pass filter; pass band* $[-\omega_c, \omega_c]$
  - *Impulse response (zero phase):*

$$h_{ip}(t) = \frac{\omega_c}{\pi} \sin c\left(\frac{\omega_c t}{\pi}\right)$$

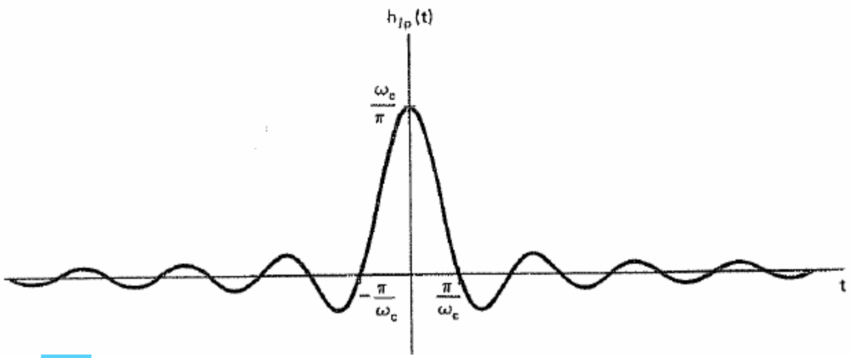
  - *Impulse response (linear phase)*

$$h_{ip}(t) = \frac{\omega_c}{\pi} \sin c\left(\frac{\omega_c (t - t_0)}{\pi}\right)$$

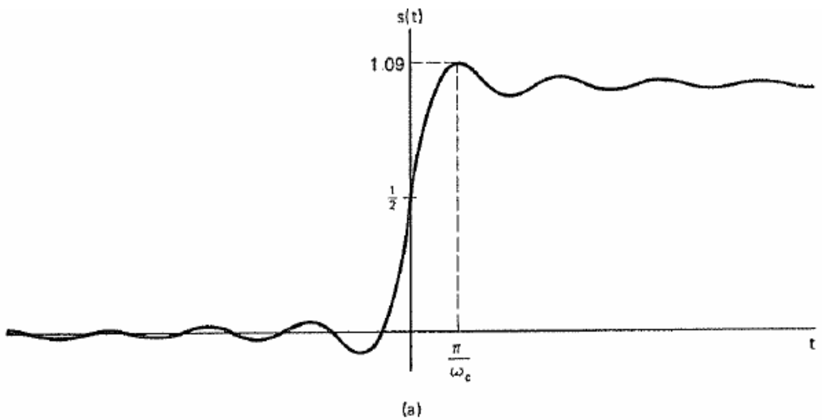
  - *Step response: overshoot and ringing*
- *Ideal freq characteristics → unacceptable time domain response*

Ideal Freq selective filters – time domain characteristics

- Ideal freq characteristics →
  - Unacceptable time domain response
  - Non-causal and hence unattainable in real time.
  - Complicated implementation



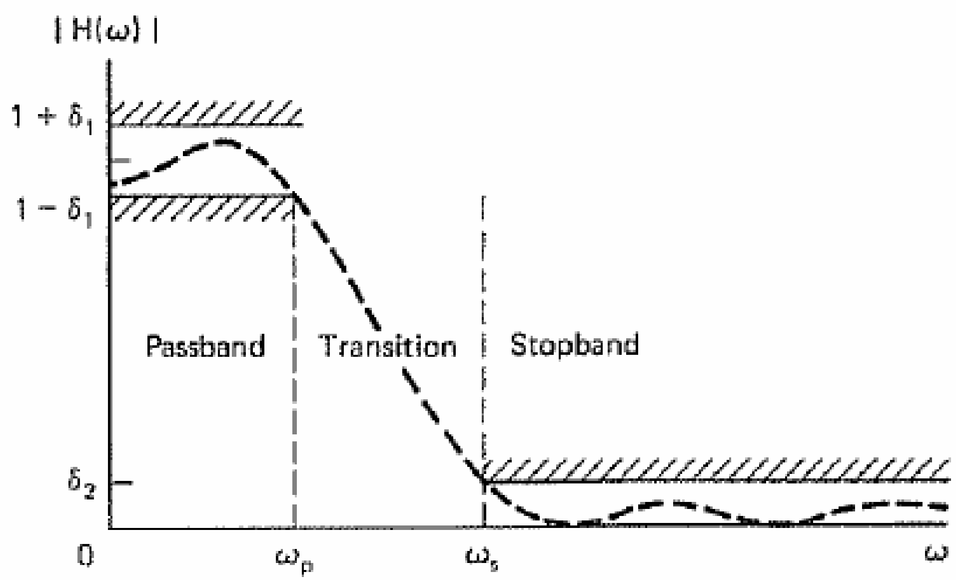
Impulse response of ideal low-pass filter



Step response of ideal low-pass filter

NonIdeal Freq selective filters – freq domain characteristics

- *Non-ideal freq domain characteristics:*
  - *Non-unity gain in pass-band*
    - *pass-band ripple*
  - *Non-zero gain in stop-band*
    - *stop-band ripple*
  - *Transition band: pass-band edge, stop-band edge*



NonIdeal Freq selective filters – example

- *Example: automotive suspension system*
  - *No sharp division between pass and stop band*
  - *Time domain response important*
  - *Cost and ease of implementation also important*

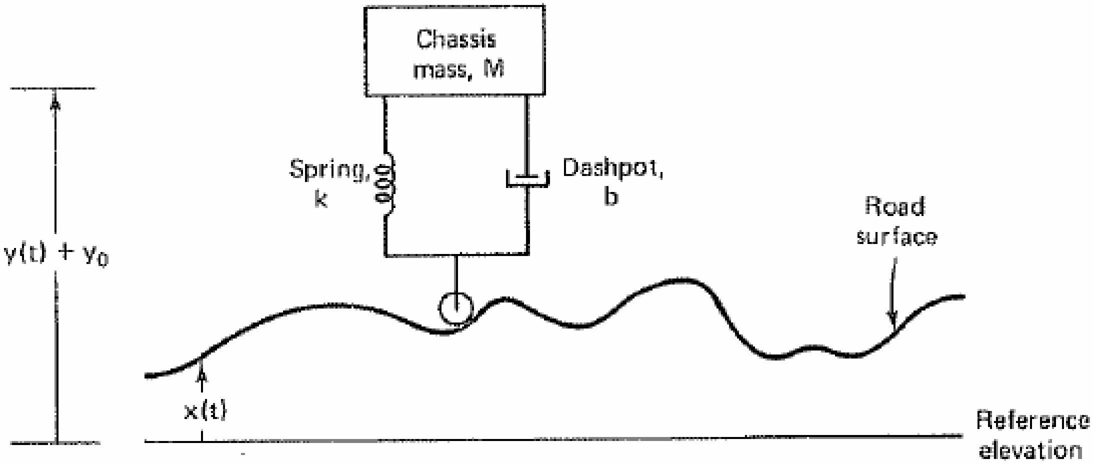
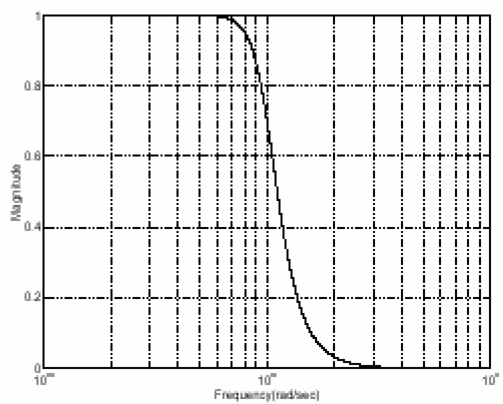
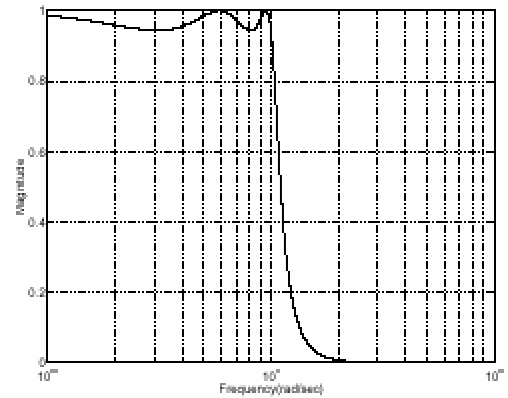
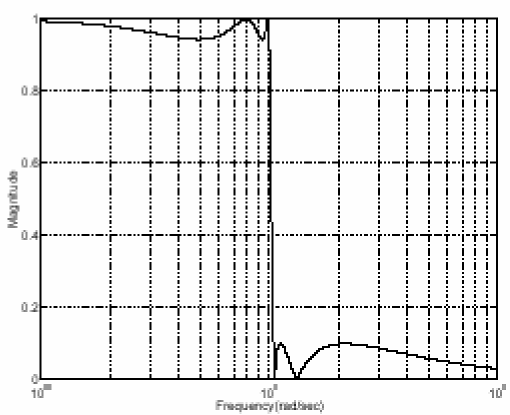


Figure 6.14 Diagrammatic representation of an automotive suspension system. y_0 represents the distance between the chassis and the road surface when the automobile is at rest, $y(t) + y_0$ the position of the chassis above the reference elevation, and $x(t)$ the elevation of the road above the reference elevation.

Filter Specifications







1. Slope

2. Attenuation

3. Ripple in pass (stop) band

Discrete time freq selective filters described by Diff Eq.

- *Non-recursive diff eq. → FIR*
  - *Always stable*
  - *Can have linear phase*
  - *Higher order usually needed*

$$y_i = \sum_{n=0}^{M} b_n x_{i-n}$$

- *Recursive diff eq. → IIR*
  - *Facilitate low-order freq implementation*

$$y_i = \sum_{n=1}^{N} -a_n y_{i-n} + \sum_{n=0}^{M} b_n x_{i-n}$$

Discrete time freq selective filters Alternative descriptions

- Difference eq. $$y_i = \sum_{n=1}^{N} -a_n y_{i-n} + \sum_{n=0}^{M} b_n x_{i-$$

- The z-transform $$Y(z) \equiv \sum_{k=-\infty}^{\infty} y_k z^{-k}$$

- The Fourier Transform is the z-transform for

$$z = \exp(j\omega T) \qquad Y(\omega) = Y(z)\Big|_{z=\exp(j\omega T)} = \sum_{k=-\infty}^{\infty} y_k e^{-jk\omega T}$$

$$Y(z) = [-a_1 z^{-1} - a_2 z^{-2} - \ldots a_N z^{-N}]Y(z) + [b_0 + b_1 z^{-1} + \ldots\ldots b_M z^{-M}]\, X($$

Discrete time freq selective filters Transfer Function

- *Difference eq.*

$$y_i = \sum_{n=1}^{N} -a_n y_{i-n} + \sum_{n=0}^{M} b_n x_{i-}$$

- *The z-transform*

$$Y(z) = [-a_1 z^{-1} - a_2 z^{-2} - \ldots a_N z^{-N}]\,Y(z) + [b_0 + b_1 z^{-1} + \ldots\ldots b_M z^{-M}]\,X($$

- *Transfer Function*

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} \ldots. + b_M z^{-M}}{1 + a_1 z^{-1} + \ldots\ldots\ldots a_N z^{-N}}$$

$$H(\omega) = H(z)\Big|_{z=\exp(j\omega T)}$$

Non- recursive Discrete time filters

- *Low pass filtering: Smoothing operation*
- *Discrete time Smoothing:* $y_i = \sum_{n=0}^{M} b_n x_{i-n}$

moving average (MA)

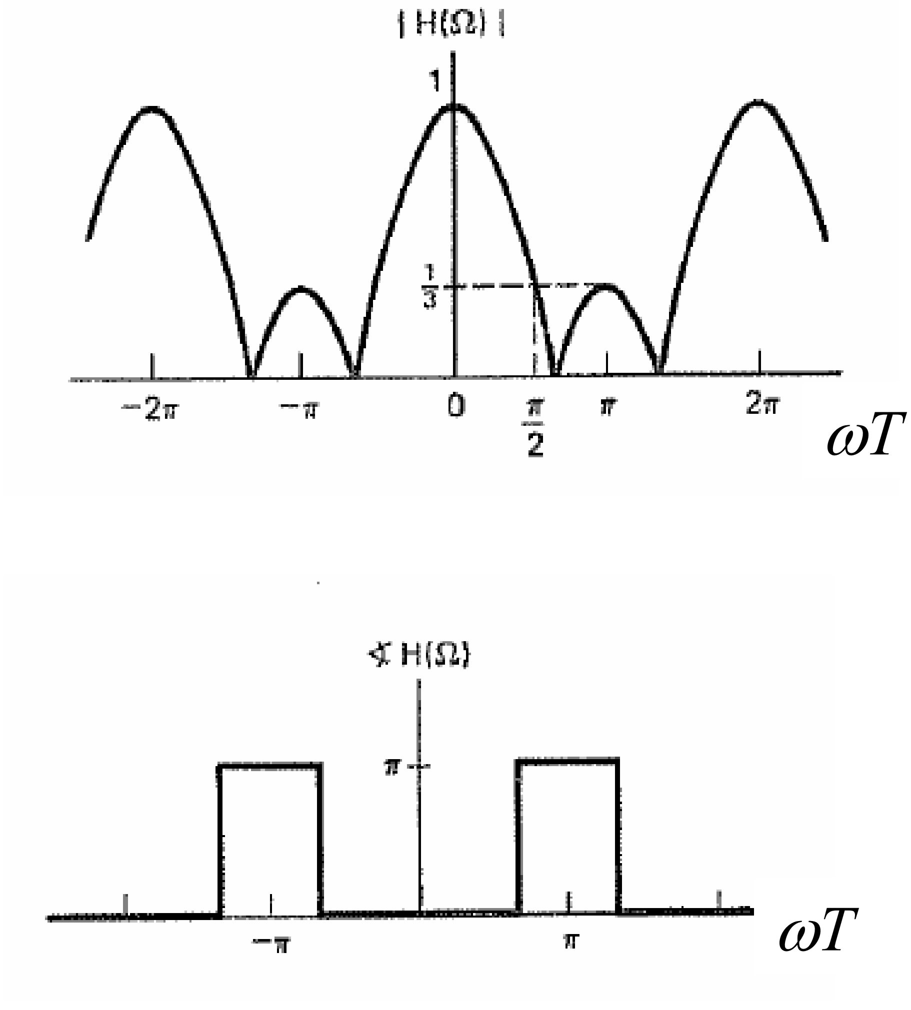
- *Example: 3-point MA* $y_i = 1/3(x_{i-1} + x_i + x_{i+1})$
  - *Impulse response: rectangle [.... 0 1 1 1 0 ....]*
  - *Transfer function (FT of impulse response or:*

$$Y(z) = 1/3(z^{-1} + 1 + z)X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = 1/3(z^{-1} + 1 + z)$$

$$H(\omega) = H(z)\Big|_{z=\exp(j\omega T)} = 1/3(1 + 2\cos(\omega T))$$

Freq. response of 3-point low-pass moving average



Non- recursive Discrete time filters General-length moving average

- Generalization: N+M-1 neighboring points

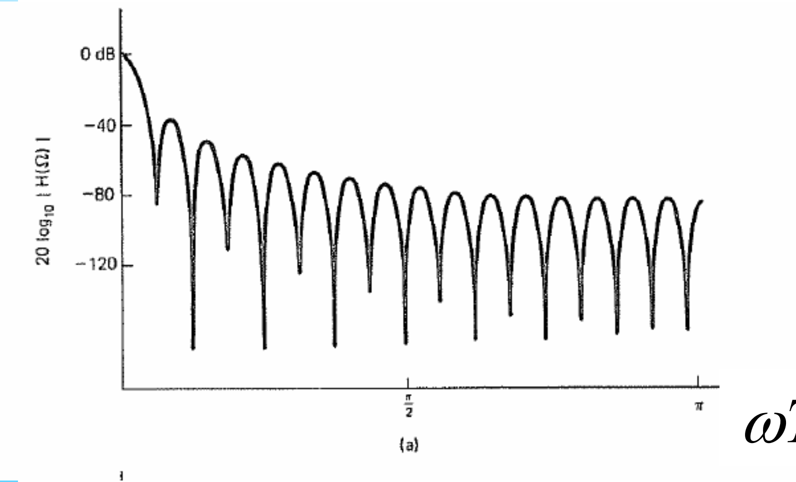
$$y_i = \frac{1}{N+M-1}\sum_{k=-N}^{M} x_{i-k}$$

  - Impulse response is rectangular
  - If $N \le 0$ the filter is causal

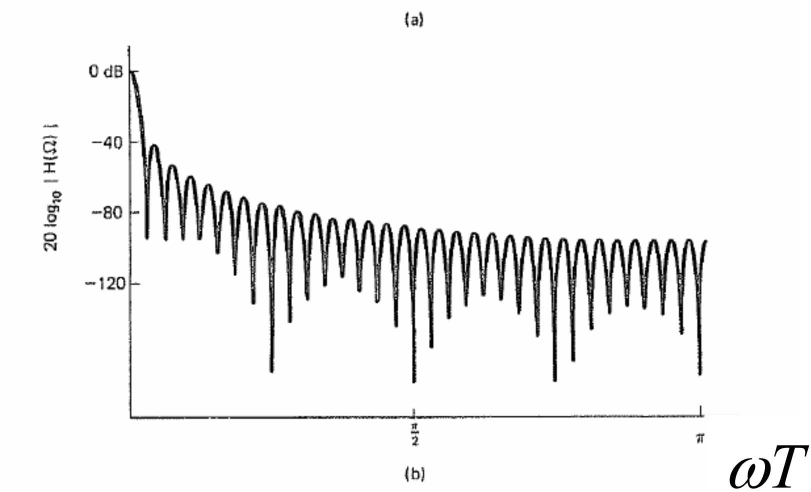
$$H(z) = \frac{1}{N+M-1}\sum_{k=-N}^{M} e^{-jk\omega T}$$

$$= \frac{1}{N+M-1} e^{j[(N-M)/2]\omega T} \frac{\sin[\omega T(\frac{N+M-1}{2})]}{\sin[\omega T/2]}$$

Fewq. Response of M-N low-pass moving average



ωT



ωT

M=N=16

M=N=32

Non- recursive Discrete time filters weighted moving average

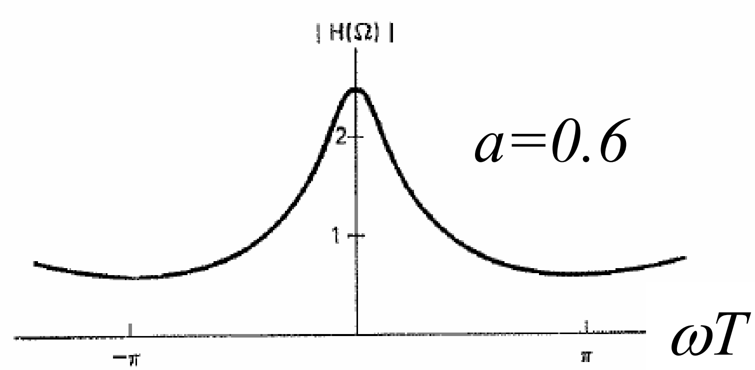
- *Generalization: weighted average over N+M-1 neighboring points*

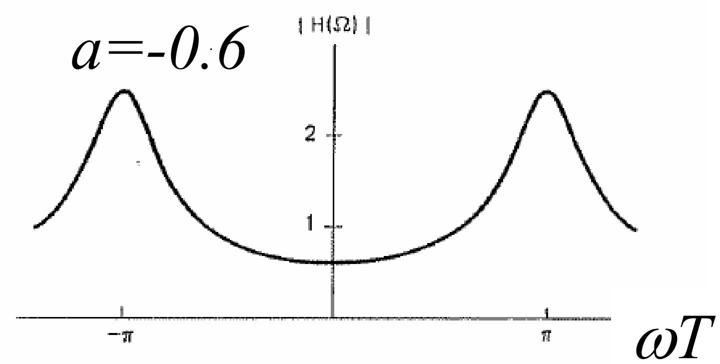
$$y_i = \sum_{k=-N}^{M} b_k x_{i-k}$$

  - *General non-recursive filter*
  - *Coeff bk are selected to achieve prescribed filter characteristics*

Recursive Discrete time Filters – simple example

- *Simple example* $y_i = a y_{i-1} + x(i)$
  - *Transfer function* $$H(\omega) = \frac{1}{1 - a\exp(-j\omega T)}$$
  - *Behaves like low/high pass for positive/negative a*





  - *Higher order recursive diff eq → provide*
    - *sharper filter characteristics*
    - *Flexibility in balancing time domain and freq domain requirements*

Butterworth Frequency Selective Filters – transfer function

- *Recall: The Transfer function of a (continuous) first order system (with time constant tau)*

$$H(\omega) = \frac{1}{j\omega\tau + 1}$$

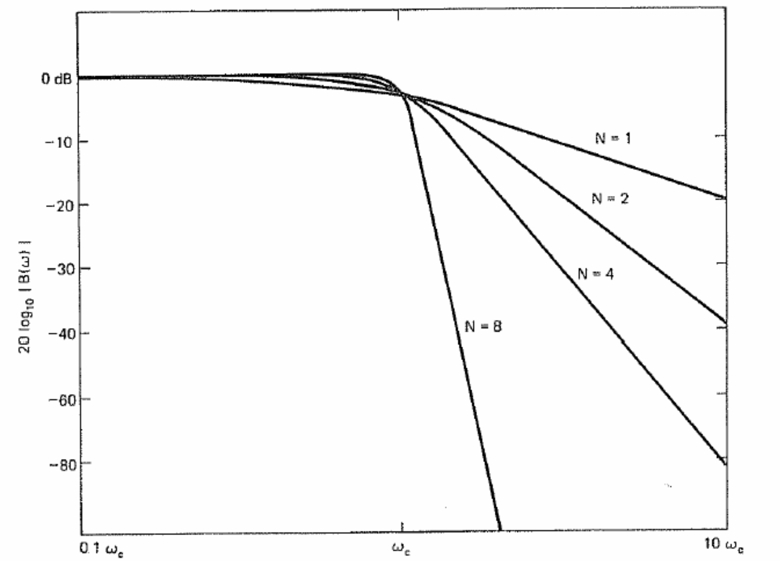
- *Class of Butterworth filters: The slope is controlle by the power of ω in the denominator*

$$\left|B(\omega)\right|^2 = \frac{1}{1 + (\omega / \omega_c)^{2N}}$$

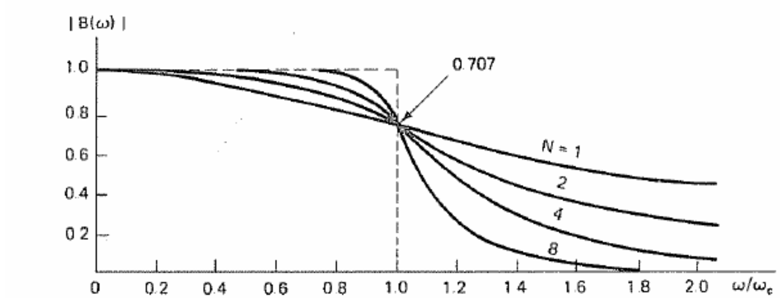
N=filter order (order of the associated diff eq)

Gain of Butterworth filters

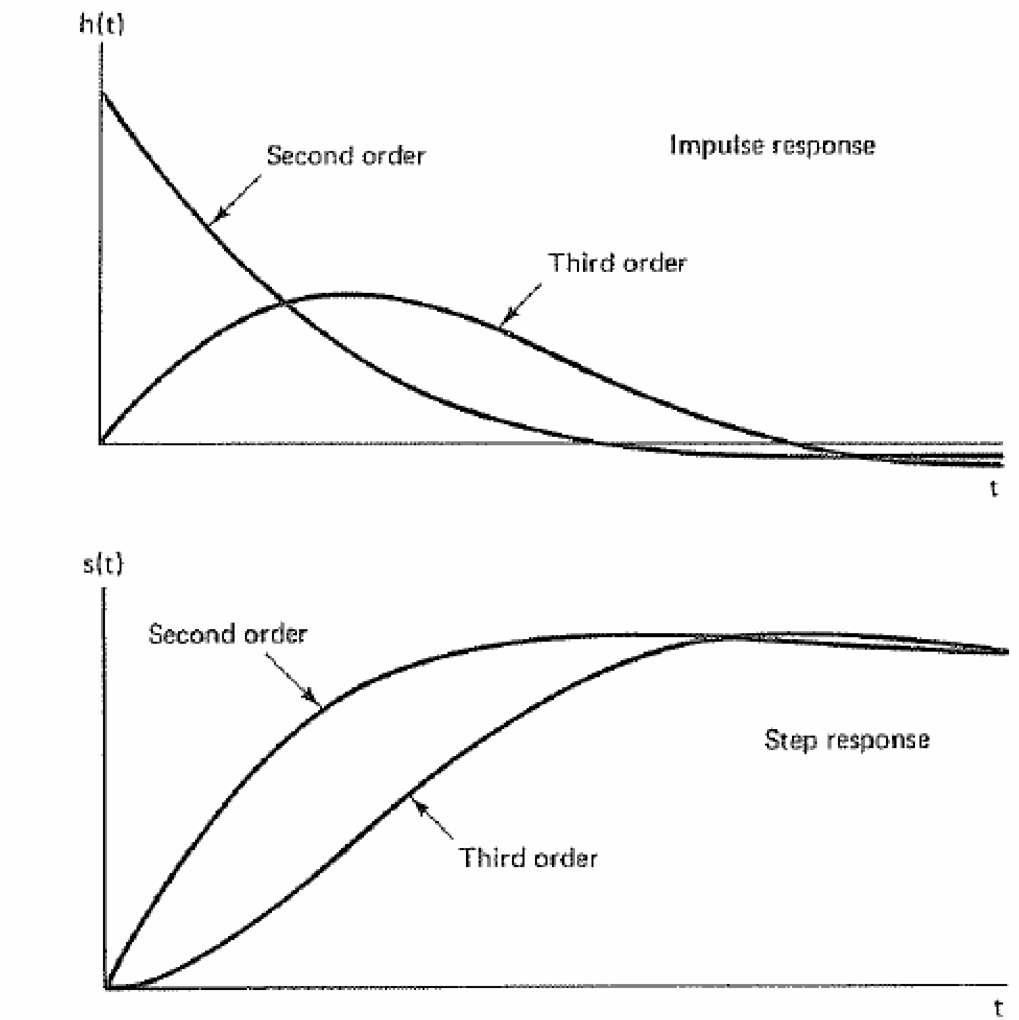
Log scale



Linear scale



Impulse and step response of Butterworth filters



Butterworth Frequency Selective Filters - poles

- Using the Laplace Transform

$$|B(\omega)|^2 = B(j\omega)B^*(j\omega) = B(j\omega)B(-j\omega) = \frac{1}{1+(j\omega/j\omega_c)^{2}}$$

$$\downarrow$$

$$B(s)B(-s) = \frac{1}{1+(s/j\omega_c)^{2N}}$$

- The roots of the denominator $\quad s_p = (-1)^{1/2N}(j\omega$

$$s_p = \omega_c \exp\left\{ j\left[\frac{\pi}{2} + \frac{\pi(2k+1)}{2N}\right] \right\}$$

$$|s_p| = \omega_c$$

$$\angle s_p = \frac{\pi}{2} + \frac{\pi(2k+1)}{2N}$$

Maximal flatness

$$G^2(\omega) = |H(j\omega)|^2 = \frac{G_0^2}{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}$$

- *Gain:*
- *Derivative of the gain with respect to frequency:*

wc=1; G0 = 1

$$\frac{dG}{d\omega} = -nG^3\omega^{2n-1}$$

  - *monotonically decreasing for all ω (G>0)→ no ripple.*
- *The series expansion of the gain is given by:*

wc=1; G0 = 1

$$G(\omega) = 1 - \frac{1}{2}\omega^{2n} + \frac{3}{8}\omega^{4n} + \ldots$$

  - *All derivatives of the gain up to but not including the 2n th derivative are zero (at w=0), resulting in "maximal flatness".*

High Frequency Roll Off

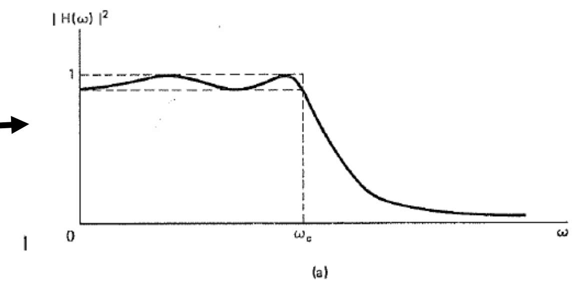
- *The slope of the log of the gain for large ω is:*

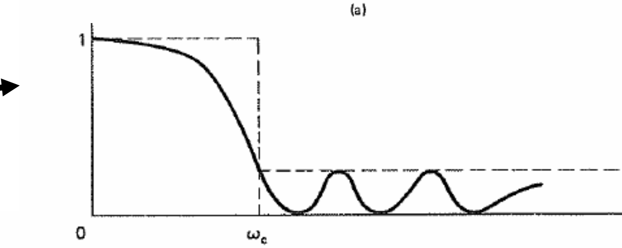
$$\lim_{\omega \to \infty} \frac{d \log(G)}{d \log(\omega)} = -n$$

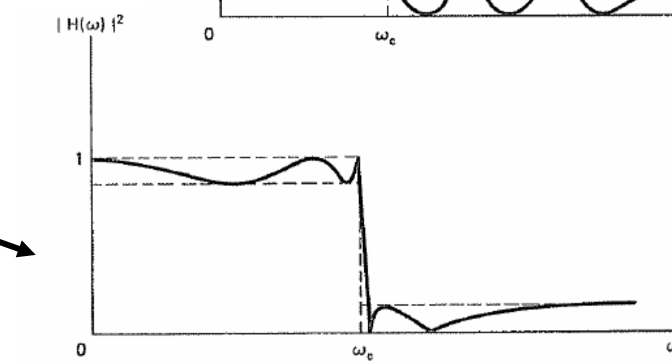
- *The high frequency roll off is therefore 20n dB/decade.*

Filter families

- *Specified by certain design criteria which give general rules for specifying the transfer function of the filter:*
  - *Butterworth filter - no gain ripple in pass band and stop band, slo cutoff (shown previousely)*
  - *Chebyshev filter(Type I) –*
    no gain ripple in stop band, moderate cutoff
  - *Chebyshev filter(Type II) –*
    no gain ripple in pass band, moderate cutoff
  - *Elliptic filter –*
    gain ripple in pass and stop band, fast cutoff
  - *Others*



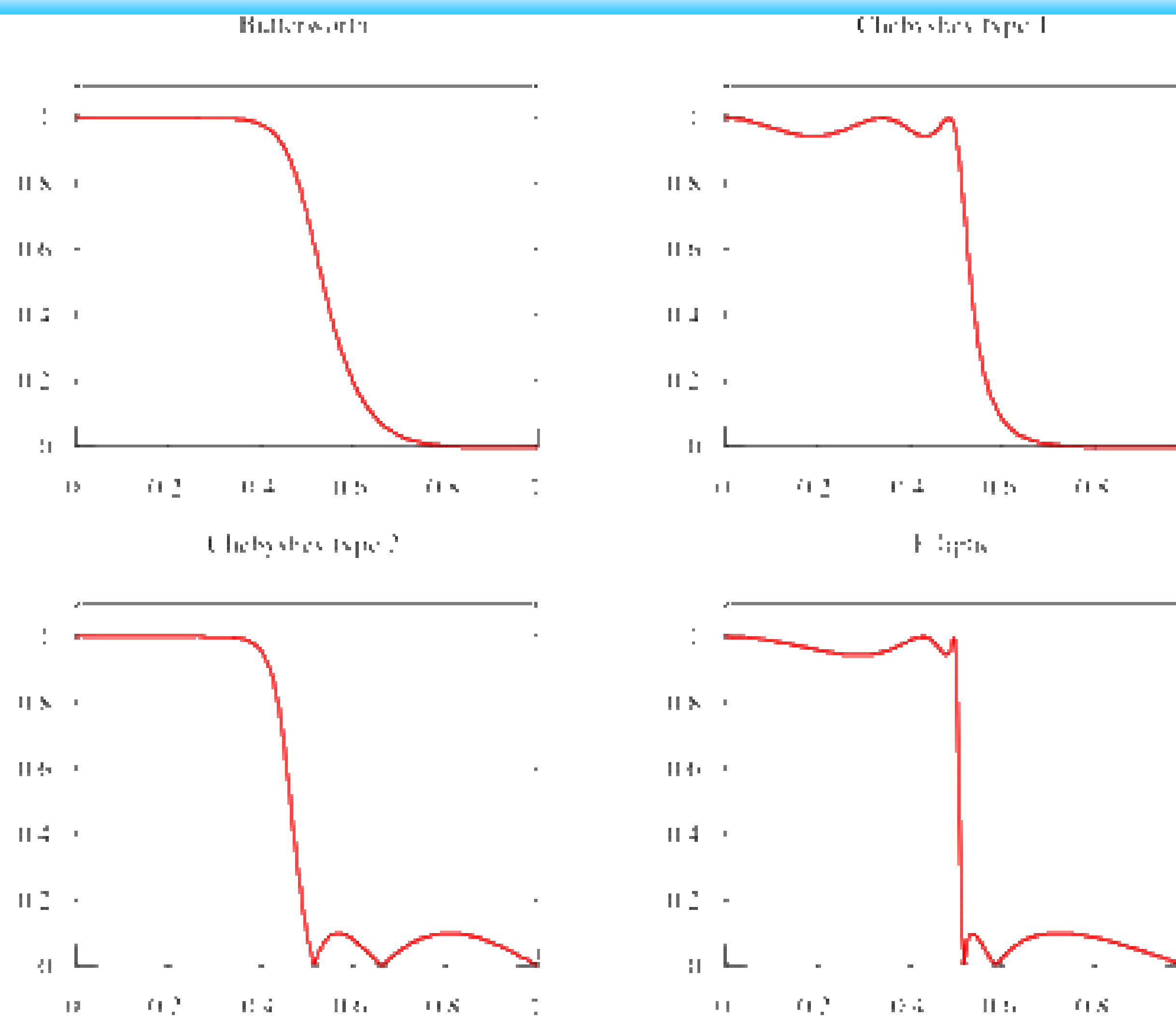




Comparison (All filters are fifth order)

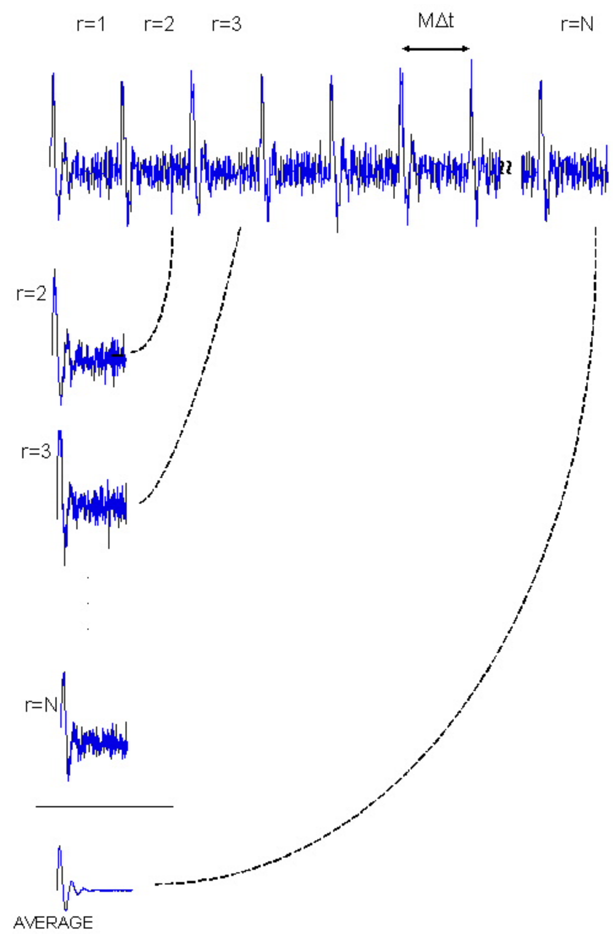
All filters are of 5th order, → all filters roll off by 5 20=100 dB per decade.*

The Butterworth filter rolls off more slowly around the cutoff frequency than the others,
but shows no ripples



Time Domain Averaging

$$y(n\Delta t) = \frac{1}{N}\sum_{r=0}^{N-1} x(n\Delta t - rM\Delta t)$$



Formulation as a Linear filter:
r – the running discrete index
y – the output

$$y_r(n\Delta t) = y_{r-1}(n\Delta t) + \frac{x_r(n\Delta t) - y_{r-1}(n\Delta t)}{r}$$

TDA – Freq response characteristics

- *Zeros for* $\omega/\omega_p = k/N \quad k = 1,2,\ldots,N/2$
- *For Large (N>10) and even N:*
  - *Approximate maxima at*

$$\omega/\omega_p = (k-0.5)/N \quad k = 1,2,\ldots,N/2$$

  - *Corresponding maxima (as the nominator is 1)*

$$\left|H_{side-lobe\ peaks}\right| = \left[N\sin(\pi\omega/\omega_p)\right]^{-1}$$

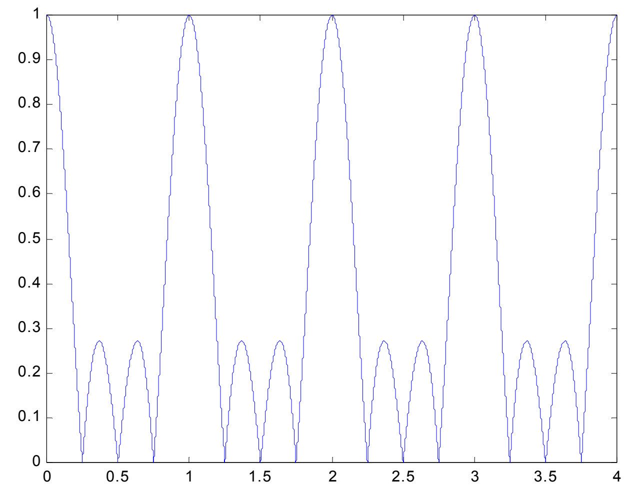
    - → *attenuation increases with N*

$$H(z)=\frac{1}{N}\frac{1-z^{-MN}}{1-z^{-N}}$$

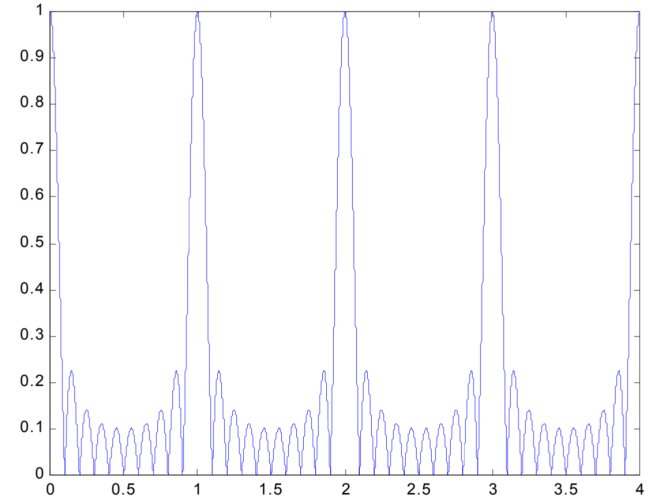
$$H(f/f_p)=\frac{1}{N}\frac{\mathrm{Sin}(\pi N f/f_p)}{\mathrm{Sin}(\pi f/f_p)}$$

$$\phi_H(f/f_p)=-\pi(N-1)f/f_p$$

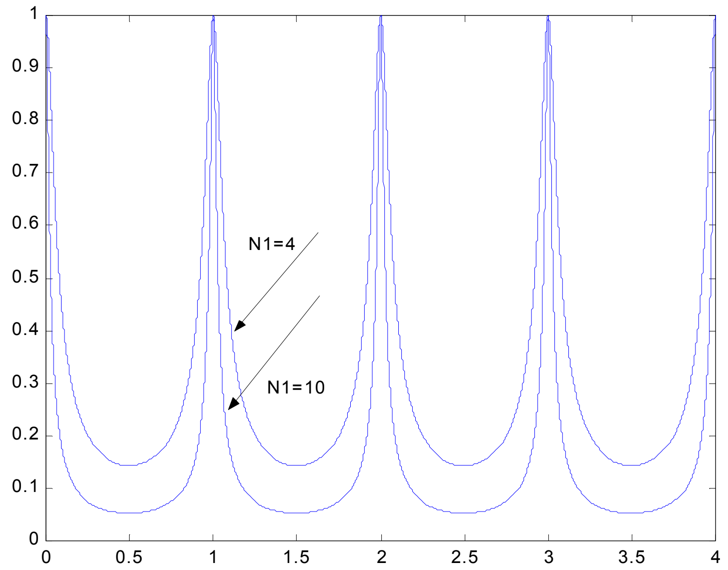
$$f_p=\frac{1}{MN\Delta t}$$



N=4



N=10



$$y_r(n\Delta t) = y_{r-1}(n\Delta t) + \frac{x_r(n\Delta t) - y_{r-1}(n\Delta t)}{N_1}$$

$$y_r(n\Delta t) = y_{r-1}(n\Delta t) + \frac{x_r(n\Delta t) - y_{r-1}(n\Delta t)}{r}$$

Spectral Analysis

Rationale

* Physical insight
* Orthogonality (no crossproducts)
* Pattern recognition
* Algebraic (closed form) solutions, as with many transform based methosds

Frequency domain presentation

Depends on signal class

* Transient
* Periodic
* Random

All computations via basic FFT

For a signal

$$x(i\Delta t) \qquad\qquad i=0,1\ldots\ldots N$$

We have computed the FFT

$$X(k\Delta f) = \sum_{i=0}^{N-1} x(i)\exp(-j\frac{2\pi}{N})^{ik} \qquad k=0,1\ldots\ldots N\text{-}1$$

Frequency scale

$$\Delta f = 1/(N\Delta t)$$

$$f(k) = k\Delta f \qquad\qquad k=0,1\ldots\ldots N\text{-}1$$

Transients:

The DFT approximates samples of the continuous FT.

For EU

$$X_{EU}\ (k\Delta f) = \Delta T\ \ X(k\Delta f) \quad [V - \sec]$$

Periodic signals:

$$X_{DFS}(f_k) = \frac{1}{N} X(k) \qquad [V]$$

The computation equals that of the DFT except for the factor of N. The results are directly in [Volts]

Assume that p integer periods are spanned by the signal length $N\Delta t$. Denoting the number of samples in the period by M

$$T_p = M\Delta t$$

with T_p the actual period of the physical signal. The total signal length

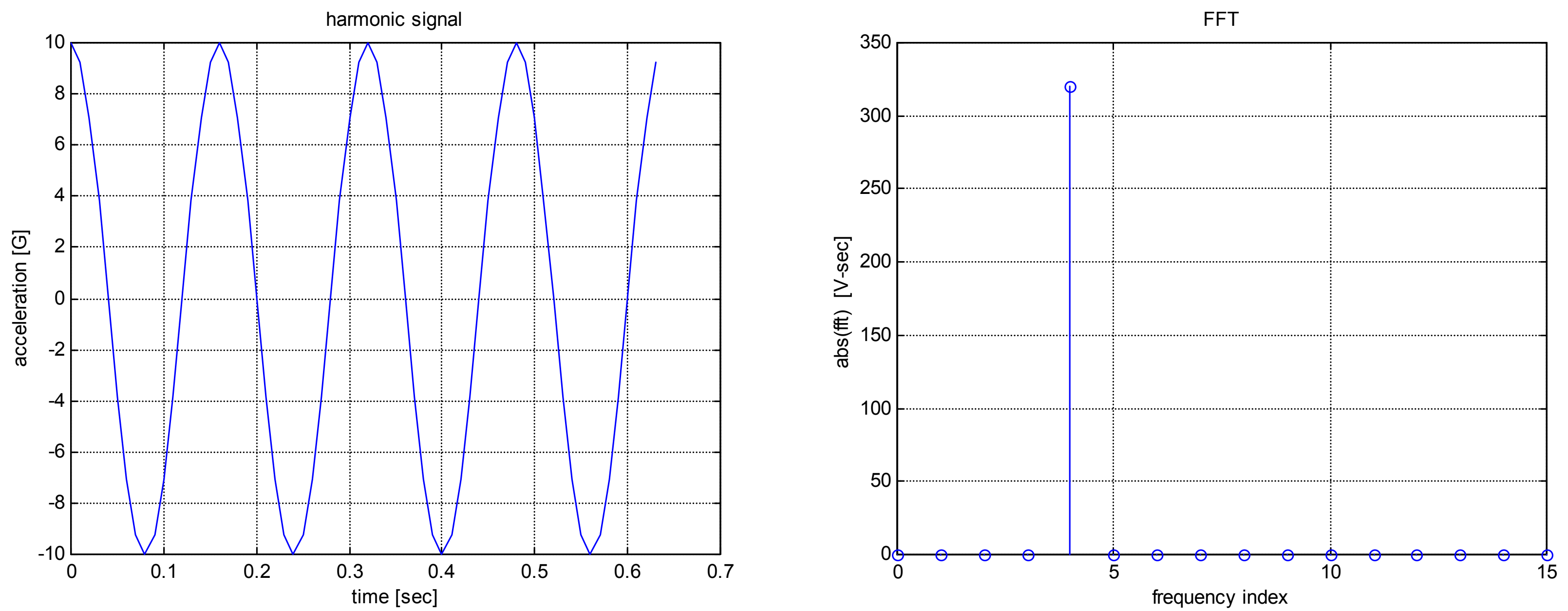
$$N\Delta t = pM\Delta t$$

and the location of f_p on the frequency scale is

$$k\Delta f = f_p = 1/(M/\Delta t)$$

$$k=1/(M \Delta t \Delta f)=N/M=p$$

when an integral number of periods are spanned , the physical frequency coincide with one of the frequencies at which the DFS is computed.

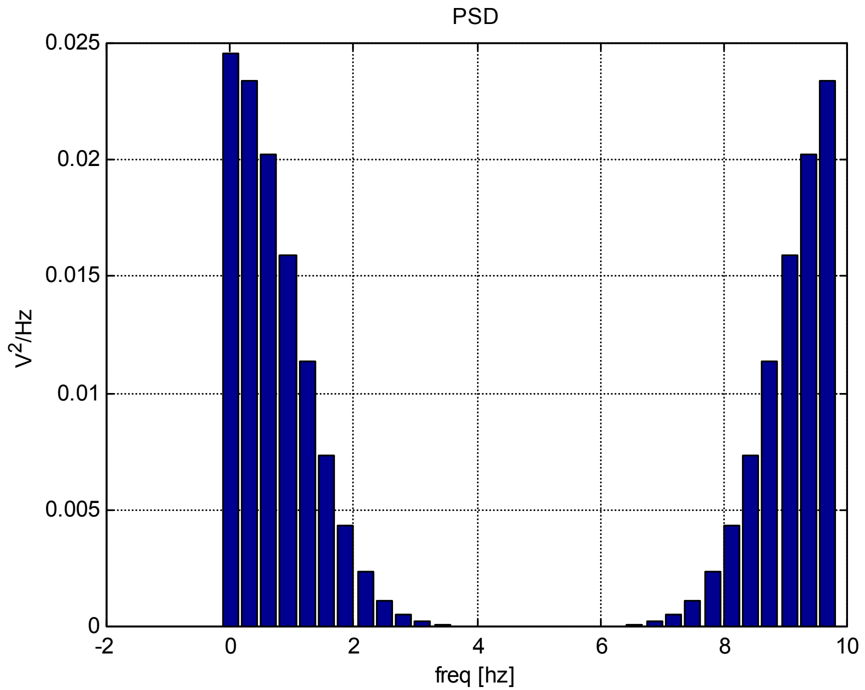


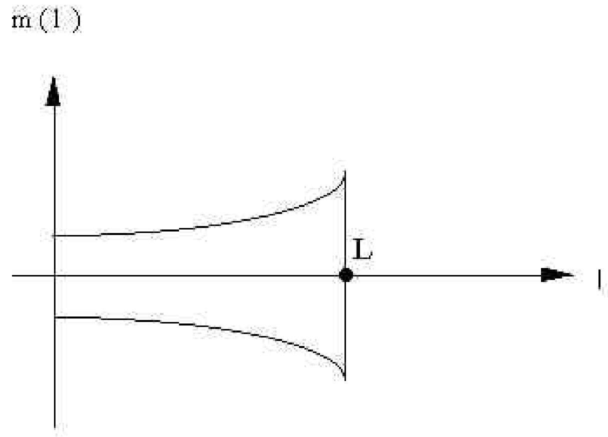
Signal Processing: Spectral Analysis

Random continuous signals:

$$P = \int_{-\infty}^{\infty} S(f)\,df$$

$$P_{f_1-f_2} = \int_{f_1}^{f_2} S(f)\,df \qquad\qquad S(k\Delta f) = \frac{\Delta T}{N}\left|X(k\Delta f)\right|^2$$




m(l)
L
l

Spectral Analysis
Engineering Units

- x( ) en Volts

Power	**PSD**	**Energy**	**ESD**
V^2	V^2/Hz	$V^2.sec$	$V^2.sec/Hz=V^2.sec^2$

- Voltage units

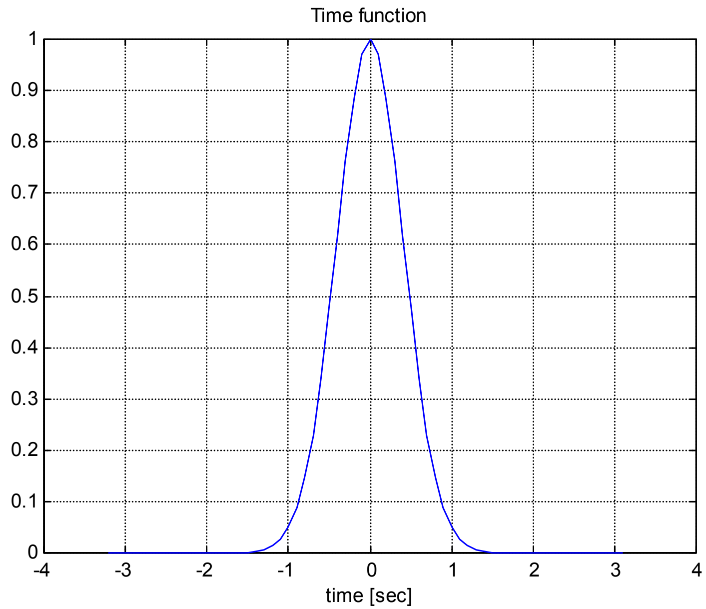
V	$V/\sqrt{Hz}$	$V.\sqrt{sec}$	V.sec

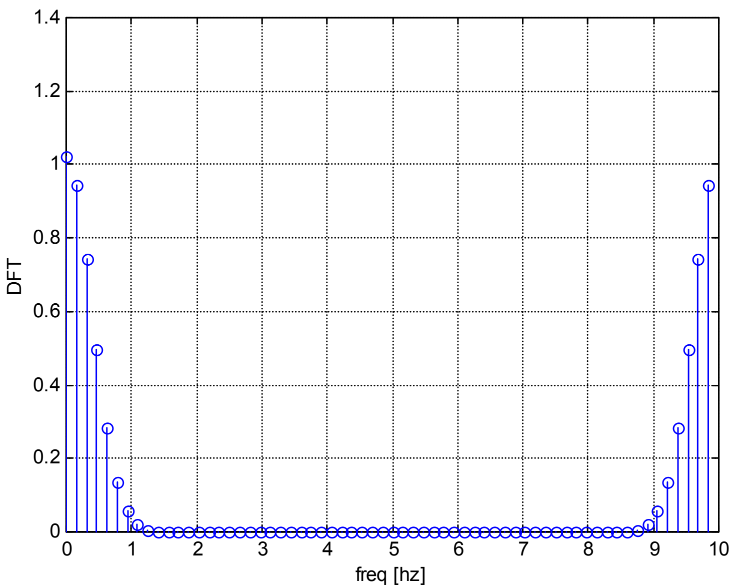
Frequency scales, one and two sided presentations :

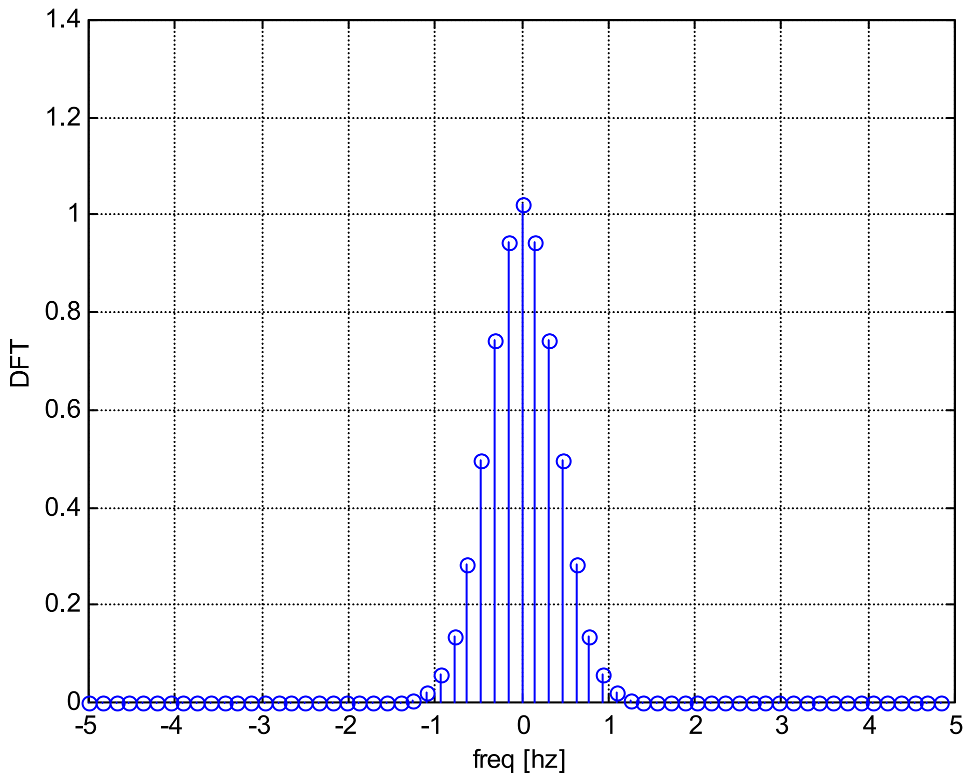
The (computational) resolution is $\Delta f=1/(N/\Delta t)$, the frequency scale is $f(k) = k\Delta f$

the locations $k = N-1, N, \ldots\ldots. N/2$

correspond to negative frequencies according to: $X(-k)=X(N-k)$



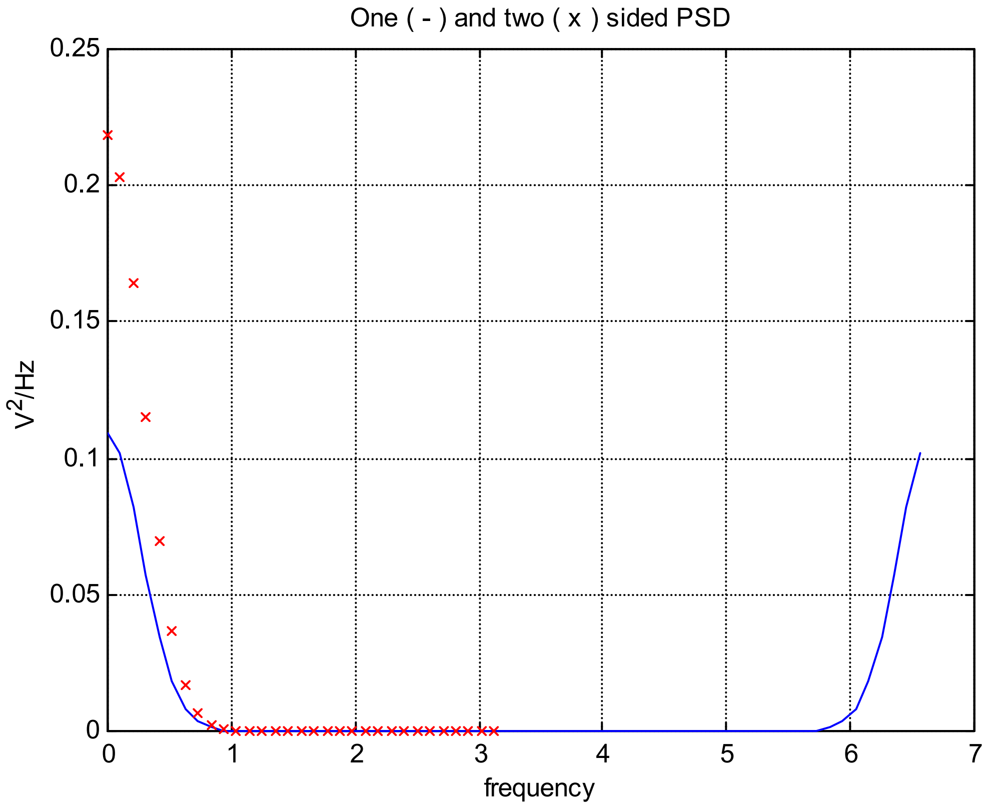




1.4
1.2
1
0.8
0.6
0.4
0.2
0
DFT
-5
-4
-3
-2
-1
0
1
2
3
4
5
freq [hz]

$$X_1(k) = \begin{cases} X(k) & k = 0,\ k = N/2 \\ 2X(k) & k = 1, 2 \ldots N/2 - 1 \end{cases}$$

$$G(k) = \begin{cases} S(k) & k = 0,\ k = N/2 \\ 2S(k) & k = 1, 2 \ldots \dfrac{N}{2} - 1 \end{cases}$$



3 Performance, errors and controls:

The main error mechanisms in spectral analysis

- Alias errors
- Leakage errors
- Random errors
- Bias errors

The uncertainty principle :

For any reasonable definition

$$(\text{Time duration}) \times (\text{Frequency bandwidth}) > C$$

given a signal length T_t, two components separated by $f_2 - f_1 = 1/T_t$ will not be separated by *any* signal processing techniques.

Transient analysis and zero padding

The frequency resolution is $\Delta f_{transient} = \dfrac{1}{n\Delta T}$

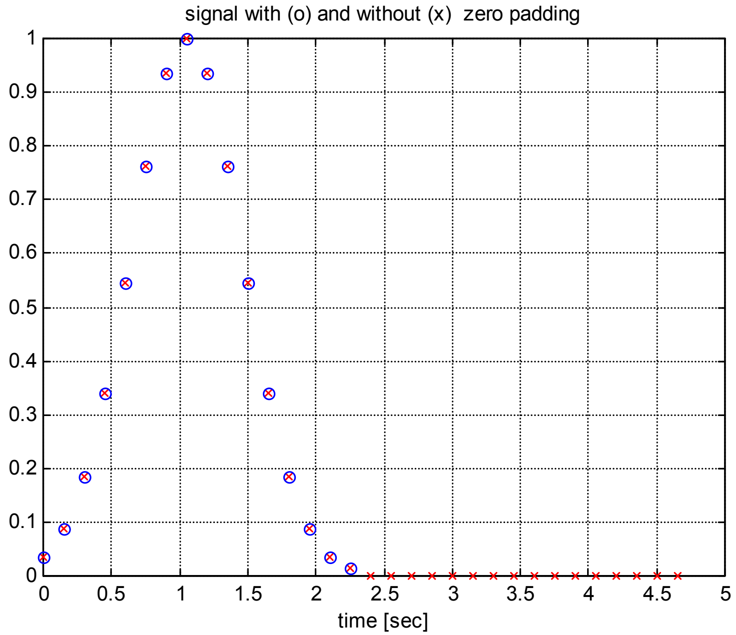
N-n zeros are now appended to the signal, resulting in total span of N points. The computational resolution is

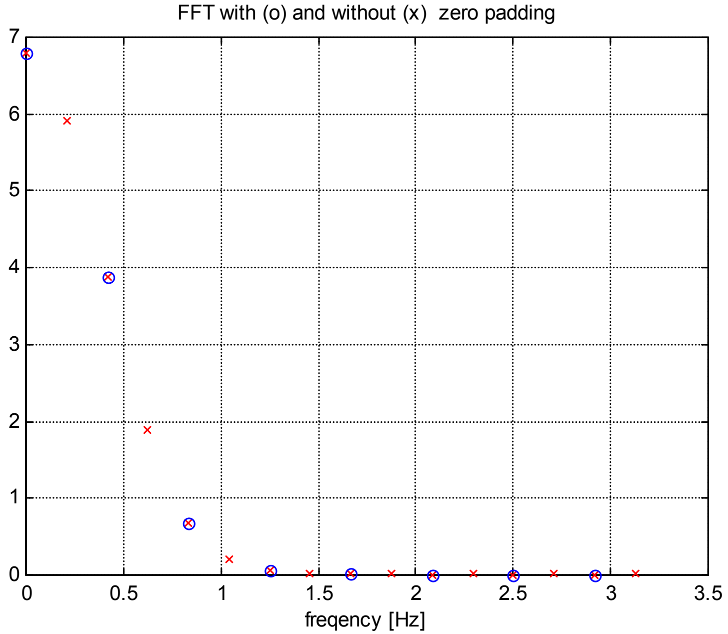
$$\Delta f = \frac{1}{N\Delta T} < \Delta f_{transient}$$

The DFT

$$X(k) = \sum_{i=0}^{n-1} x(i)\exp(-j\frac{2\pi}{N})^{ik} + \sum_{i=n}^{N} 0 \ \exp(-j\frac{2\pi}{N})^{ik}$$

Nothing is contributed to X(k) by the second term.


signal with (o) and without (x) zero padding
time [sec]

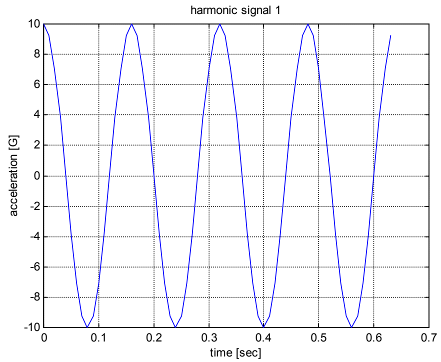

FFT with (o) and without (x) zero padding
freqency [Hz]

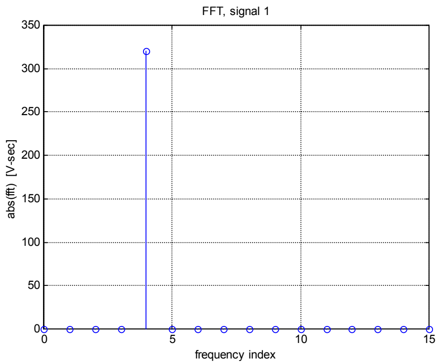
Periodic signals :

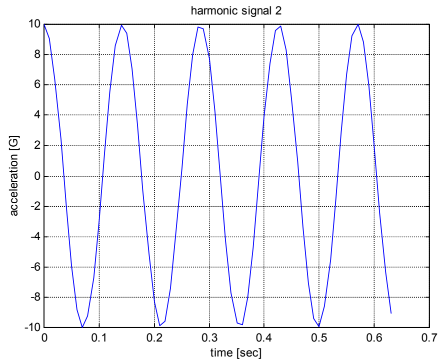
For periodic signals , with period T_p, the physical frequencies are

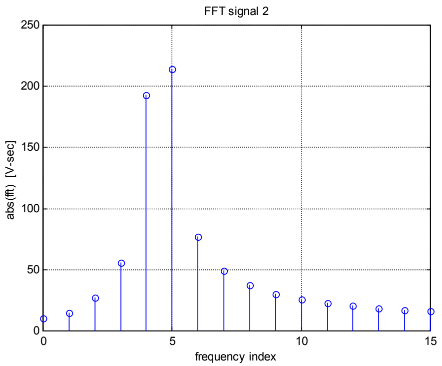
$$k/\,T_p,\ \text{with}\ k=1,2\ldots$$

The computed frequencies will be located at $k\Delta f = k/(N\Delta t)$

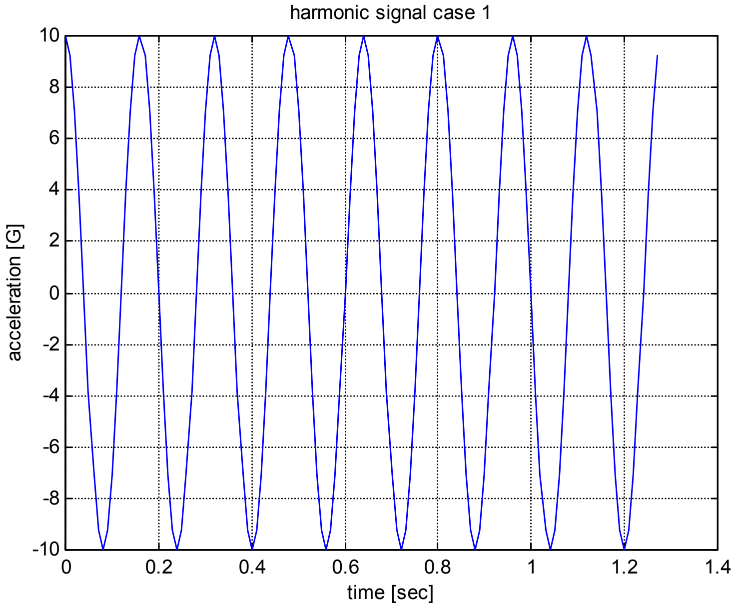


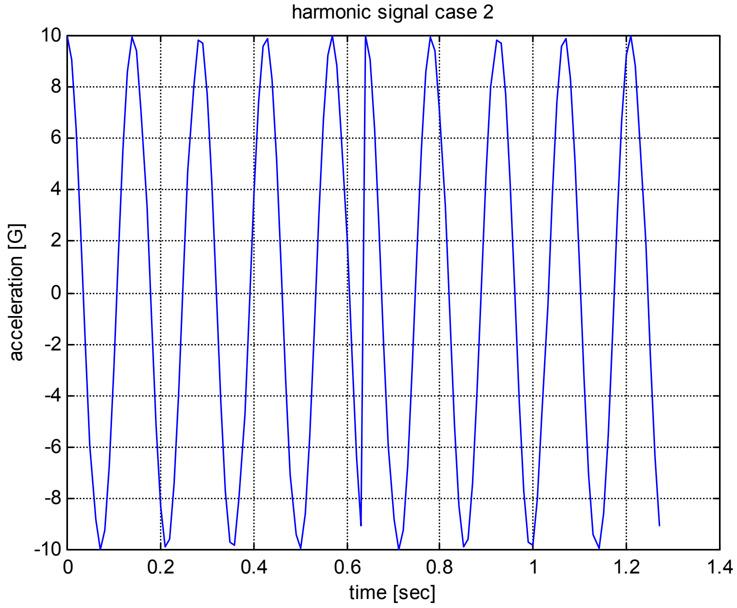




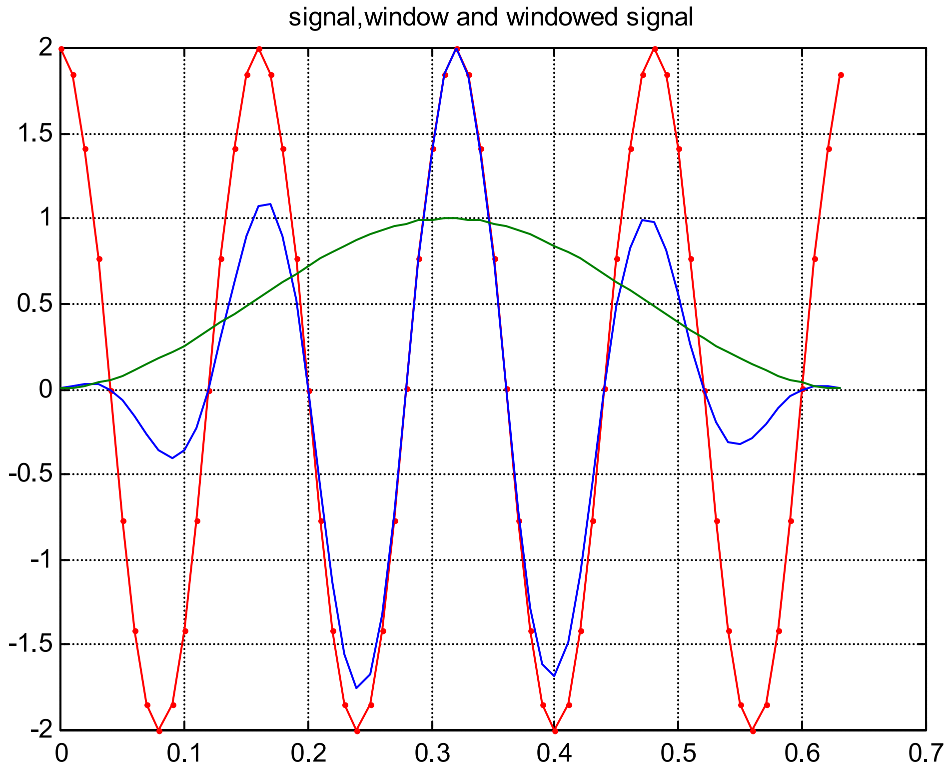


No discontinuities exist when extending periodically an harmonic (or any periodic) signal composed of an integer number of periods.





Give diminishing weights to the signals beginning and end by means of suitable windows.



The windowing is undertaken by multiplying the time signal by the appropriate window function.

$$x'(i)=w(i)x(i)$$

Fourier Transform – complex polar notation

- Generally X(w) is a complex-valued function (even if x is real)
  - Complex notation $X(\omega) = A(\omega) - jB(\omega)$

$$A(\omega) = \left|X(\omega)\right| \cos\theta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)\cos(\omega t)dt$$

$$B(\omega) = \left|X(\omega)\right| \sin\theta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)\sin(\omega t)dt$$

  - Complex polar notation

$$X(\omega) = \left|X(\omega)\right| \exp(-j\theta(\omega))$$

Spectral density – via Fourier Transform of the Auto-correlation

- Spectral density as Fourier Transform of the Auto-correlation

$$S_x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_x(\tau)\exp(-j\omega\tau)d\tau$$

  - A function of the frequency w, both positive and negative
- Symmetry of the Auto-correlation (Rx is an even function) → Spectral density is even

$$S_x(-\omega) = S_x(\omega) = A(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_x(\tau)\cos(\omega\tau)d\tau$$

  - So since the Auto-correlation is real so is the spectral density
- Summary: spectral density is real even and non-negative (not shown here, but see next)

Area under the spectral density

- Inverse:

$$R_x(\tau) = \int_{-\infty}^{\infty} S_x(\omega)\exp(j\omega\tau)d\omega$$

$$= \int_{-\infty}^{\infty} S_x(\omega)\cos(\omega\tau)d\omega$$

Since Sx is even

- Note that:

$$R_x(\tau = 0) = \int_{-\infty}^{\infty} S_x(\omega)d\omega$$

- So

$$\int_{-\infty}^{\infty} S_x(\omega)d\omega = E[x^2] = \psi_x^2 = \mu_x^2 + \sigma_x^2$$

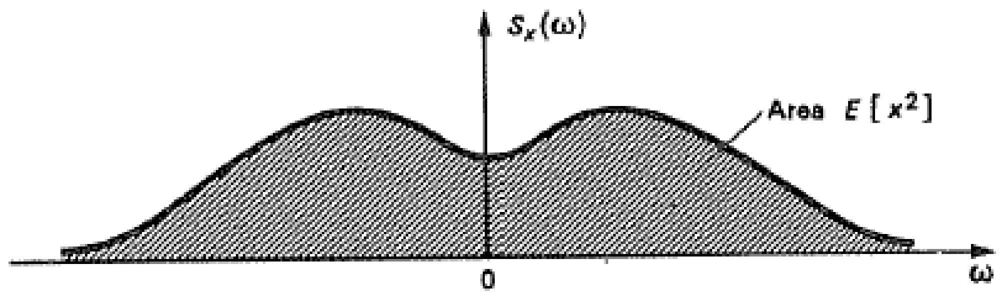
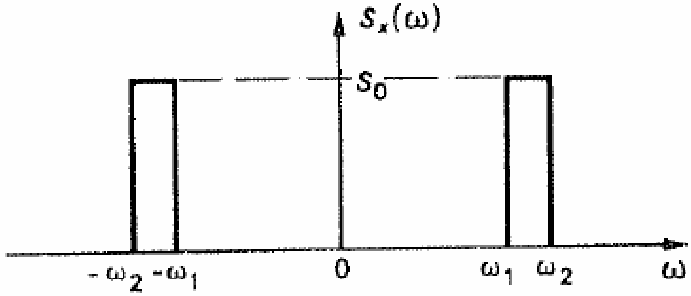


Fig. 5.1 The area under a spectral density curve is equal to $E[x^2]$

Examples – Band Limited

- Mean Square

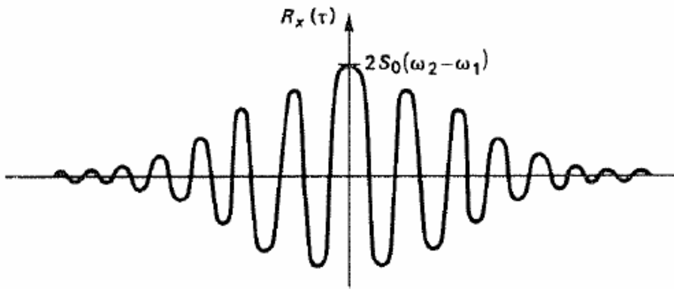
$$E[x^2] = \int_{-\infty}^{\infty} S_x(\omega)d\omega$$
$$= 2S_0(\omega_2 - \omega_1)$$



Spectral density of Narrow band random process

- Auto-correlation

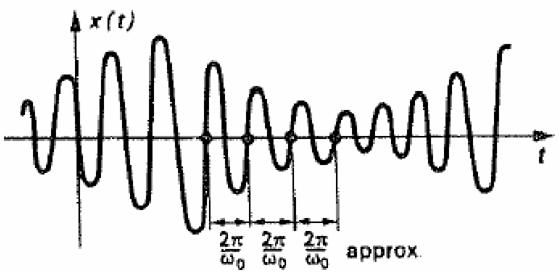
$$R_x(\tau) = \int_{-\infty}^{\infty} S_x(\omega)\exp(j\omega\tau)d\omega$$
$$= \int_{-\infty}^{\infty} S_x(\omega)\cos(\omega\tau)d\omega$$
$$\frac{4S_0}{\tau}\cos\left(\frac{\omega_1+\omega_2}{2}\tau\right)\sin\left(\frac{\omega_2-\omega_1}{2}\tau\right)$$

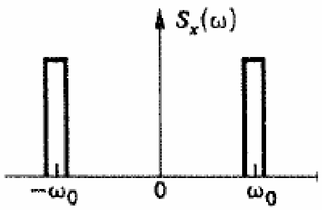


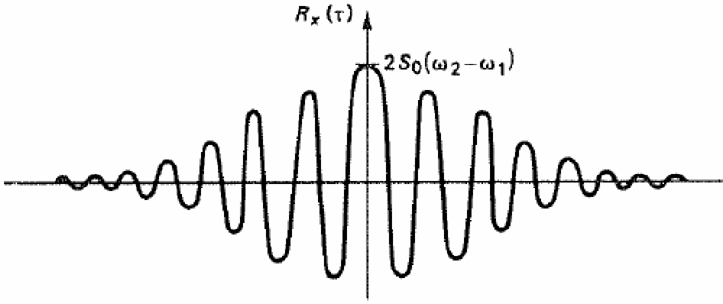
Auto-correlation

Narrow and broad band processes

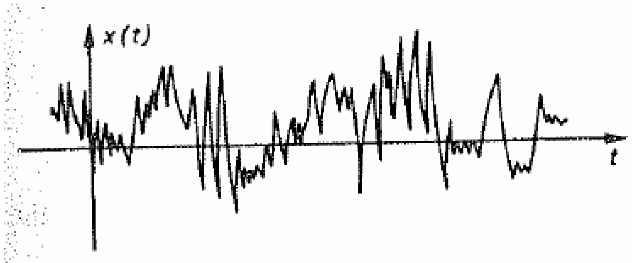
- Narrow band process:
  - Sample history; spectrum; Autocorrelation

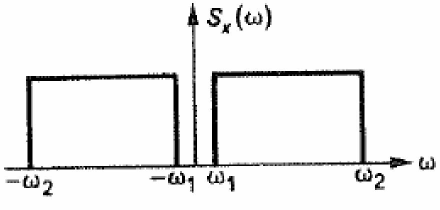


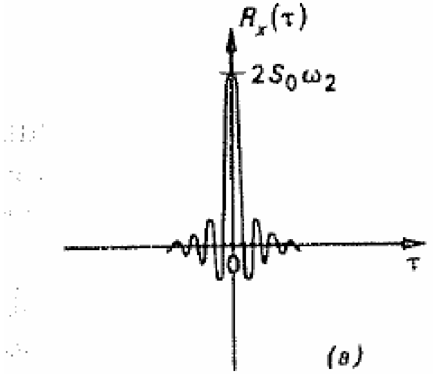




- Wide band process
  - Sample history; spectrum; Autocorrelation







One-sided spectral density function

- In practice it is common to:
  - Use the one-sided transform, defined only for positive frequencies w>0,

$$W_x(\omega) = 2\frac{1}{2\pi}\int_0^{\infty} R_x(\tau)\cos(\omega\tau)d\tau = 2S_x(\omega)$$

  - Express it as a function of the frequency f>0 instead of the angular frequency w.

$$W_x(f) = 2\int_0^{\infty} R_x(\tau)\cos(2\pi f\tau)d\tau = 4\pi S_x(\omega = 2\pi f)$$

One-sided spectral density function

- Shaded areas in the
  - Two sided Sx(w) OR -- One sided Wx(f)

  Have the same contribution to the mean square in this frequency band, So

$$2S_x(\omega)d\omega = W_x(f)df = W_x(f = \frac{\omega}{2\pi})\frac{d\omega}{2\pi}$$

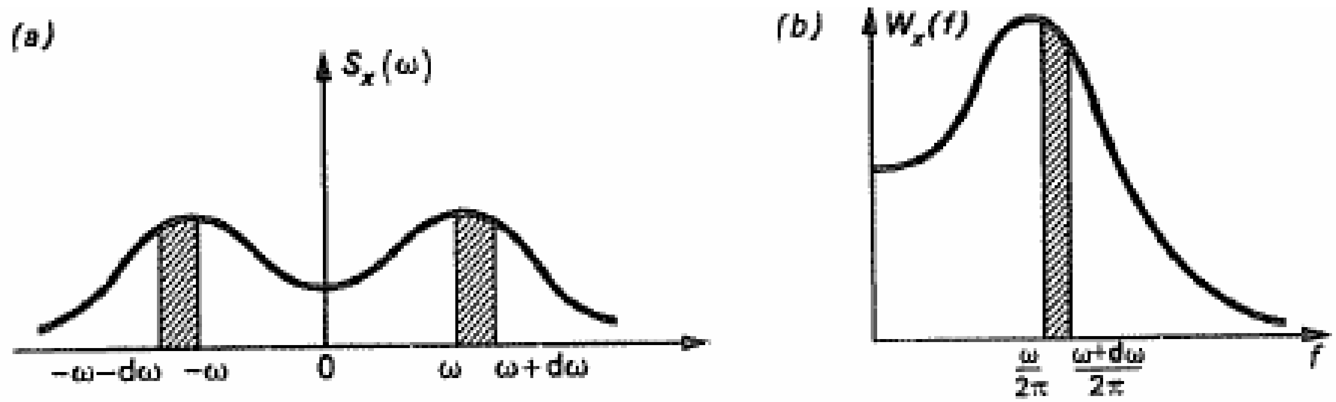


Fig. 5.9 Illustrating the relationship between alternative spectral density parameters

Examples

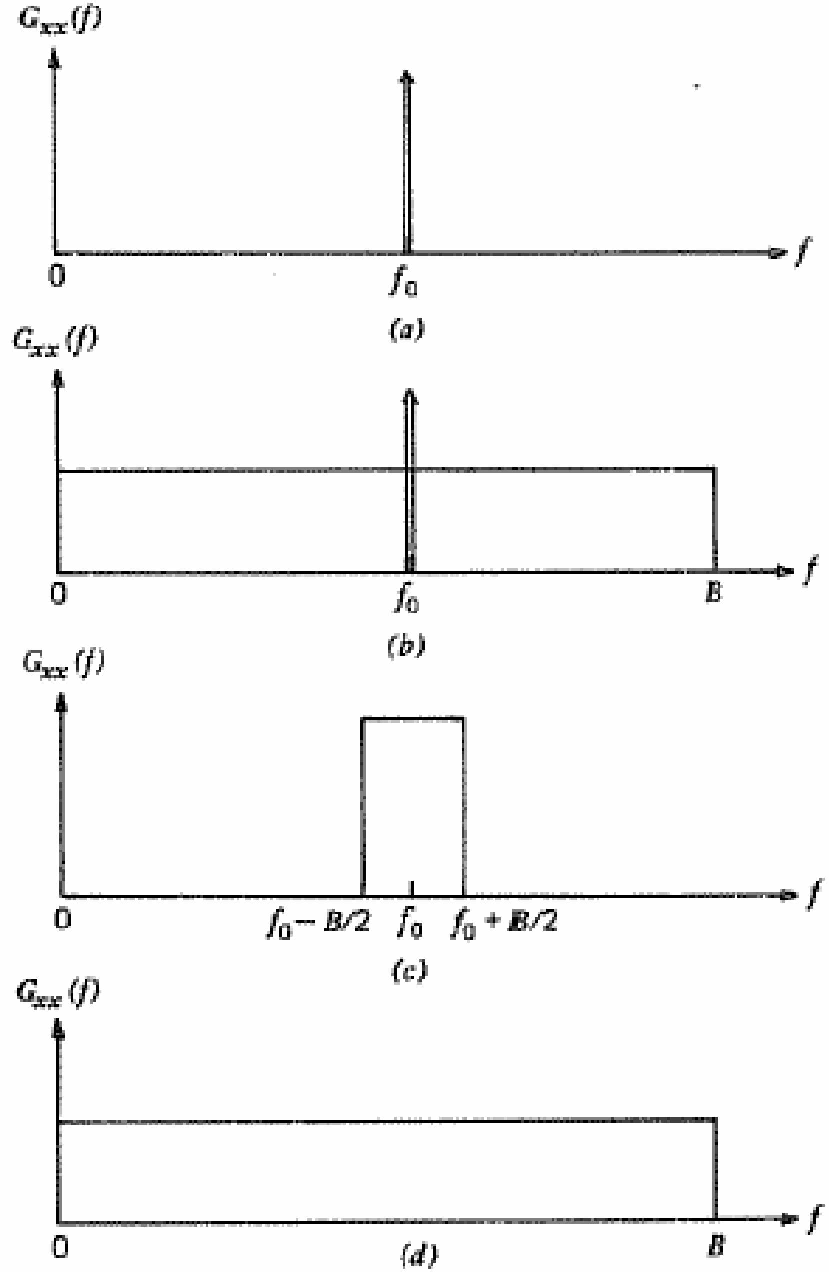
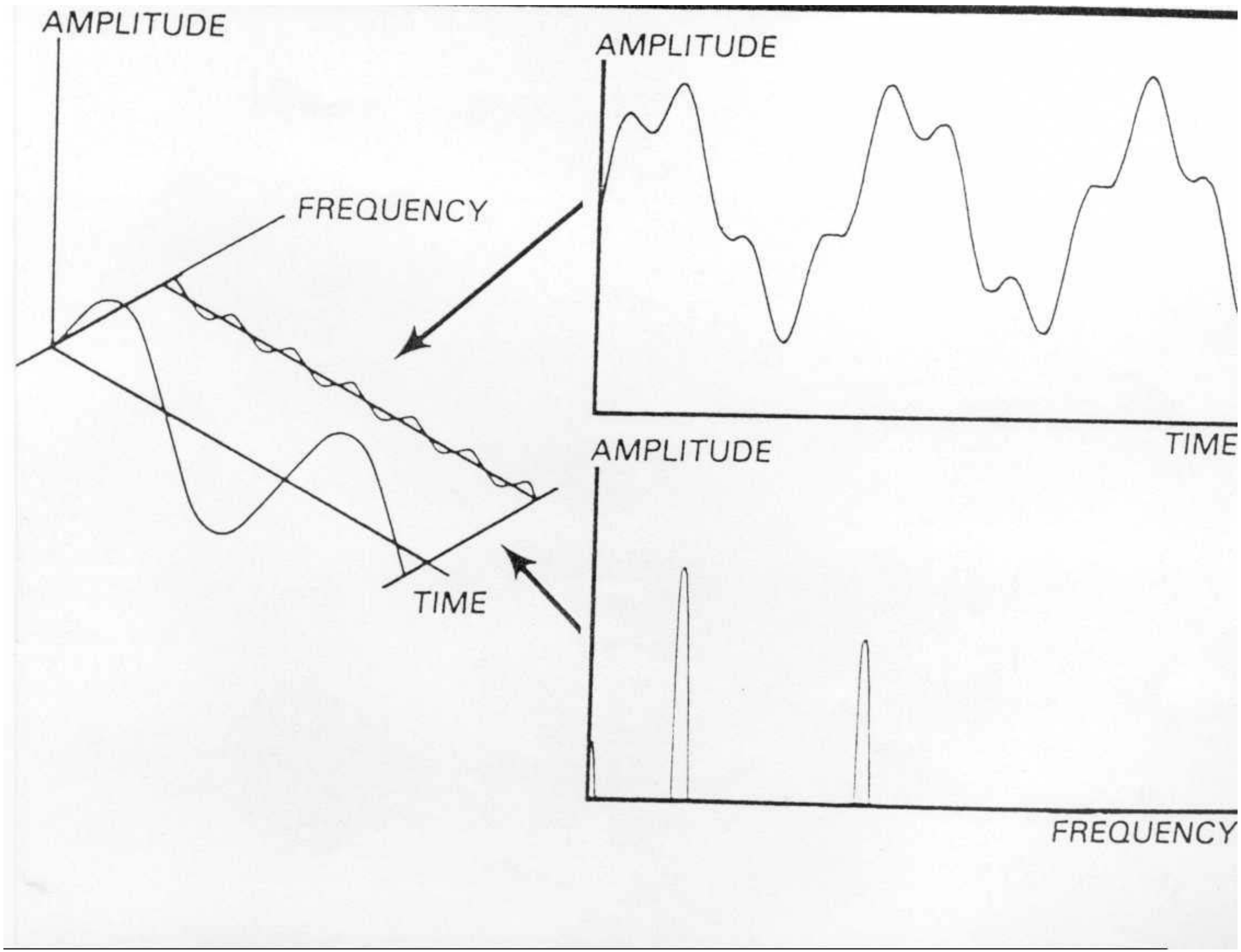


Figure 3.10 Idealized autospectral density functions. (*a*) Sine wave. (*b*) Sine wave plus random noise. (*c*) Narrow-band random noise. (*d*) Wide-band random noise.

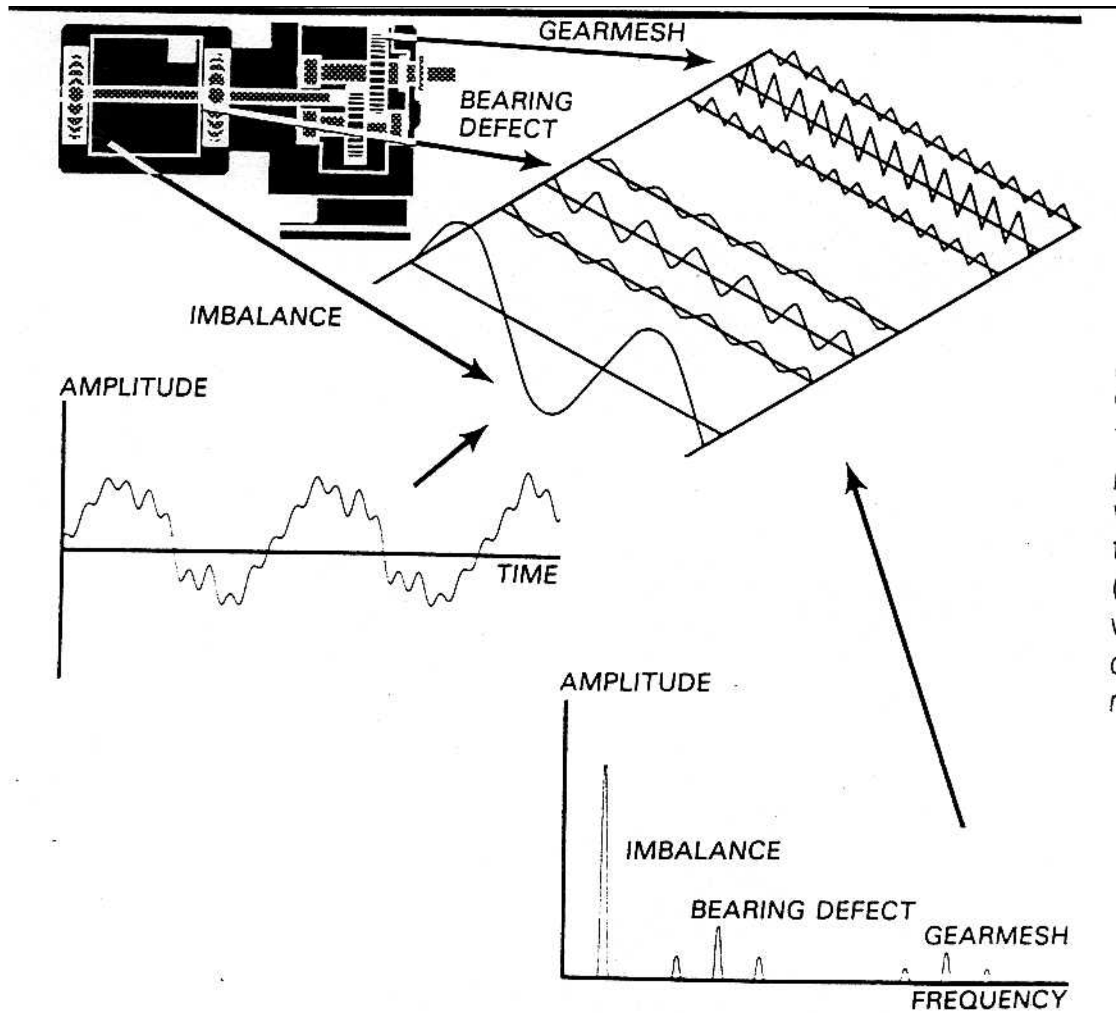
Time – Frequency representation

Time domain: *No spectral information*

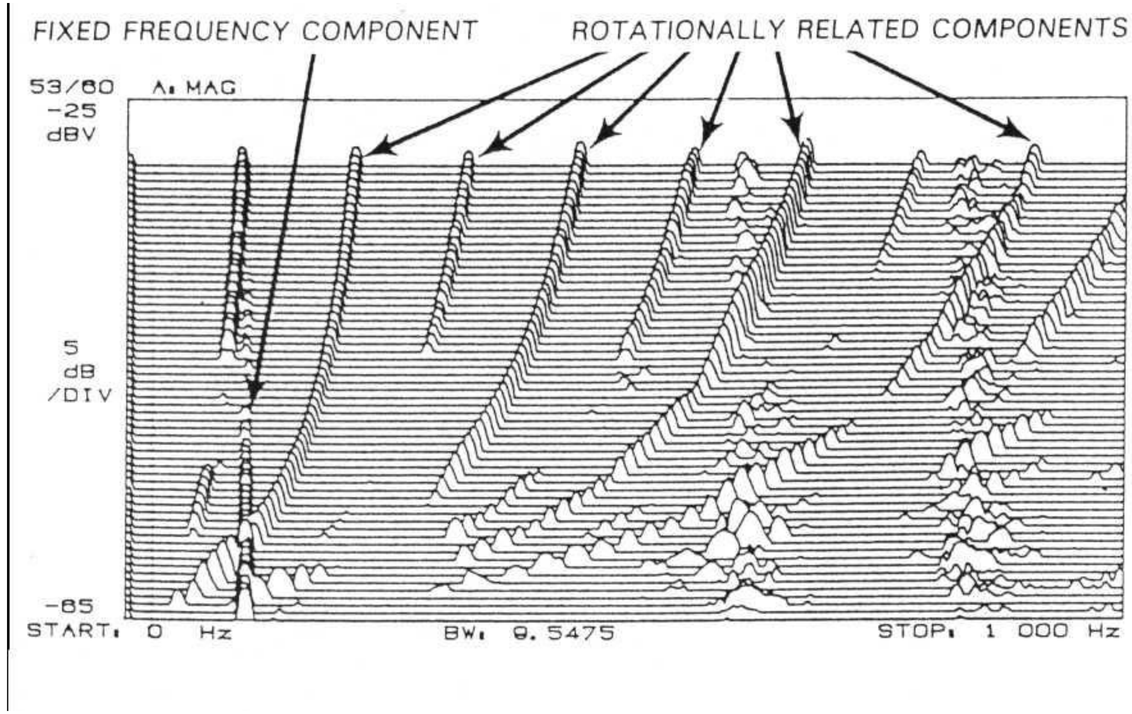
Frequency domain: *No temporal information*



Signal Processing: Time_frequency

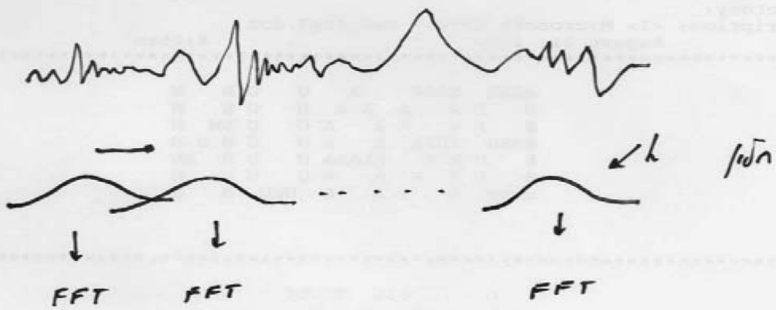


Signal Processing: Time_frequency



Signal Processing: Time_frequency

STFT - Short Time Fourier Transform



$$X(f) = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi f t)\,dt$$

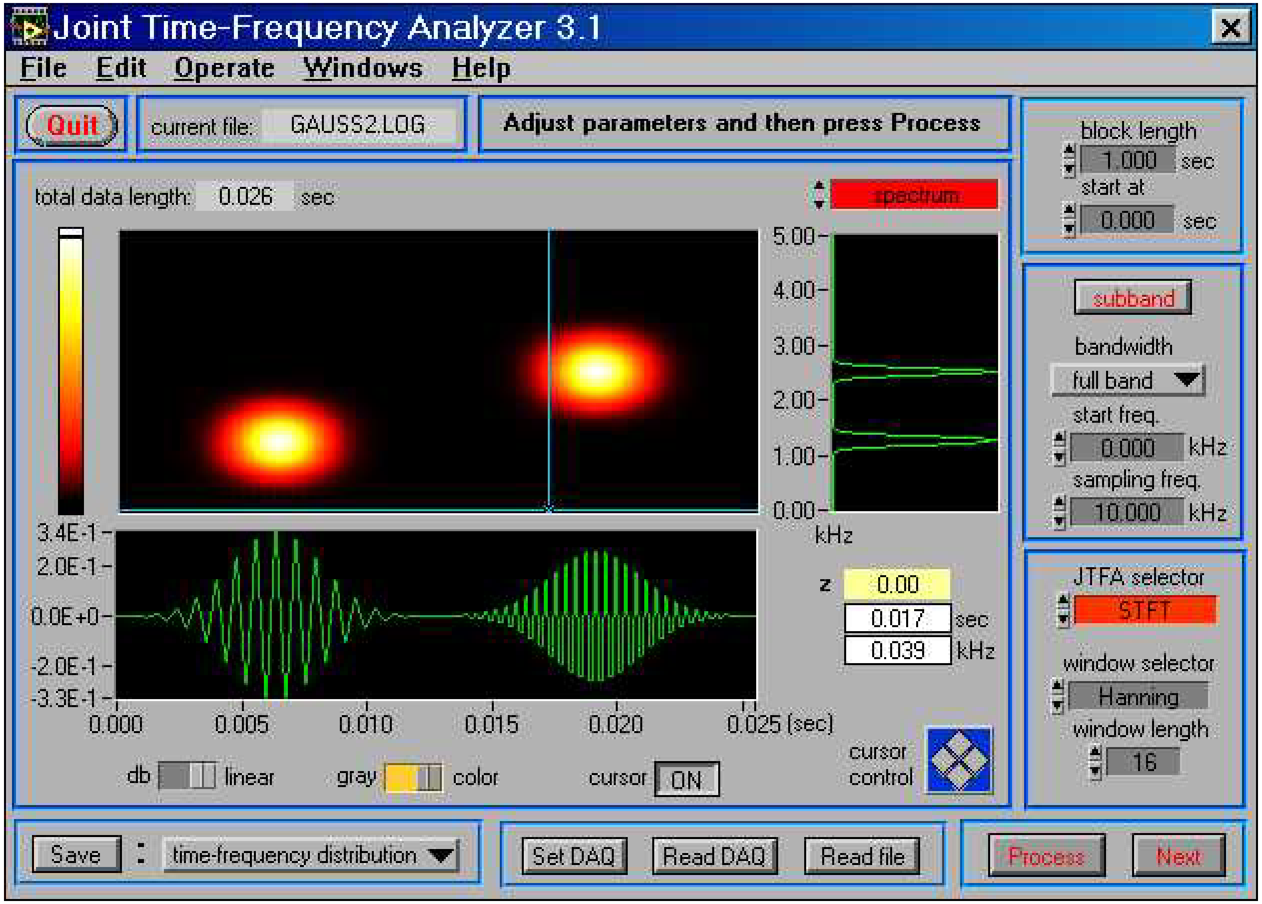
$$F_x(t,f) = \int_{-\infty}^{\infty} x(t')\,h(t'-t)\exp(-j2\pi f t')\,dt'$$

STFT Short Time Fourier Transform

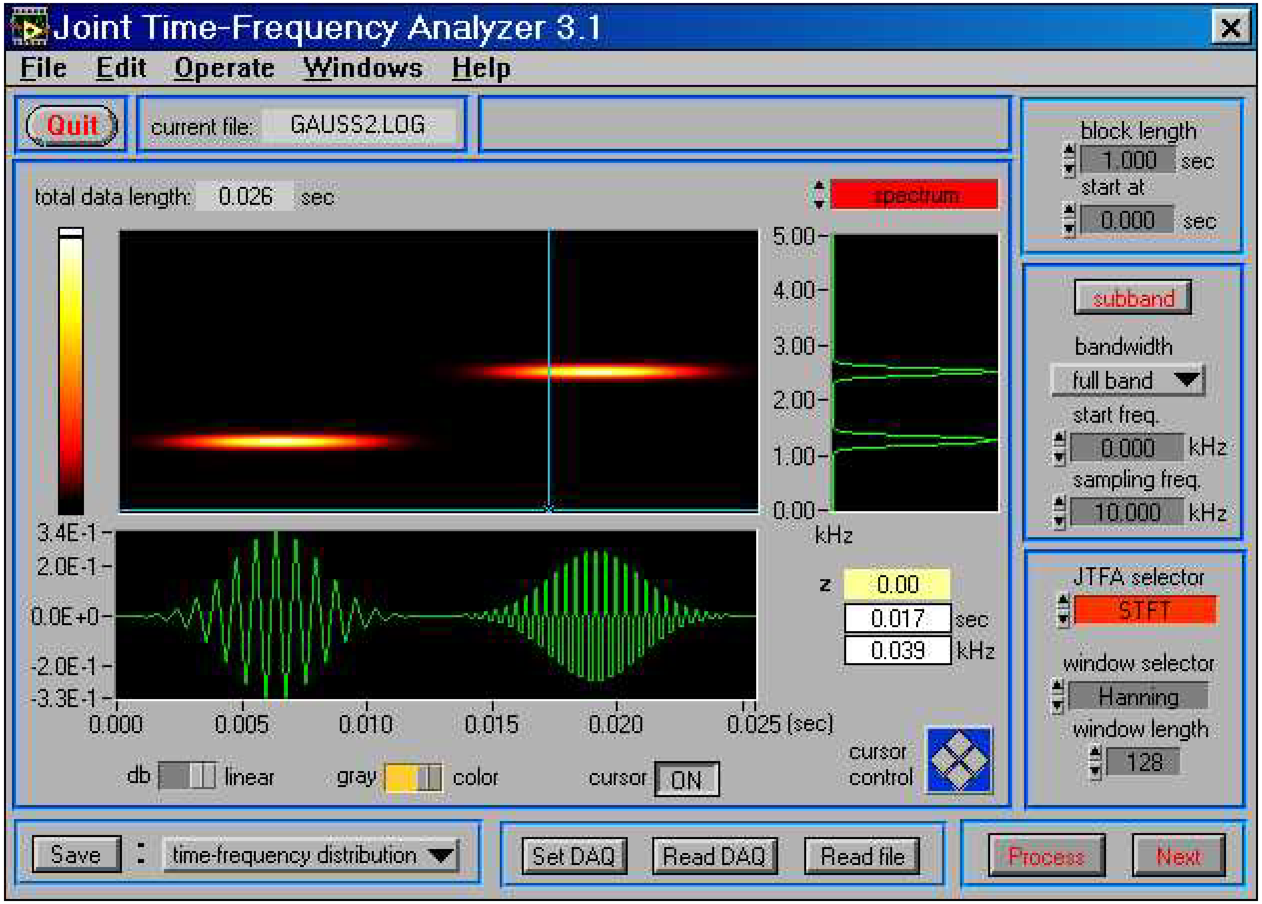
$$F_x(t,f) = \int_{-\infty}^{\infty} x(t')h(t'-t)\exp(-j\ 2\pi f t')dt'$$

$$x(t) = \frac{1}{E_h}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_x(t',f')h(t-t')\exp(j2\pi t\ f')df'dt'$$

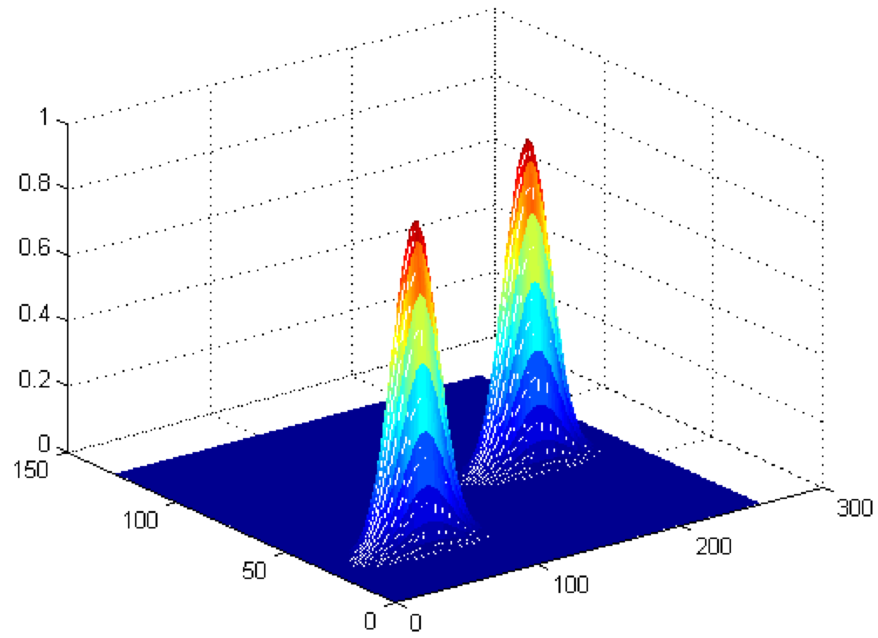
$$E_h = \int_{+\infty}^{\infty} |h(t)|^2\ dt$$



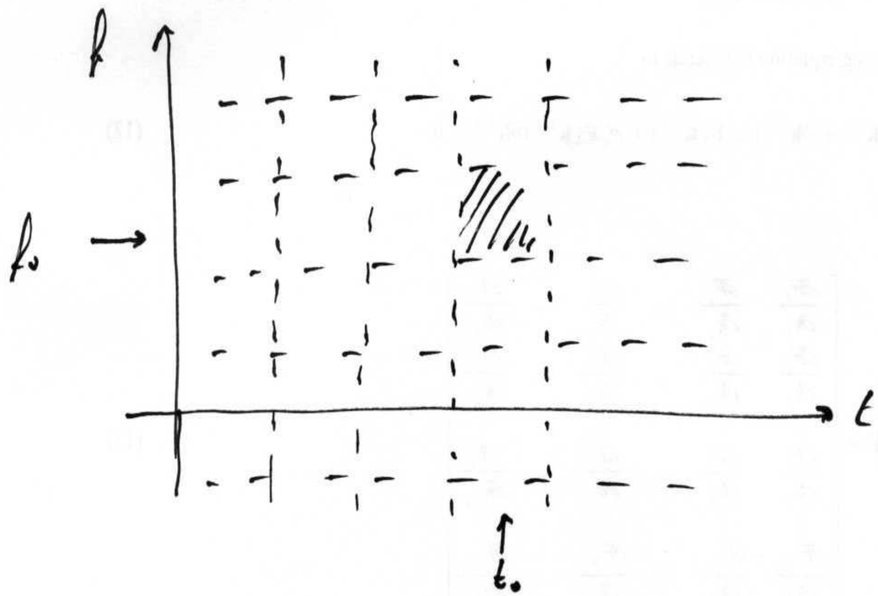
Signal Processing: Filters



Signal Processing: Time_frequency


1
0.8
0.6
0.4
0.2
0
150
100
50
0
0
100
200
300

DISCRETE STFT



$t = i\Delta T \quad ; \; f = k\Delta f$

$$F_x(t,f)=\int_{-\infty}^{\infty} x(t')h(t'-t)\exp(-j\ 2\pi f t')dt'$$

$$x(t)=\frac{1}{E_h}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_x(t',f')h(t-t')\exp(j2\pi t\ f')df'dt'$$

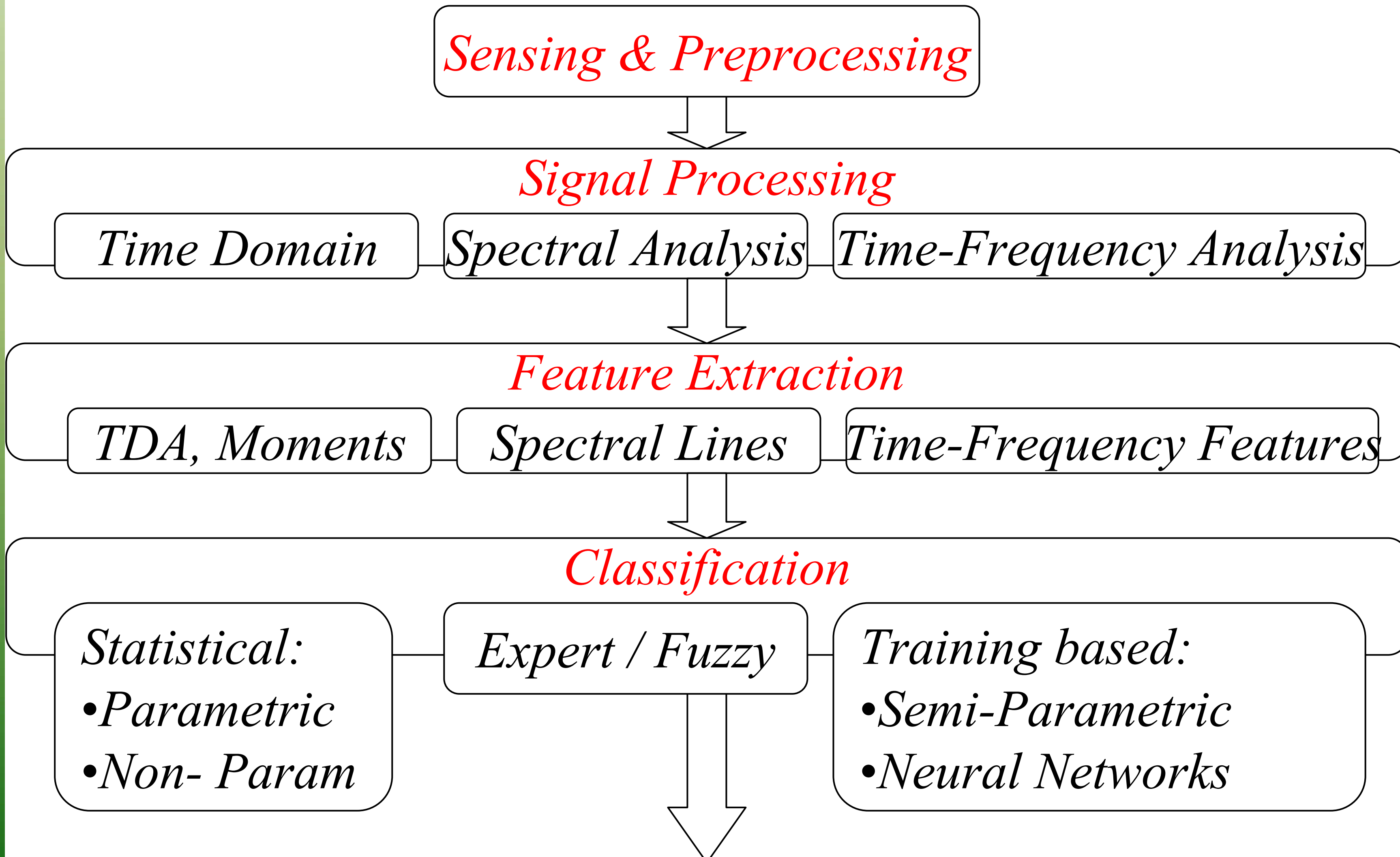
$$E_h=\int_{+\infty}^{\infty} |h(t)|^2\ dt$$

Spectogram

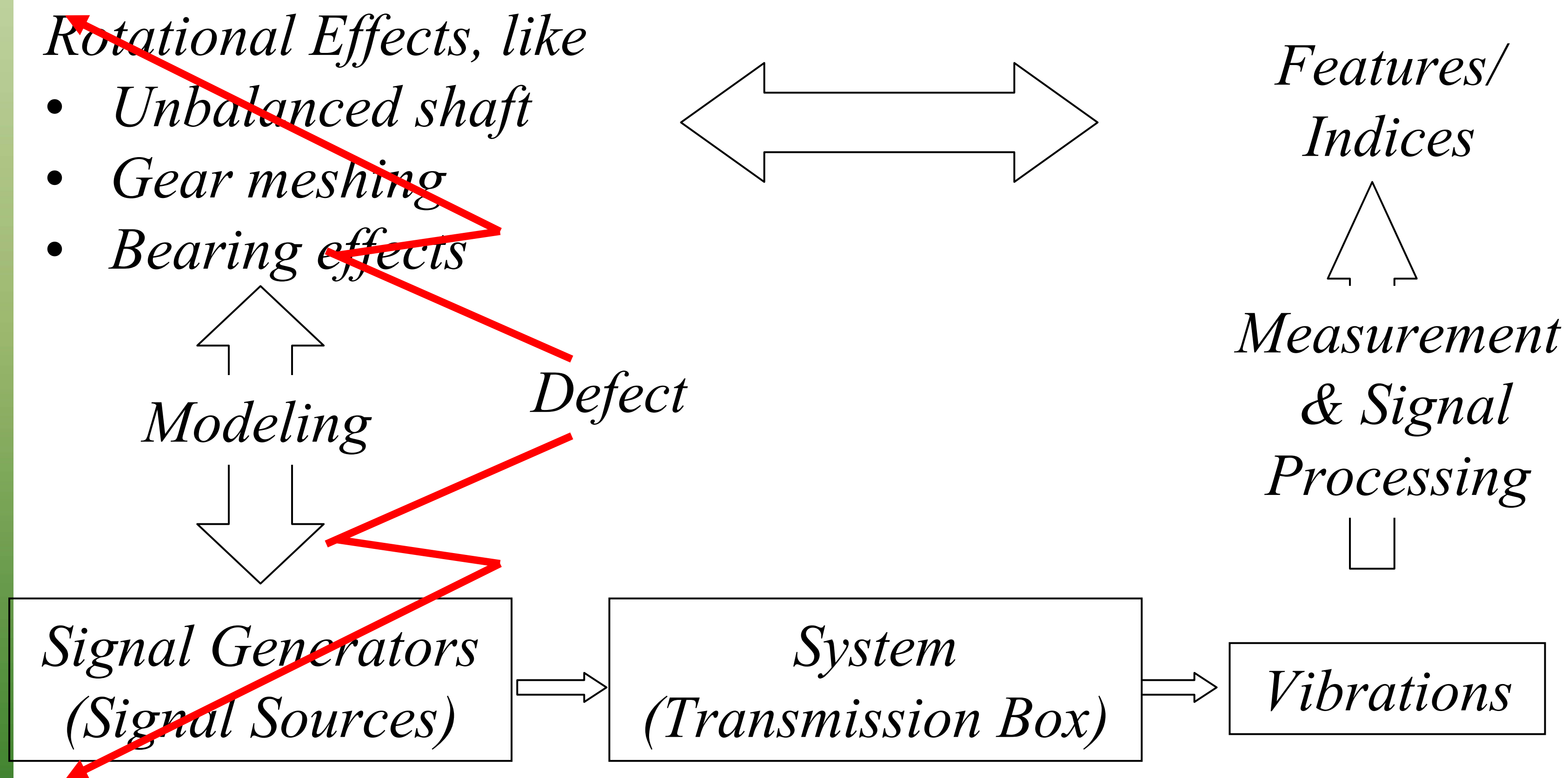
$$|\,F_x(t,f)\,|^2$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\,F_x(t,f)^2 dt\ df = E_x = |\,x(t)\,|^2\ dt$$

General Approach to HUMS



Diagnostic Approaches based on Signal Generation Models



Sources of vibrations in gears

- *Gear transmission error:*
  - *Tooth meshing*
    - *Tooth deflection under load*
    - *Individual tooth imperfections*
    - *Localized defects on gears*
  - *Shaft rotation*
    - *Eccentricities of the driving/ driven gears*
    - *Instantaneous speed variations*

Main gear troubles and their causes

The real condition of manufacturing and using of gears are never perfect. This implies some quality faults are always present.
Even if not out of service by complete failure, gears can suffer from many different troubles, the most frequent of which are indicated in this table together with their causes.

Type of trouble	Description	Causes
Eccentricity and similar faults	Wheel and shaft geometrical centres not coincident.	Manufacturing errors.
Looseness of gears or bearings on the shaft	Excessive backlash between gear and shaft or between bearings and shaft.	Manufacturing errors.
Misalignment	Axes of mating gears not parallel (cylindrical gears) or coplanar (bevel gears).	Manufacturing or assembly errors.
Excessive backlash	Excessive distance between the non-working flanks of two meshing gears.	Manufacturing or assembly errors.
Machining signs	Gear tooth profile shaped like a broken line enveloping an involute curve.	Manufacturing methods.

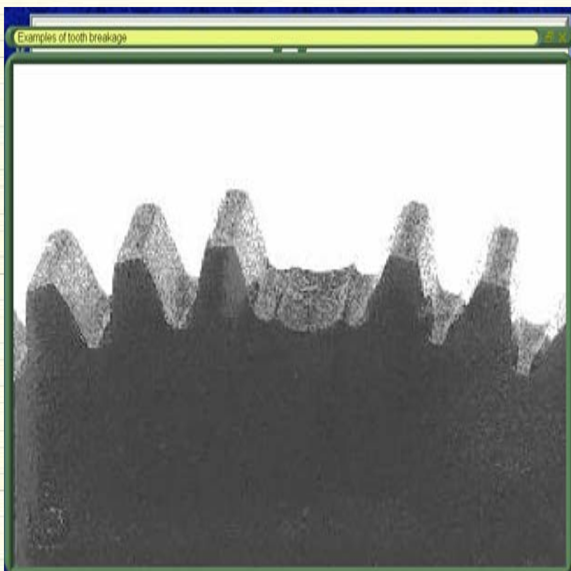
For more details on manufacturing or assembly errors, see the corresponding section of this chapter.

Gear failures and their causes (local faults)

The local faults of gears concern a damage on one or more teeth. The both kinds of damage can be spalling or cracking of tooth, which finally can produce breakage of the tooth.

Except in the extreme case of breakage, this kind of fault is recognisable with two symptoms :

- An amplitude and/or phase modulation of the meshing signal (present in the spectrum with sidebands separated by the wheel rotational frequency around the gearmesh frequency and its harmonics).
- Periodic shocks are visible in the time history. This particularity is important since is enables the differentiation of the local faults of using and some quality faults also generating modulations.



Type of failure	Description	Causes
Spalling	Breakaway of relatively large bits of tooth surface, typically in the case hardened gears	Too abrupt a transition between hard case and soft material underneath; local metallurgical defects; development of pitting
Cracking	Failure due to the propagation of microscopic flaws in the material under cyclic loading; more common in hardened gears	Generally it results from incorrect processing — grinding, queching — and usually leads to breakage
Breakage	Fracture of an entire gear tooth or a substantial portion of it	Overstress, fatigue

Faults and symptoms 1/4

Fault	Vibration Symptoms	Remarks
Static Unbalance	•Frequency: 1×rpm. •Direction: Radial. •Phase difference on end bearings: approx. 0°.	•Vibration level constant at constant rotational speed. •Amplitude proportional to square of angular speed below critical speed. •High vibration level at rotor critical speed.
Couple Unbalance	•Frequency: 1×rpm. •Direction: Radial. •Phase difference on end bearings: approx. 180°.	•Vibration level constant at constant rotational speed. •Amplitude proportional to square of angular speed below critical speed. •High vibration level at rotor critical speed.
Dynamic Unbalance	•Frequency: 1×rpm. •Direction: Radial. •Phase difference on end bearings: between 0° and 180°.	•Vibration level constant at constant rotational speed. •Amplitude proportional to square of angular speed below critical speed. •High vibration level at rotor critical speed.

Fault	Vibration Symptoms	Remarks
Angular Misalignment	•Frequency: 1, 2, and also 3×rpm. •Direction: Axial (and Radial). •Phase: approx. difference of 180° across the coupling.	•Vibration characteristics strongly depend of the type of coupling and bearings.
Parallel Misalignment	•Frequency: 1×rpm and harmonics; 2×rpm usually dominates. •Direction: Radial. •Phase: approx. difference of 180° across the coupling.	•Vibration characteristics strongly depend of the type of coupling and bearings.
Rolling Bearing Defects	•Frequency: impact rate related to each defect and harmonics, proportional to the rotor angular speed; excitation of structural resonances in the high frequency range.	•Vibration usually localised near bearing location.
Oil Film Bearings: Oil Whirl	•Frequency: 0.4 to 0.5×rpm.	•Unsteady vibration level.

Fault	Vibration Symptoms	Remarks
Gear Defects	•Frequency: gearmesh frequency (#teeth×rpm) and harmonics, usually modulated by rotor angular speed (sidebands are present).	•Very high frequency for fast gears with many teeth. •Gear faults generally produce increment in the amplitude of gearmesh harmonics or sidebands.
Mechanical Looseness	•Frequency: 1×rpm and harmonics; sometimes sub-harmonics or half-frequency harmonics, depending on the type of looseness.	•Random vibration level.
Electrical Defects	•Frequency: 1×rpm or 1 or 2×electrical system frequency.	
Resonance	•Frequency: rotor angular speed.	•High vibration level at each rotor critical speed.

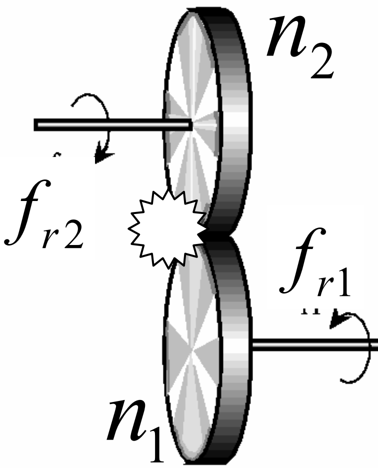
Fault	Vibration Symptoms	Remarks
Eccentricity	•Frequency: 1×rpm of eccentric component. •Direction: Radial.	•Balancing eccentric rotors can reduce vibration level in one direction, but increasing it in the orthogonal one.
Bent Shaft	•Frequency: 1×rpm; also 2×rpm if bent near the coupling. •Direction: Axial. •Phase: approx. difference of 180° at the end bearings.	

NOTES :

In general, $n \times$ **rpm** refers to the vibration component at the frequency of n times the rotor angular speed (n-per-revolution).

Direction indicates which usually is the direction of the highest vibration.

Signal Generation Models - Basic Models



Gear meshing excites periodic vibrations at the Meshing frequency $f_m = f_{r1} n_1 = f_{r2} n_2$

$$s(t) = \sum_k x_k \sin(2\pi k f_m t + \phi_k)$$

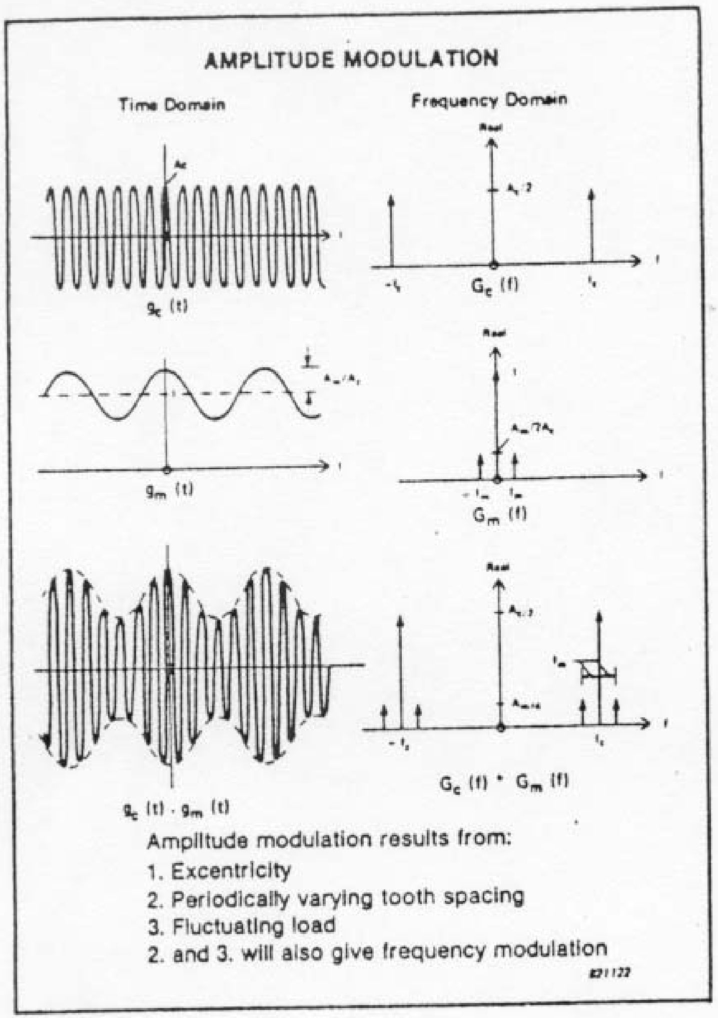
Modulated by shaft induced vibrations and tooth-variability:

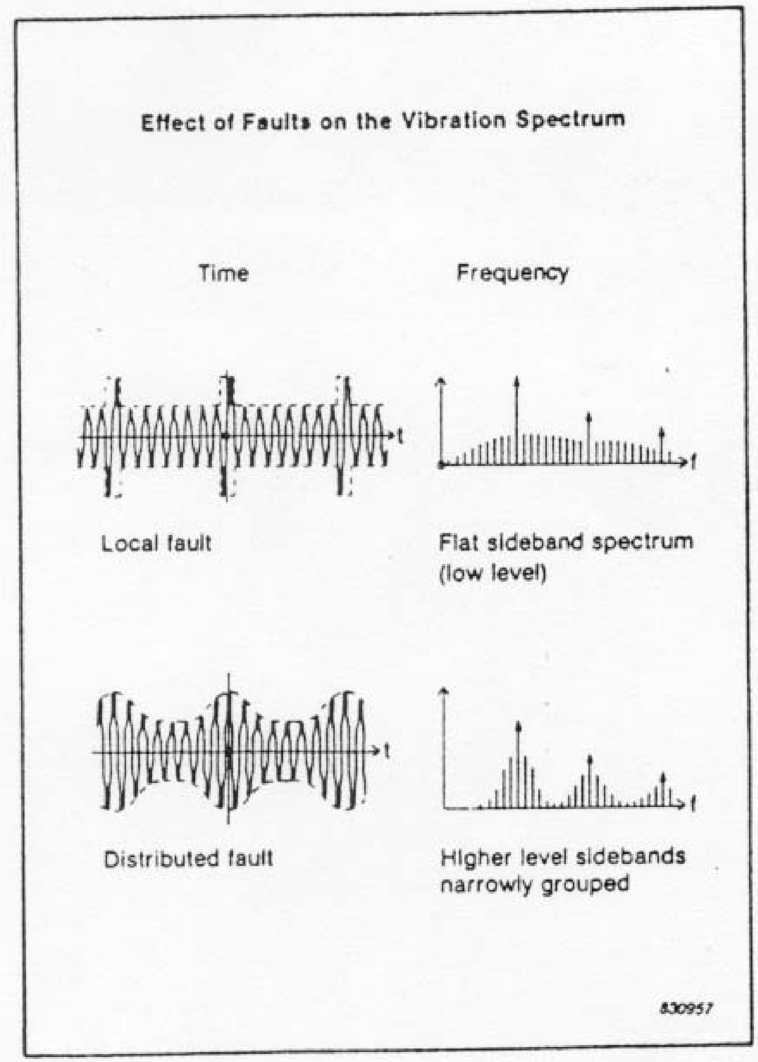
$$s(t) = \sum_k x_k \left[1 + a_k(t)\right] \sin\left(2\pi k f_m t + \phi_k + b_k(t)\right)$$

$$a_k(t) = \sum_p a_{kp} \sin\left(2\pi p f_{r1} t + \alpha_{kp}\right) + \sum_q a_{kq} \sin\left(2\pi q f_{r2} t + \beta_{kq}\right)$$

$$b_k(t) = \sum_p b_{kp} \sin\left(2\pi p f_{r1} t + \alpha_{kp}\right) + \sum_q b_{kq} \sin\left(2\pi q f_{r2} t + \beta_{kq}\right)$$

Amplitude modulations





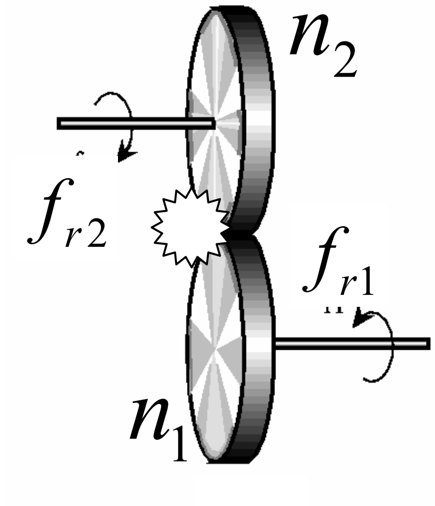
Signal Generation Models - Example: Collector Gear

Predicted spectral lines:

$$f(k,q,p) = kf_m \pm qf_{r1} \pm pf_{r2}$$

Expected spectral lines

16	*f(1,0,0)*	3155.75
17;18	*f(1,1,0)*	3029.5; (3282.0)[3]
19;20	*f(1,0,1)*	3113.1; 3198.4
21;22	*f(1,2,0)*	2903.3; 3408.2
23;24	*f(1,0,2)*	3070.5; 3241.1
25;26 27;28	*f(1,1,1)*	2986.9; (3324.6)[4] (3072.2)[5];(3239.3)[6]



$$n1 = 25 \quad fr1 = 126.23$$

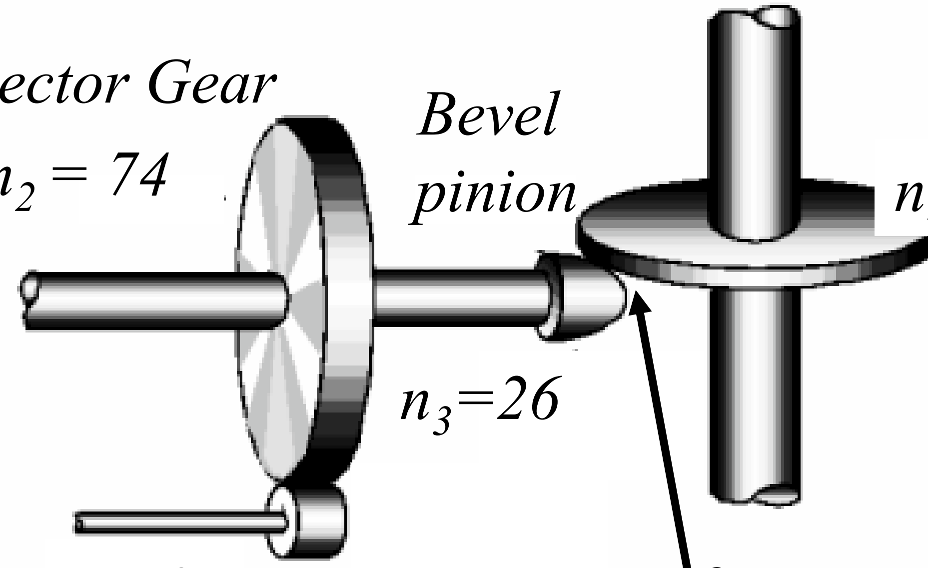
$$n2 = 74 \quad fr1 = 42.645$$

$$fm = 3155.75$$

Example (Con't); CH-64 Transmission Gear Box?

Expected spectral lines
Bevel Gear

1	*f(1,0,0)*	1108.9
2	*f(2,0,0)*	2217.8
3	*f(3,0,0)*	3326.7
4;5	*f(1,1,0)*	1066.2; 1151.6
6;7	*f(2,1,0)*	2175.2; 2260.5
8;9	*f(3,1,0)*	3284.0; 3369.4
10;11	*f(1,0,1)*	(1093); (1126.5)
12;13	*f(2,0,1)*	2200.2; 2235.4
14;15	*f(3,0,1)*	(3309.1); (3344.3)

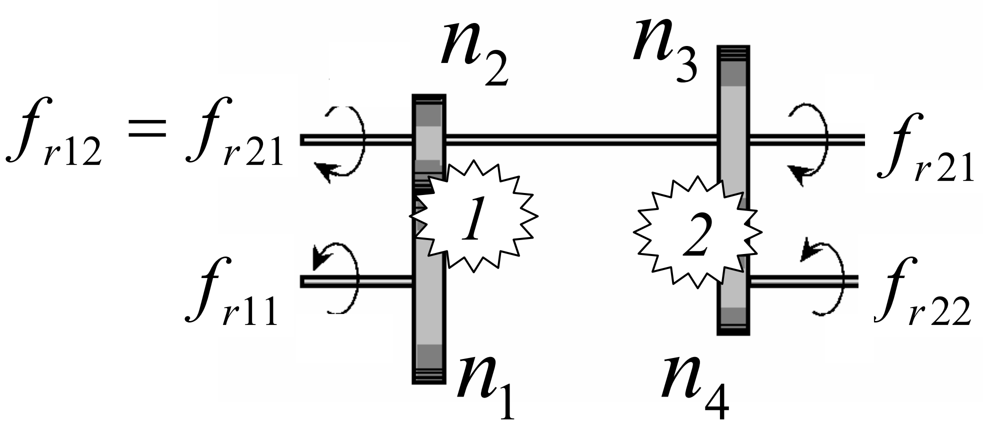


$n3 = 26 \quad fr3 = 42.645$

$n4 = 63 \quad fr4 = 17.6$

$fm2 = 1108.9$

Signal Generation Models Multiple Gears

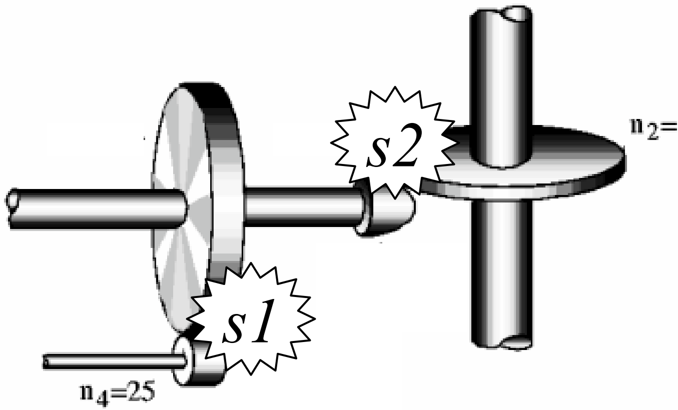


Small Signal Assumption:
The vibrations generated by different sources, which are associated with different gears, add up

$$s(t) = s_1(t) + s_2(t)$$

Cross Gear-pair Interaction

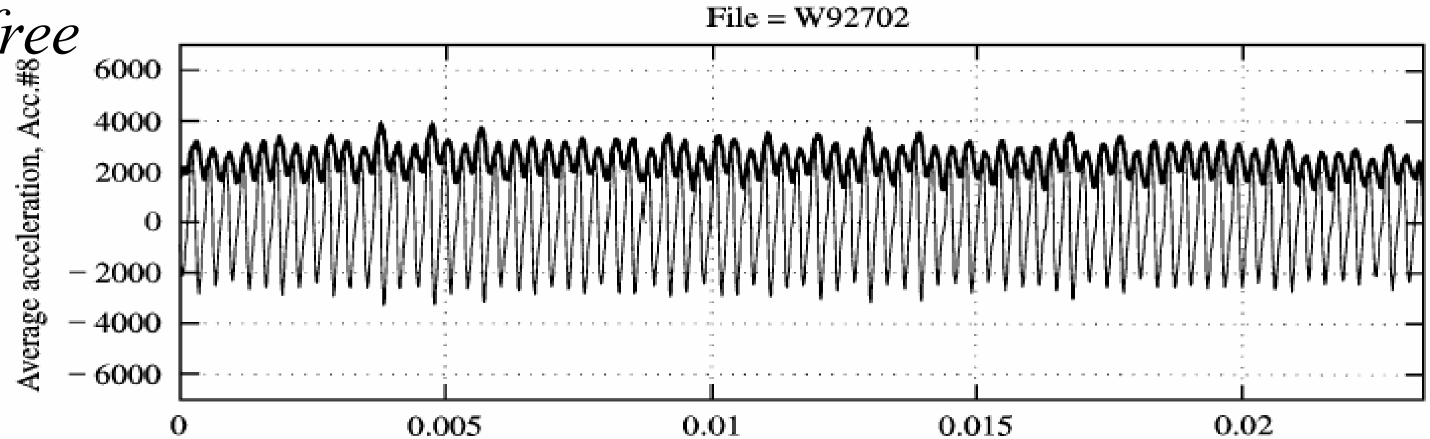
- *Modulation effect:*
  - *Each gear pair generates vibrations and torque variations*
    $T_1 \sim s_1 \qquad T_2 \sim s_2$
  - *Modulates by torque variations induced by the other gear-pair*
    $S = s_2(1+Ks_1)$



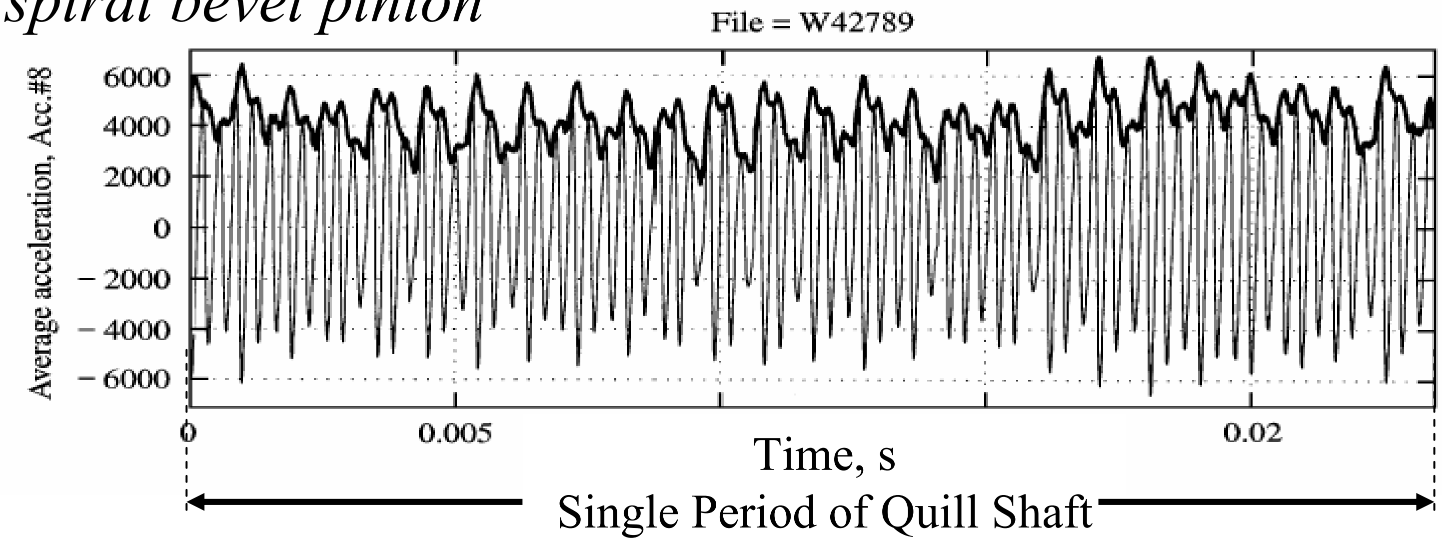
Cross-gear pair modulation

Time-Averaged Signals

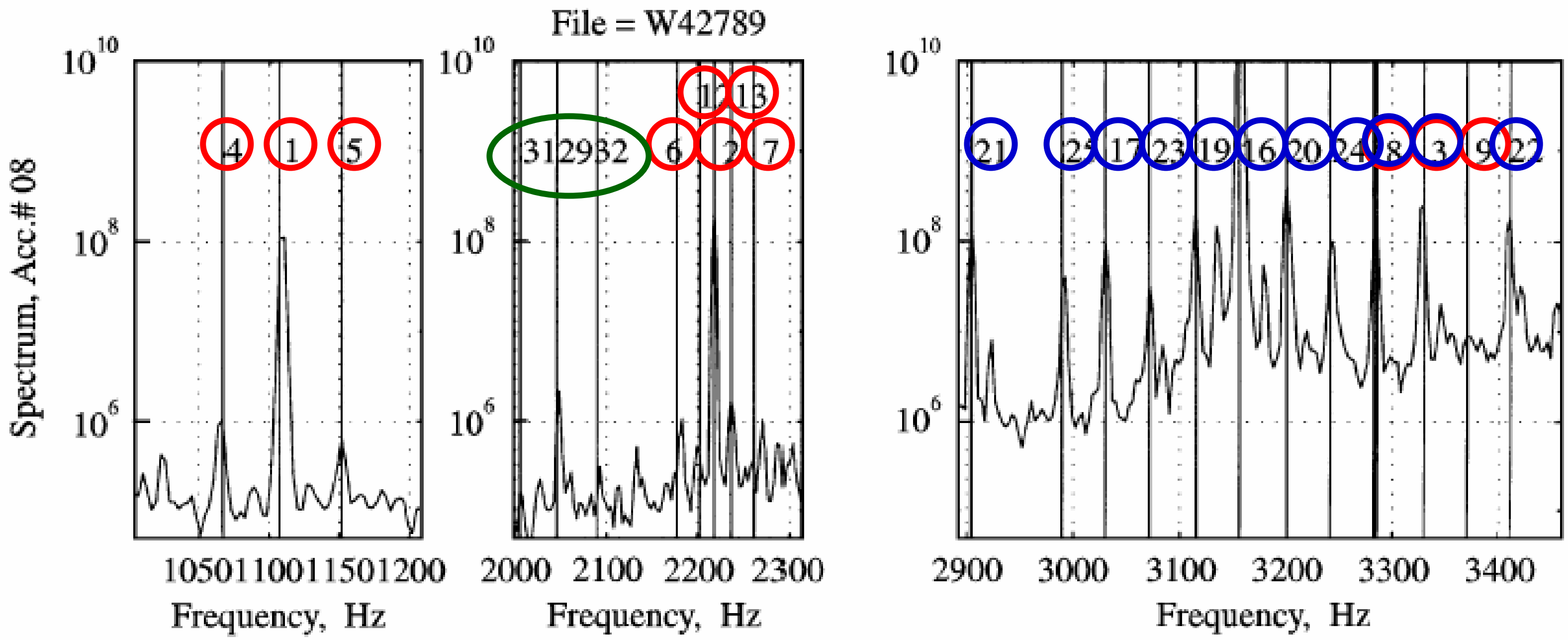
Fault free



Faulty spiral bevel pinion



Signal Generation Model
Predicted Spectral Lines



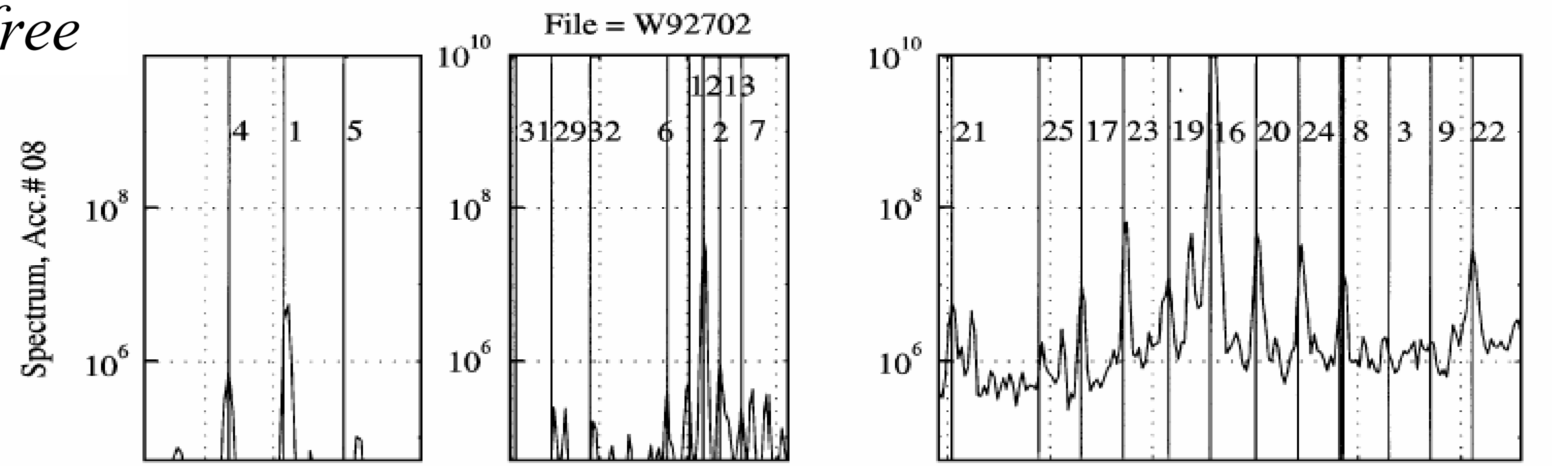
Vibrations induced by spiral bevel gear tooth meshing (1-15)

Vibrations induced by the collector gear tooth meshing (16-28)

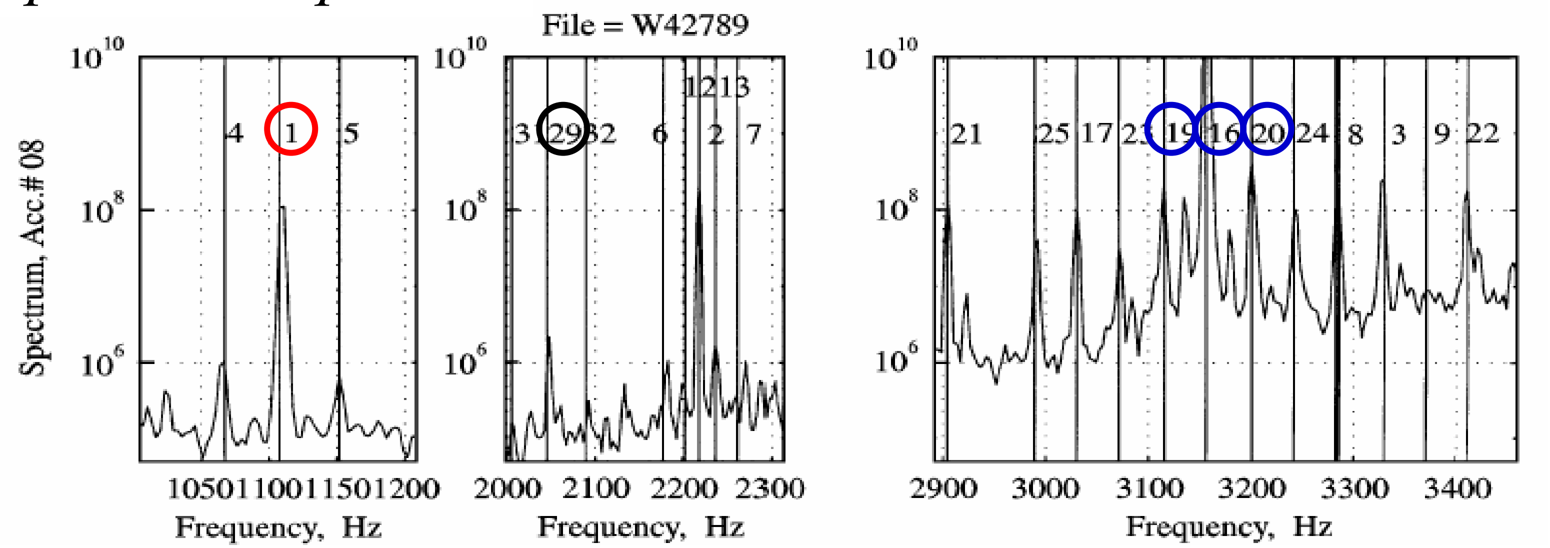
Vibrations induced by cross gear pair interaction (29-32)

Effect of Fault on spectral lines
(Faulty spiral bevel pinion)

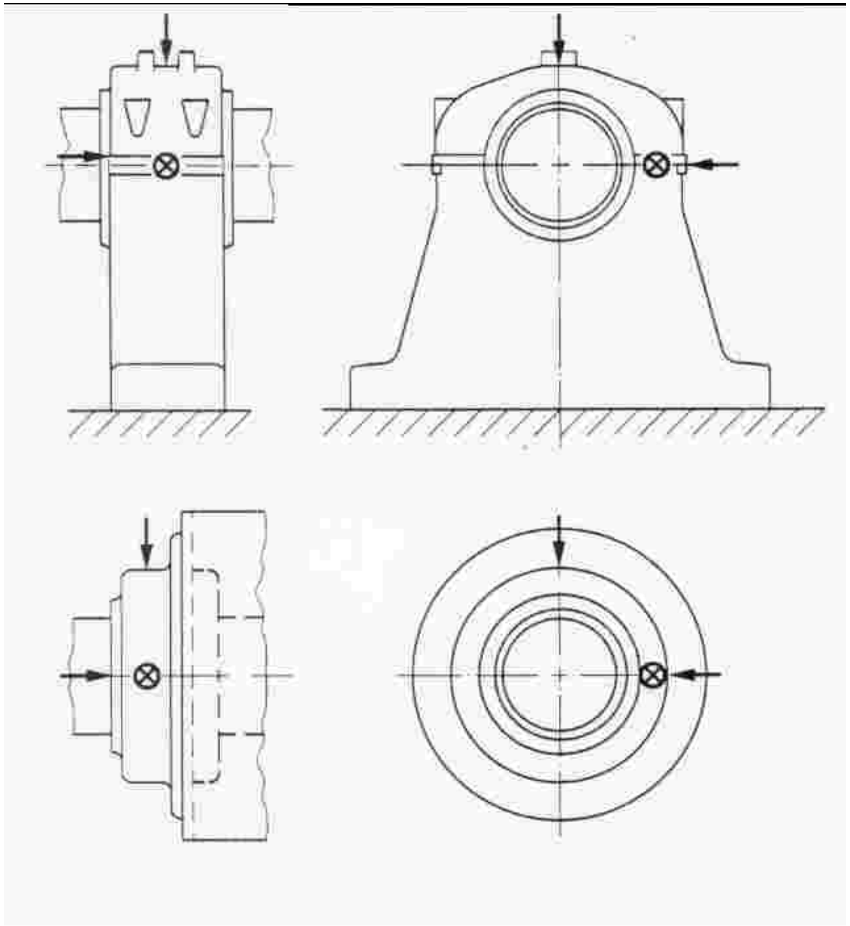
Fault free

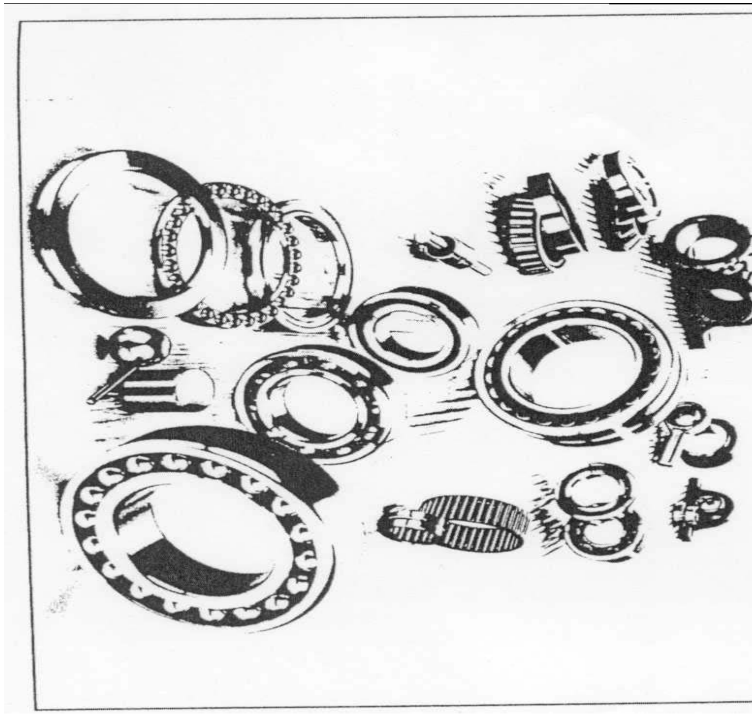


Faulty spiral bevel pinion



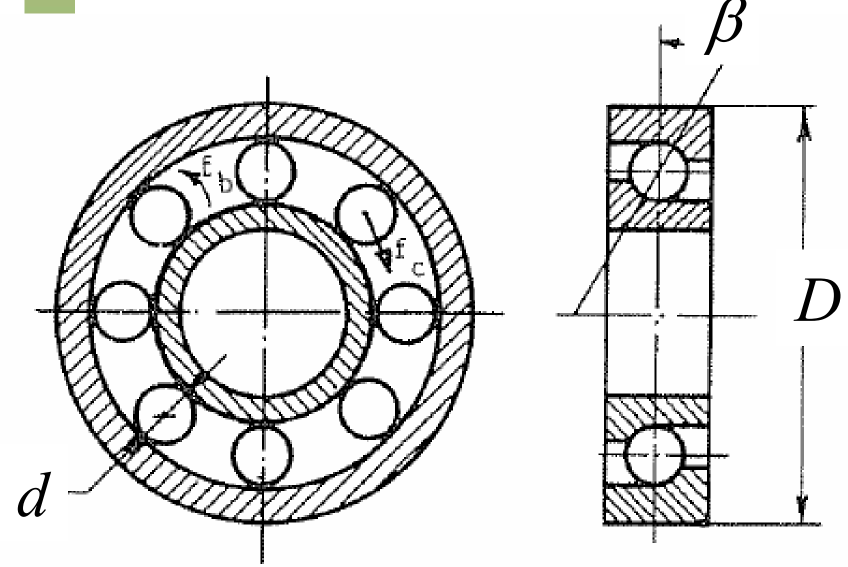
Bearings





Rotational speeds and frequencies for Roller-ball bearings

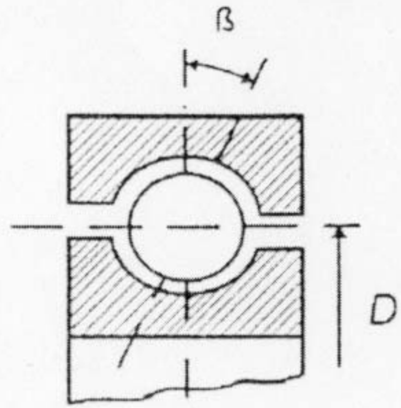
- *fir, for = rotational freq of inner/outer race*
- *fc, fb – rotational freq of the cage and rollers*



$$f_c = 0.5 f_{ir}(1 - \frac{d}{D}\cos\beta) + 0.5 f_{or}(1 + \frac{d}{D}\cos$$

$$f_c = 0.5\frac{d}{D}(1 - \frac{d}{D}\cos\beta)(1 + \frac{d}{D}\cos\beta)(f_{or} -$$

Basic freq of different defects



ß Contact angle
d Ball/roller diameter
D Ball/roller pitch diameter
n Number of balls/rollers
N Speed of shaft

$$f_c = 0.5 f_{ir}(1 - \frac{d}{D}\cos\beta) + 0.5 f_{or}(1 + \frac{d}{D}\cos\beta$$

$$f_c = 0.5\frac{d}{D}(1 - \frac{d}{D}\cos\beta)(1 + \frac{d}{D}\cos\beta)(f_{or} - f$$

Damaged outer race: $f_o = \frac{n * N}{2 * 60}(1 - \frac{d}{D}\cos ß)$ ⟵ $n * f_c(f_{or} =$

Damaged inner race: $f_i = \frac{n * N}{2 * 60}(1 + \frac{d}{D}\cos ß)$ ⟵ $n * f_c(f_{ir} =$

Damaged ball/roller: $f_r = \frac{D * N}{d * 60}(1 - \left[\frac{d}{D}\right]^2 \cos^2 ß)$ ⟵ f_b

Damaged cage: $f_c = \frac{N}{2 * 60}(1 - \frac{d}{D}\cos ß)$ ⟵ $f_c(f_{or} = 0$

Vibration generation model for Roller Bearing – I Spectral Lines

- A single defect with repetitive freq fi would generate a signal $x_i(t) = \sum_j g_i(t - j/f_i)U(t - j/f$
- Where gi is the impact response to the defect
- The FT of the periodic signal x(t)

$$X_i(f) = 2\pi \sum_k X_{ik}(f)\delta(f - kf_i)$$

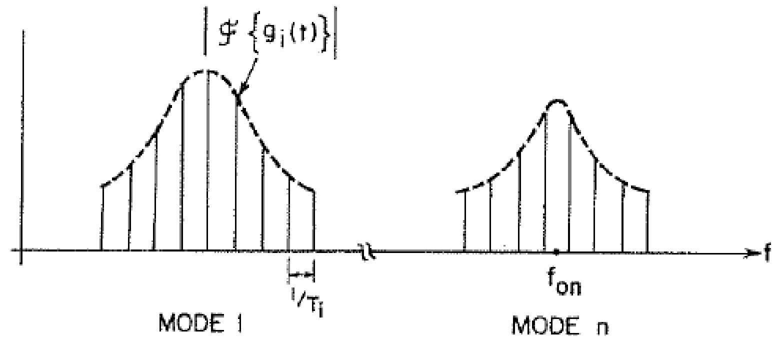
$$X_{ik} = FT[g_i(t)] = \frac{1}{T_i} \int_{-T_i/2}^{T_i/2} g_i(t)\exp(-j2\pi k f_i t)dt$$

  - Xi(f) consists of spectral lines at k*fi
  - The envelope of the spectrum dictated by the structural freq response

Vibration generation model for Roller Bearing – II Spectral Envelope

- Assuming impulse response of damped sinusoids (one per mode): $g_i(t) = \sum_n A_{in} \exp(-\alpha t) \sin(2\pi f_0$
  - Modes at resonance freq of the races
  - Approximating the outer ring as a thin ring

Spectrum of localized defect



- In summary:
  - Localized defects generate periodic excitations at the freq of impacting.
  - The effect is enhanced around structural resonant freq

Cepstrum Analysis

- *Periodic character of Bearing and Gear signature spectra results in periodic spectrum*
- *Cepstrum can reveal the periodicity in the spectrum*

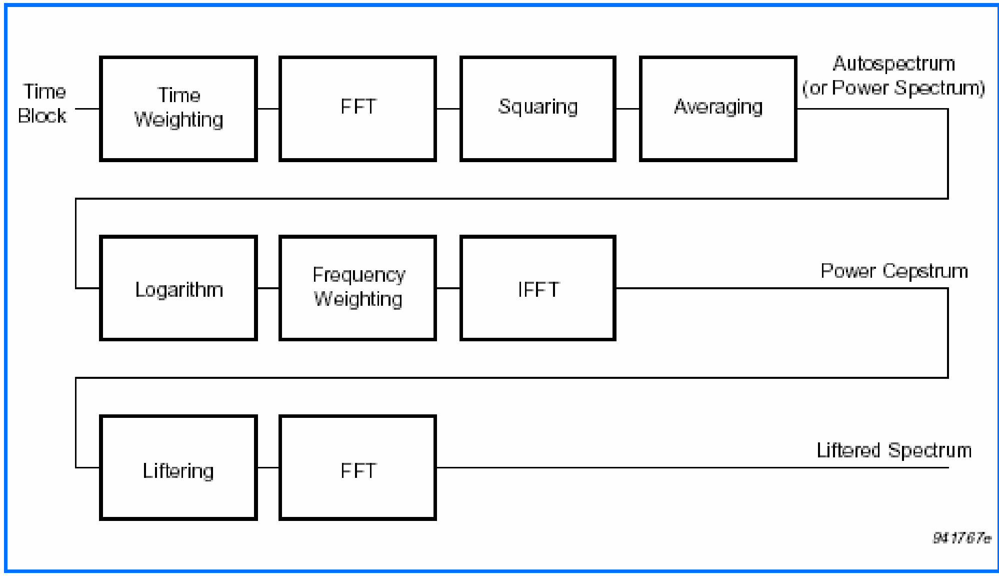
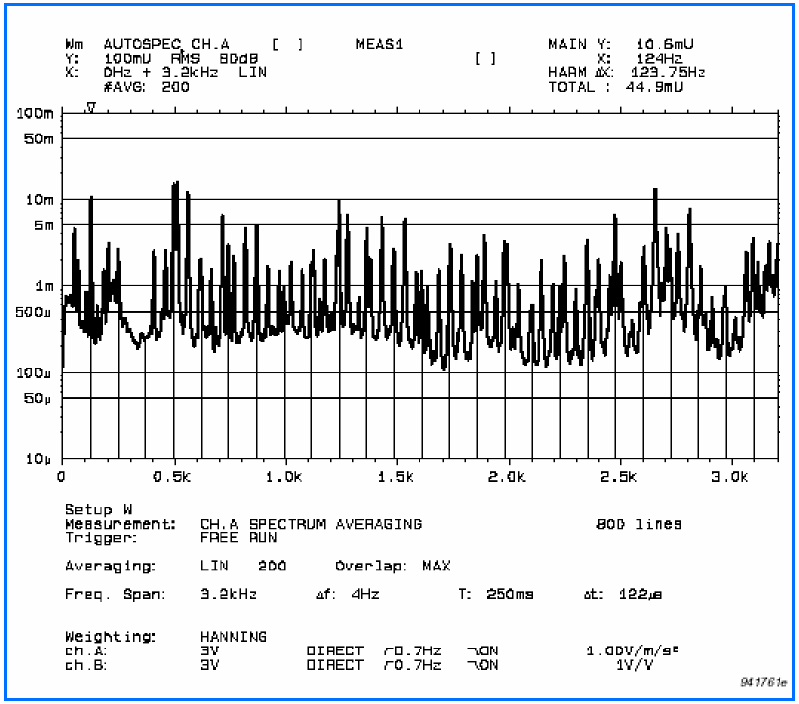
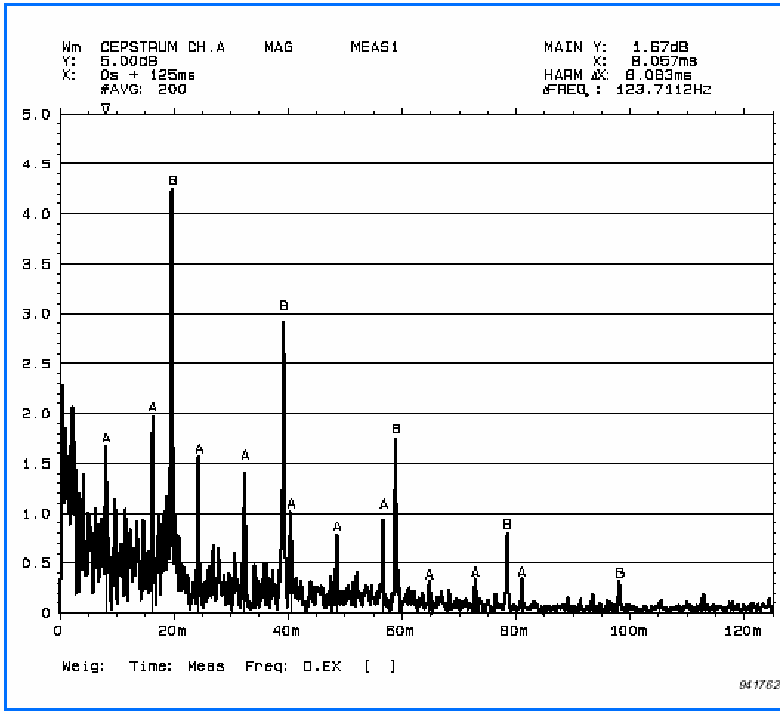


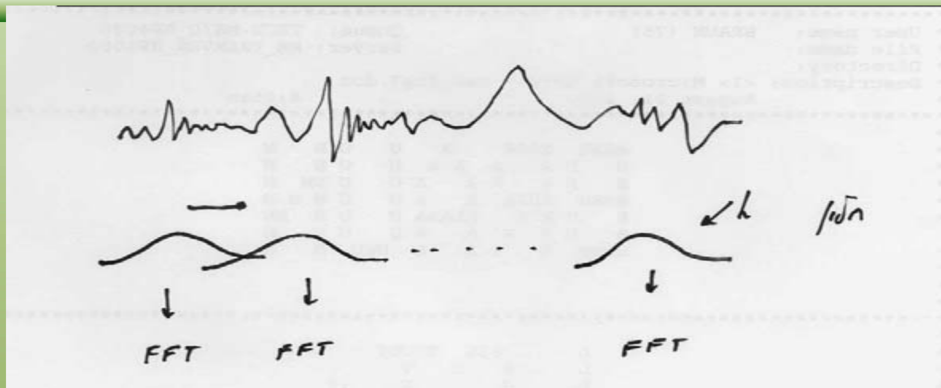
Fig.1 Block diagram showing the data flow when performing cepstrum analysis

Example – power spectrum and Cepstrum of gearbox vibration signal





STFT – Short Time Fourier Transform



$$F_x(t, f) = \int_{-\infty}^{\infty} x(t')h(t' - t)\exp(-j\ 2\pi f t')dt'$$

$$x(t) = \frac{1}{E_h}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_x(t', f')h(t - t')\exp(j2\pi t\ f')df'dt'$$

$$E_h = \int_{+\infty}^{\infty} |h(t)|^2\ dt$$

Spectrogram – time/freq analysis

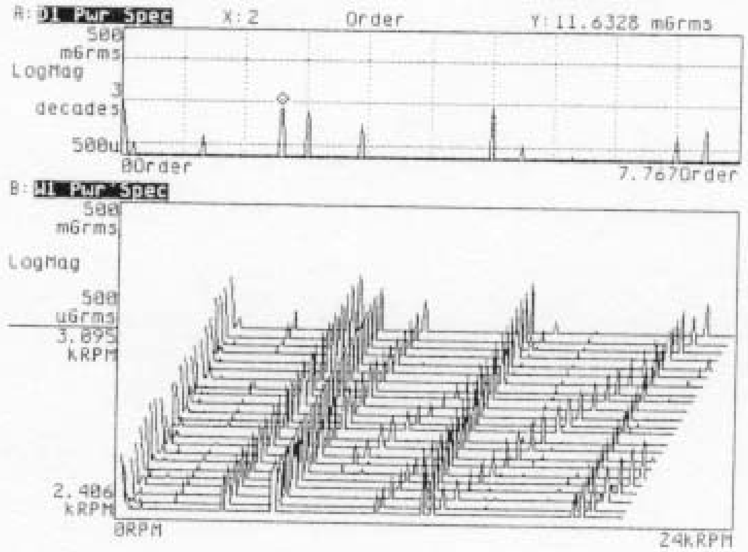
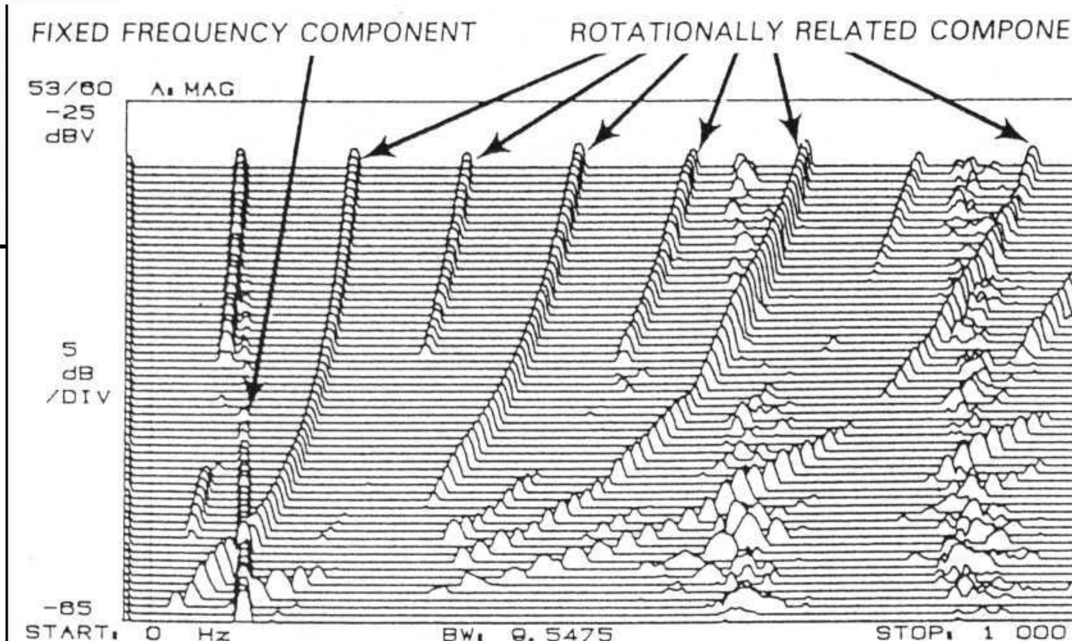


Figure 3.3-1 Spectral maps show variation in the vibration spectrum with time or rpm.



Spectral density of derived processes

- *Relationship between the spectral density of a stationary process x and its derivatives:*
  - *First derivative of Auto-correlation*

$$R_x(\tau) = E[x(t)x(t+\tau)] \Rightarrow \quad \frac{d}{d\tau} R_x(\tau) = E[x(t)\dot{x}(t+\tau)]$$

  - *Second derivative of Auto-correlation*

$$\frac{d^2}{d\tau^2} R_x(\tau) = -R_{\dot{x}}(\tau)$$

  - *From Spectrum*

$$R_x(\tau) = \int_{-\infty}^{\infty} S_x(\omega)\exp(i\omega\tau)d\omega \Rightarrow$$

$$\frac{d^2}{d\tau^2} R_x(\tau) = -\int_{-\infty}^{\infty} \omega^2 S_x(\omega)\exp(i\omega\tau)d$$

Spectral density of derived processes – con't

- Relationship between the spectral density of a stationary process x and its derivatives:
  - Spectrum
    $$S_{\dot{x}}(\omega) = \omega^2 S_x(\omega)$$
  - Mean square values:
    - Mean square Velocity

$$E[\dot{x}^2] = \int_{-\infty}^{\infty} S_{\dot{x}}(\omega)d\omega = \int_{-\infty}^{\infty} \omega^2 S_x(\omega)d\omega$$

    - Mean square Acceleration

$$E[\ddot{x}^2] = \int_{-\infty}^{\infty} S_{\ddot{x}}(\omega)d\omega = \int_{-\infty}^{\infty} \omega^4 S_x(\omega)d\omega$$

Cross spectral density

- *Cross spectral density of a pair of stationary random processes*

$$S_{xy}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{xy}(\tau)\exp(-i\omega\tau)d\tau$$

  – *Inverse relationship* $R_{xy}(\tau) = \int_{-\infty}^{\infty} S_{xy}(\omega)\exp(i\omega\tau)d$

- *Anti-symmetric cross correlation implies conjugate cross spectral*

$$R_{xy}(\tau) = R_{yx}(-\tau) \Rightarrow S_{xy}(\omega) = S_{yx}^{*}(\omega)$$

תכונות דינמיות של מערכות ליניאריות - תיאורים מתמטיים שקולים

- משוואה דיפרנציאלית ליניארית עם מקדמים קבועים

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + \cdots + b_0 x \quad m < n$$

- פונקצית תמסורת – מתארת קשר כניסה/יציאה בין האותות המותמרים (עבור תנאי התחלה אפס)

$$G(s) = \frac{Y(s)}{X(s)}; \quad Y(s) = L[y(t)]; \; X(s) = L[x(t)]$$

- תגובת התדירות – מתארת תגובה לכניסה מחזורית במצב מתמיד

$$H(j\omega) = G(s)\big|_{s=j\omega} = \frac{Y(j\omega)}{X(j\omega)}; \quad Y(j\omega) = F[y(t)]; \; X(j\omega) = F[x(t)]$$

- תגובת הלם – מתארת את תגובת המערכת לאות הלם (Impulse response)
  - במערכת סטציונרית

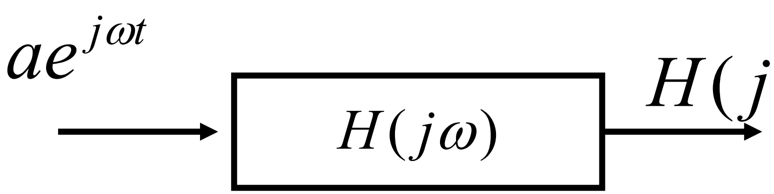
$$h(t) = F^{-1}[H(j\omega)] = g(t) = L^{-1}[G(s)]$$

משפט תגובת התדירות

- אותות טריגונומטריים:
  - אותות סינוסואידליים $u(t) = A\cos(\omega t + \phi) = \mathrm{Re}\left[e^{j\omega t + \phi}\right]$
  - אותות אקספוננציאליים מדומים $u(t) = e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$

מהווים פונקציות עצמאיות (Eigen-functions)
של מערכות ליניאריות

- במצב מתמיד:



$$H(j\omega)ae^{j\omega t} = \left|H(j\omega)\right|ae^{j\arg(H(j\omega))}e^{j\ldots}$$

- יחס אמפליטודה $\left|H(j\omega)\right|$
- היסט הפזה $\phi = \arg(H(j\omega))$

תגובה לכניסה כללית – תגובת התדירות מתגובת הלם

- תגובה של מערכת לינארית עם תגובת הלם $h(t)$ לכניסה $x(t)$ ניתנת ע"י קונבולוציה:

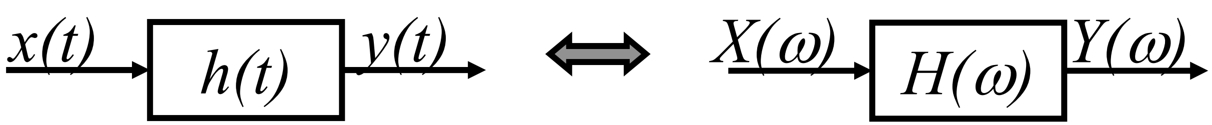
$$y(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau$$

- תכונת הקונבולוציה:

$$y(t)=h(t)\otimes x(t)\xleftrightarrow{F}Y(\omega)=H(\omega)X(\omega)$$

- משמעותה: חישוב פשוט בתחום התדר

$H(\omega)=$ תגובת התדירות



תגובה לכניסה כללית – תגובת התדירות מסופרפוזיציה

- תכונת הקונבולוציה נובעת מ:
  - ייצוג $x(t)$ כהתמרת פורייה הפוכה

    $$x(t)=\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$
  - תגובת התדירות (של פונקציות טריגונומטריות)

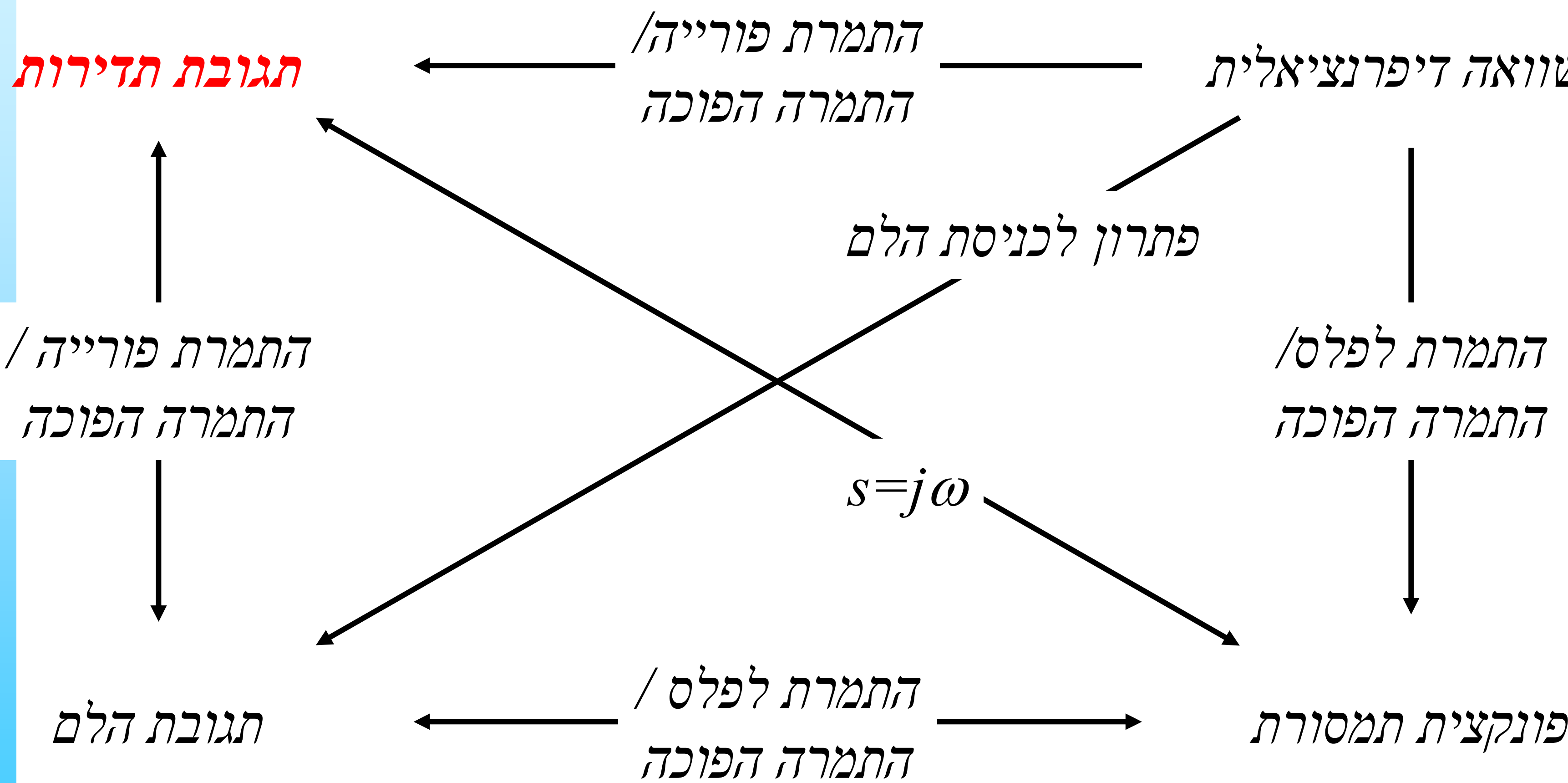
    $$x(t)=e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow y(t)=H(\omega)e^{j\omega t}$$
  - סופרפוזיציה של מערכות לינאריות

    $$x(t)=\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \longrightarrow \boxed{h(t)} \longrightarrow y(t)=\int_{-\infty}^{\infty} H(\omega)X(\omega)e^{j\omega t}d\omega$$

$H(\omega) =$ תגובת התדירות

$$\Rightarrow \quad X(\omega) \longrightarrow \boxed{H(\omega)} \longrightarrow Y(\omega)=H(\omega)X(\omega)$$

מערכות ליניאריות - ייצוגים אקוויוולנטיים



הדגמה עבור מע מסדר ראשון

מערכות ליניאריות – שיטות פתרון אקוויוולנטיות

תגובת תדירות

שוואה דיפרנציאלית

התמרת פורייה (הפוכה)

פתרון הומוגני ופרטי

פתרון בזמן

קונבולוציה

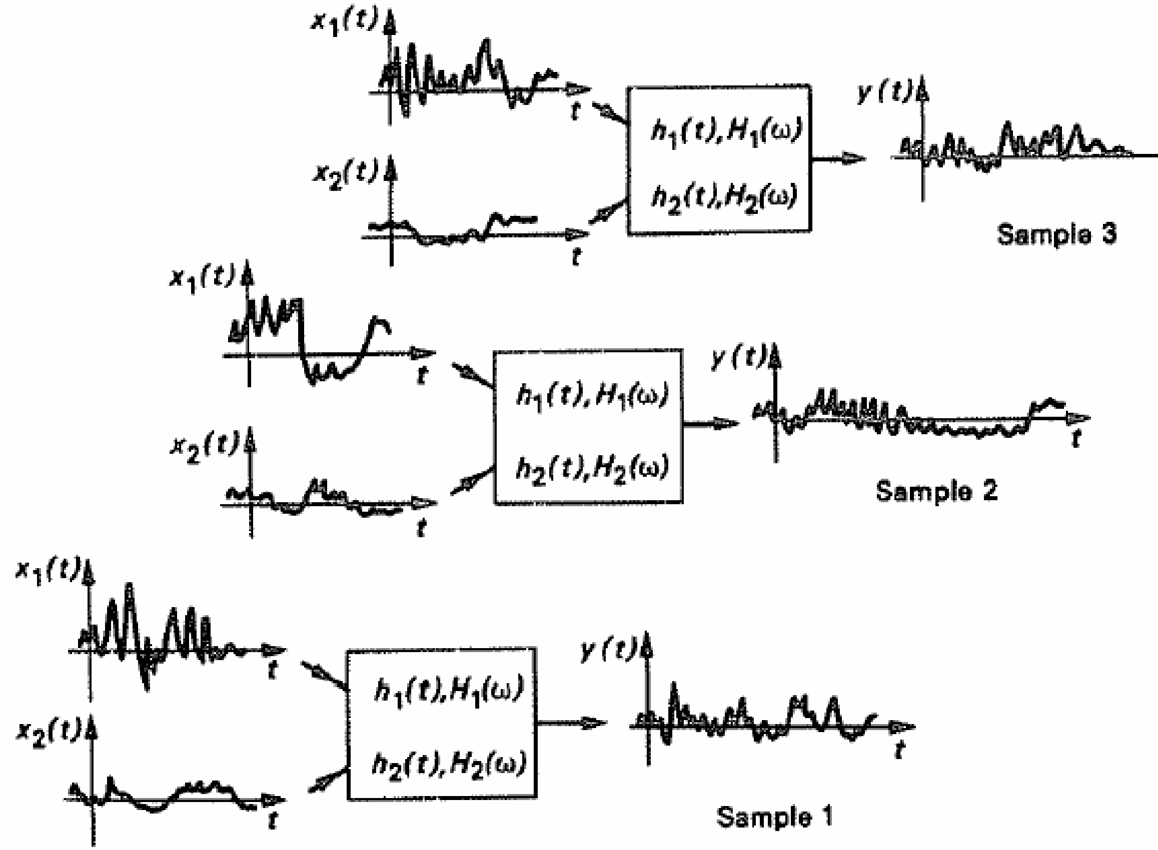
התמרת לפלס (הפוכה)

תגובת הלם

פונקצית תמסורת

Transmission of random vibrations thru stable linear systems

- *Concept of ensemble averaging for a linear system subjected to random excitation*



$$y(t)=\int_{-\infty}^{\infty}h_1(\theta)x_1(t-\theta)d\theta+\int_{-\infty}^{\infty}h_2(\theta)x_2(t-\theta)d\theta$$

Transmission of random vibrations–mean values

- *System response*

$$y(t) = \int_{-\infty}^{\infty} h_1(\theta) x_1(t-\theta) d\theta + \int_{-\infty}^{\infty} h_2(\theta) x_2(t-\theta) d\theta$$

- *Mean level (for stationary vibrations)*

$$E[y] = E[x_1] \int_{-\infty}^{\infty} h_1(\theta) d\theta + E[x_2] \int_{-\infty}^{\infty} h_2(\theta) d\theta$$

- *In terms of freq response*

$$E[y] = E[x_1] H_1(0) + E[x_2] H_2(0)$$

Transmission of random vibrations– Response Autocorrelation

- System response

$$y(t)=\int_{-\infty}^{\infty}h_1(\theta)x_1(t-\theta)d\theta+\int_{-\infty}^{\infty}h_2(\theta)x_2(t-\theta)d\theta$$

- Autocorrelation

$$R_y(\tau)=E[y(t)y(t+\tau)]$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h_1(\theta_1)h_1(\theta_2)R_{x_1}(\tau-\theta_2+\theta_1)d\theta_1d\theta_2$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h_1(\theta_1)h_2(\theta_2)R_{x_1x_2}(\tau-\theta_2+\theta_1)d\theta_1d\theta_2$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h_2(\theta_1)h_1(\theta_2)R_{x_2x_1}(\tau-\theta_2+\theta_1)d\theta_1d\theta_2$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h_2(\theta_1)h_2(\theta_2)R_{x_2}(\tau-\theta_2+\theta_1)d\theta_1d\theta_2$$

Transmission of random vibrations–Spectrum

- Spectrum

$$S_y(\omega) = H_1^*(\omega)H_1(\omega)S_{x_1}(\omega) + H_1^*(\omega)H_2(\omega)S_{x_1x_2}(\omega)$$
$$+ H_2^*(\omega)H_1(\omega)S_{x_2x_1}(\omega) + H_2^*(\omega)H_2(\omega)S_{x_2}(\omega)$$

- Single excitation
- N excitations

$$S_y(\omega) = \left|H(\omega)\right|^2 S_x(\omega)$$

$$S_y(\omega) = \sum_{r=1}^{N}\sum_{s=1}^{N} H_r^*(\omega)H_s(\omega)S_{x_rx_s}(\omega) \qquad S_{x_rx_r} = S_{x_r}$$

- Uncorrelated excitations (zero cross spectral density)

$$S_y(\omega) = \sum_{r=1}^{N}\left|H_r(\omega)\right|^2 S_{x_r}(\omega)$$

Transmission of random vibrations– Mean Square Response

- *General case* $$E[y^2] = \int_{-\infty}^{\infty} S_y(\omega)d\omega$$
- *Single excitation* $$E[y^2] = \int_{-\infty}^{\infty} |H(\omega)|^2 S_x(\omega)d\omega$$
- *Uncorrelated excitations*

$$E[y^2] = \sum_{r=1}^{N} \int_{-\infty}^{\infty} |H_r(\omega)|^2 S_{x_r}(\omega)d\omega$$

  - *Sum of mean square response due to each input separately*
- *However, when the excitations are correlated the mean square response has to be computed from Sy(w)*

Transmission of random vibrations– Cross correlation

- *Cross correlation between output and one of the inputs* $R_{x_1y}(\tau) = E[x_1(t)y(t+\tau)]$

$$R_{x_1y}(\tau) = \int_{-\infty}^{\infty} h_1(\theta)R_{x_1}(\tau-\theta)d\theta + \int_{-\infty}^{\infty} h_2(\theta)R_{x_1x_2}(\tau-\theta)d\theta$$

- *Special case: white noise x1, uncorrelated with x2, i.e.,* $R_{x_1}(\tau-\theta) = 2\pi S_0\delta(\tau-\theta)$ $R_{x_1x_2}(\tau-\theta) =$
  - *The auto-correlation is proportional to the impulse response* $R_{x_1y}(\tau) = 2\pi S_0 h_1(\tau)$
  - *Can be used to evaluate impulse response experimentally*

Transmission of random vibrations– Cross spectral density

- *Cross-spectral density between the input and the output is the FT of the cross correlation*

$$R_{x_1y}(\tau) = \int_{-\infty}^{\infty} h_1(\theta) R_{x_1}(\tau-\theta)d\theta + \int_{-\infty}^{\infty} h_2(\theta) R_{x_1x_2}(\tau-\theta)d\theta$$

$$= h_1(\tau) \otimes R_{x_1}(\tau) + h_2(\tau) \otimes R_{x_1x_2}(\tau)$$

- So $S_{x_1y}(\omega) = H_1(\omega)S_{x_1}(\omega) + H_2(\omega)S_{x_1x_2}(\omega)$
- For uncorrelated inputs $S_{xy}(\omega) = H(\omega)S_x(\omega)$

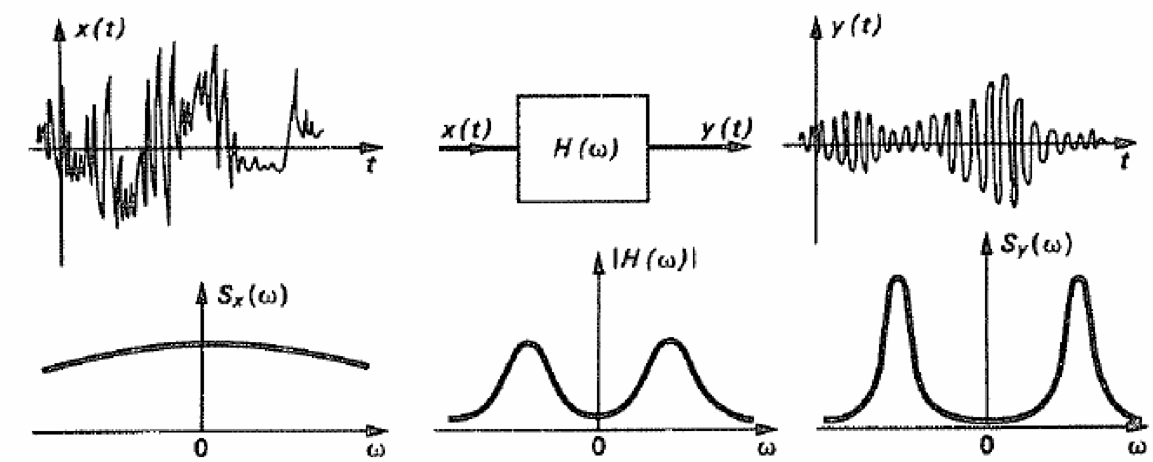
- Since Sx is real: $S_{yx}(\omega) = S^*_{xy}(\omega) = H^*(\omega)S_x(\omega)$

Transmission of random vibrations– Probability distributions

- No general method for obtaining the output pdf
- Special case: Normal pdf
  - Sum of a pair of jointly Normal RV, $y=y1+y2$ is normally dist.
  - Extended to convolution → the response of a linear system to a Normally distributed excitation is normally distributed
  - Extended to the case of multiple jointly Gaussian inputs → the output process after transmission through a linear system will also be jointly Normal
  - The derivative processes are also Normal distributed
- System response may approximate to Normal if it is a narrow band response derived from broad band excitation.
- CAUTION: poor at the tails of the distribution so predictions of maximum excursions are questionable.

Characteristics of Narrow Band Processes

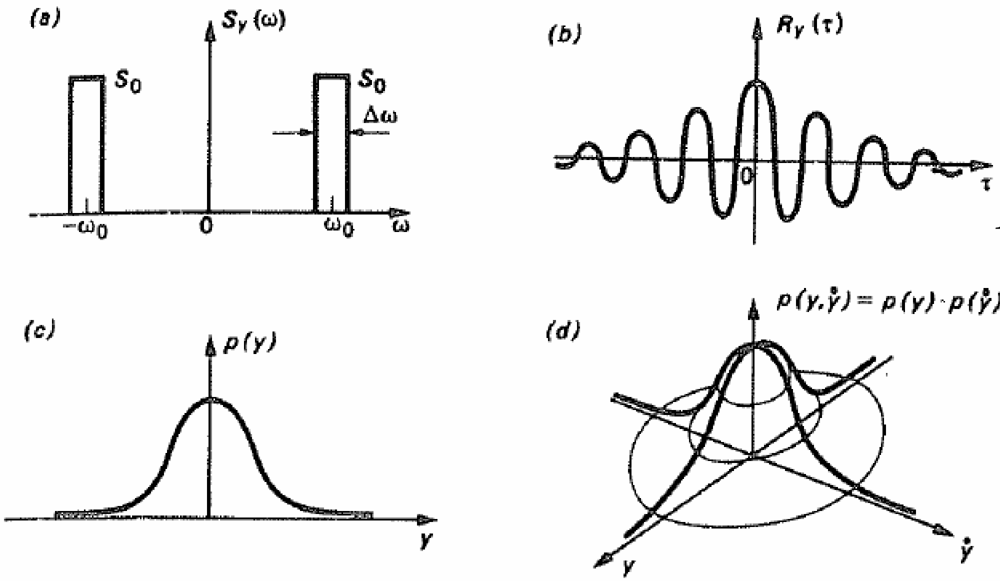
- *Narrow band processes – response of resonant systems excited by broad band noise*



  - *Amplification at resonant freq.*
  - *Attenuation at other frequencies*
  - *Vanishes at high freq since equivalent mass is high*

$$E[\dot{x}^2] = \int_{-\infty}^{\infty} S_{\dot{x}}(\omega)d\omega = \int_{-\infty}^{\infty} \omega^2 S_x(\omega)d\omega$$

Characteristics of Stationary Normal, Narrow Band Processes



-Autocorrelation

$$R_y(\tau) = 4S_0 \frac{\sin(\Delta\omega\tau/2)}{\tau} \cos(\omega_0$$

- *Mean of y* $\quad S_y(0) = 0 \Rightarrow m_y = E[y] = 0$
- *Mean of its derivative* $\quad S_{\dot{y}}(0) = 0 \Rightarrow m_{\dot{y}} = E[\dot{y}] = 0$
- *Variance of y* $\quad \sigma_y = E[y^2] = 2S_0\Delta\omega$
- *Variance of its derivative*

$$\sigma_{\dot{y}} = E[\dot{y}^2] = \int_{-\infty}^{\infty} \omega^2 S_y(\omega) d\omega \cong 2S_0\omega_0^2\Delta\omega$$

Characteristics of Stationary Normal, Narrow Band Processes

- *Coefficient of correlation*
  - *Recall*

$$\rho_{y\dot{y}} = \frac{E[y\dot{y}]}{\sigma_y \sigma_{\dot{y}}} = \frac{R_{y\dot{y}}(0)}{\sigma_y \sigma_{\dot{y}}}$$

$$\frac{d}{d\tau} R_x(\tau) = E[x(t)\dot{x}(t+\tau)]$$

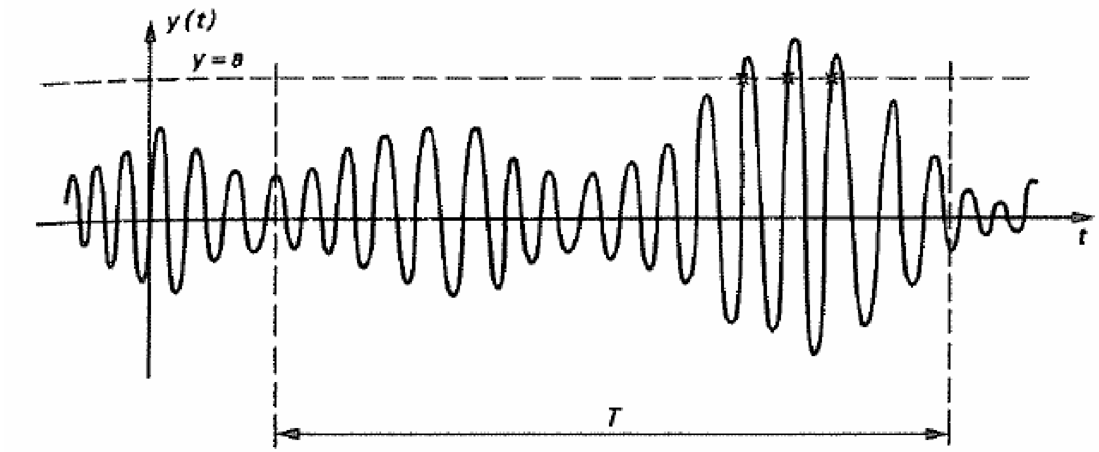
  - *But* $$E[y\dot{y}] = \frac{dR_y(\tau)}{d\tau}\Big|_{\tau=0} = i\int_{-\infty}^{\infty} \omega S_y(\omega)\,d\omega = 0$$
  - *So for any stationary random process y and its derivative are uncorrelated, and the coeff of corr is zero*
- *pdf*

$$p(y) = \frac{1}{\sqrt{2\pi}\sigma_y}\exp\left(-\frac{y^2}{2\sigma_y^2}\right)$$

$$p(y,\dot{y}) = \frac{1}{2\pi\sigma_y\sigma_{\dot{y}}}\exp\left(-\frac{1}{2}\left(\frac{y^2}{\sigma_y^2}+\frac{\dot{y}^2}{\sigma_{\dot{y}}^2}\right)\right) = p(y)p(\dot{y})$$

Narrow Band Processes – Crossing Analysis

- *Positive slope crossing of y=a in time T for a typical sample – denoted by* $n_a^+(T)$



- *Mean for all samples:* $N_a^+(T) = E[n_a^+(T)]$
- *For stationary process, the mean is proportional to T,* $N_a^+(T) = \nu_a^+ T$ *where* ν_a^+ *is the average freq*

Narrow Band Processes – Crossing Analysis con't

- *Conditions of positive slope crossing of y=a in time interval dt*
  - *Since a narrow band function is smooth, the conditions are* $y < a \quad \frac{dy}{dt} > \frac{a-y}{dt}$
- *The average freq. of positive crossing of y=a*

$$\nu_a^+ = \int_0^\infty p(a,\dot{y})\dot{y}\,d\dot{y}$$

- *Gaussian processes*

$$\nu_a^+ = \frac{1}{2\pi}\frac{\sigma_{\dot{y}}}{\sigma_y}\exp(-a^2/2\sigma_y^2$$

- *Special case of a=0 → statistical (ensemble) average freq for the processes (along time for ergodic processes)*

System identification – complementary to the problem of analysis

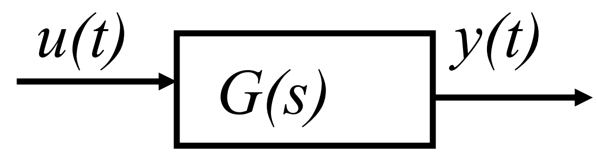
- *Analysis: response predictions based on*
  - *Input*
  - *Frequency response function FRF H(w)*
- *System identification: determine system FRF from input output measurements*
  - *Based on experimentally acquired data,*
  - *Measurement aspects have to be considered*
  - *Actual identification methods utilize sampled data*
  - *Usually presumes linear system*

Types of identification methods

- *Parameterization*
  - *Non-parameteric – no physical or mathematical description of the system is obtained*
  - *Parameteric – estimates the parameters of a presumed mathematical model*
- *Excitation signal*
  - *Rich –enabling identification of the complete system response*
    - *Transients, periodic, random, stepped sine*
  - *Harmonic (sine excitation)*
- *Assumptions about noise (uncertainties in the measurements)*
  - *Additive white noise added to the response (output)*

Parametric Identification based on system realization

- System realization:



$$\dot{x} = Ax(t) + Bu(t)$$

$$y = Cx(t) + Du(t)$$

- Laplace Transform (zero initial conditions

$$X(s) = [sI - A]^{-1} BU(s)$$

$$Y(s) = \{C[sI - A]^{-1} B + D\} U(s)$$

- So, The Transfer function is

$$G(s) = C[sI - A]^{-1} B + D \qquad H(\omega) = G(s = j\omega)$$

Frequency response identification – noiseless case

- Based on the input output relationship:

$$X(\omega) \longrightarrow \boxed{H(\omega)} \longrightarrow Y(\omega)$$

$$Y(\omega) = H(\omega)X(\omega) \Rightarrow H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

  - FRF: identified separately at each frequency
  - Impulse response: computed using inverse FT.
- Effect of signal class
  - Transient excitation:
    - Both the excitation and response would be broadband,
    - A complete FRF can then be extracted from a single test..
    - The frequency range of the identification is dictated by the richness of the excitation, i.e., by the region where excitation energy exists.

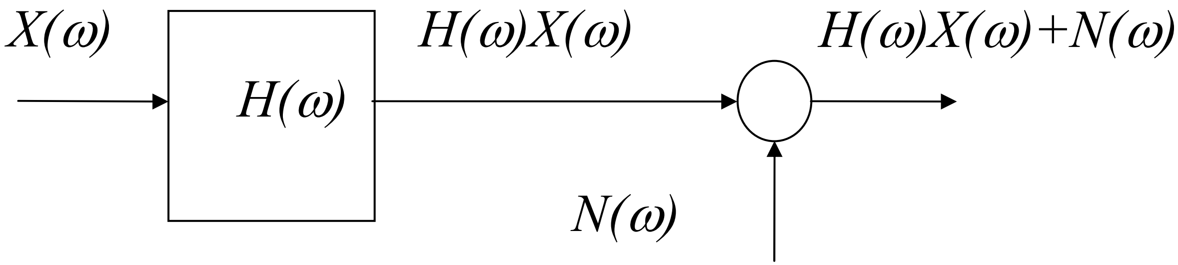
Frequency response identification – noiseless case Con't

- Effect of signal class
  - Sine excitation:
    - FRF is computed at a single frequency.
    - Stepped sine (pointer) → FRF computed step by step.
  - Random excitation:
    - Usually broadband, i.e. rich, so complete FRF be identified.
    - The computed FT $X(w)$ and $Y(w)$ are RV (with complex pdfs with significant variance.
    - The resulting variance in the FRF can be reduced by averaging different FRF estimates (and not the ratio of the averaged FT as $E[X(w)]=E[Y(w)]=0$)

$$\hat{H}(\omega) = \frac{1}{L}\sum_{l=1}^{L}\frac{Y_l(\omega)}{X_l(\omega)}$$

Frequency response identification – output additive noise

- *popular model, (even if not always justified*



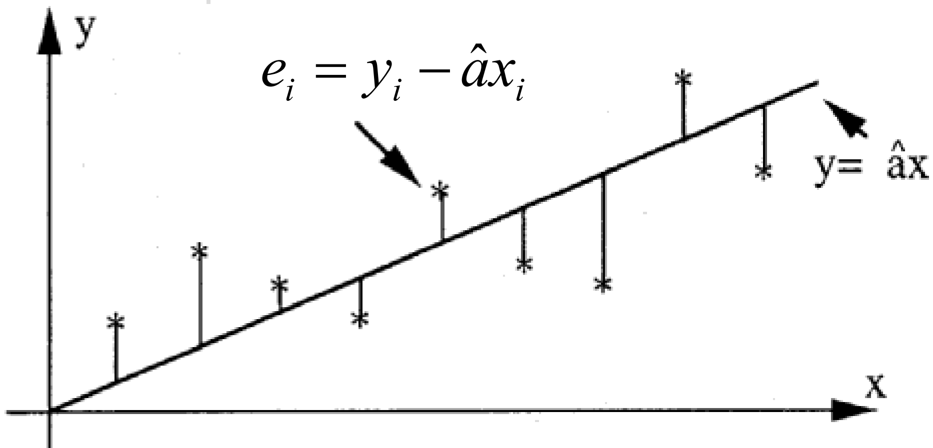
- *In analogy with least-squares regression*

$$\hat{H}_{op}(\omega)=\frac{\sum_{l=1}^{L}X_l^*(\omega)Y_l(\omega)}{\sum_{l=1}^{L}X_l^*(\omega)X_l(\omega)}=\frac{\sum_{l=1}^{L}S_{xy\,l}(\omega)}{\sum_{l=1}^{L}S_{xx\,l}(\omega)}=\frac{\bar{S}_{xy}}{\bar{S}_{xx}}$$

  – *Ratio of cross-spectrum to auto-spectrum*

קירוב ליניארי - בשיטת מינימזציה של סכום ריבועי השגיאה

$\hat{y} = ax$



דרוש: קירוב ליניארי דרך האפס

נתון: סט המדידות (x_i,y_i);
$i=1 \ldots N$

ללא רעש יציאה:

– שיערוך ממדידה בודדת $\hat{a} = \frac{y_i}{x_i}$

עם רעש יצאה: $y_i = ax_i + e_i$

– מינימום ריבועי שגיאה

- סכום ריבועי השגיאה:

$$J = \sum_{i=1}^{N} e_i^2 = \sum_{i=1}^{N} (y_i - \hat{a}x_i)^2$$

- מינימזציה:

$$\frac{dJ}{d\hat{a}} = 2\sum_{i=1}^{N} (y_i - \hat{a}x_i)x_i = 0$$

$$\hat{a} = \frac{\sum_{i=1}^{N} y_i x_i}{\sum_{i=1}^{N} x_i^2}$$

קירוב ליניארי (דרך האפס) – שגיאת השערוך

המשערך $\hat{a}$ מתאר את הרגישות

המשערך הוא גודל אקראי שתלוי בסט נקודות האימון, ומהווה פונקציה ליניארית של המדידות y_i

$$\hat{a} = \frac{\sum_{i=1}^{N} y_i x_i}{\sum_{i=1}^{N} x_i^2}$$

- הממוצע של המשערך:
  - משערך ללא-היסט unbiased

$$E[\hat{a}] = \frac{\sum_{i=1}^{N} x_i E[y_i]}{\sum_{i=1}^{N} x_i^2} = a$$

- השונות של המשערך:
  - בהנחה שהשונות של השגיאות:
    - קבועה (לא תלויה בנקודת המדידה)

$$Var[e_i] == \sigma_e^2$$

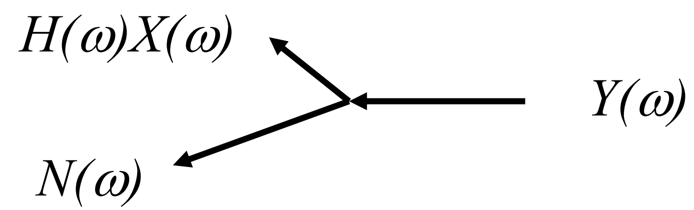
    - בלתי תלויות אחת בשניה

$$Cov[e_i e_j] == 0$$

$$Var[\hat{a}] = \frac{\sum_{i=1}^{N} x_i^2 Var[y_i]}{\left[\sum_{i=1}^{N} x_i^2\right]^2} = \frac{\sigma_e^2}{\sum_{i=1}^{N} x_i^2}$$

Frequency response identification – output additive noise - interpretation

- *Y(w) is decomposed into two orthogonal components:*
  - *Coherent part of the response*
  - *Residual part*



Frequency response identification – output additive noise - coherence

- *Analog to Coeff of determination in linear regression*

$$\gamma^2(\omega) = \frac{|S_{xy}|^2}{S_{xx}S_{yy}} \qquad 0 \le \gamma^2(\omega) \le 1$$

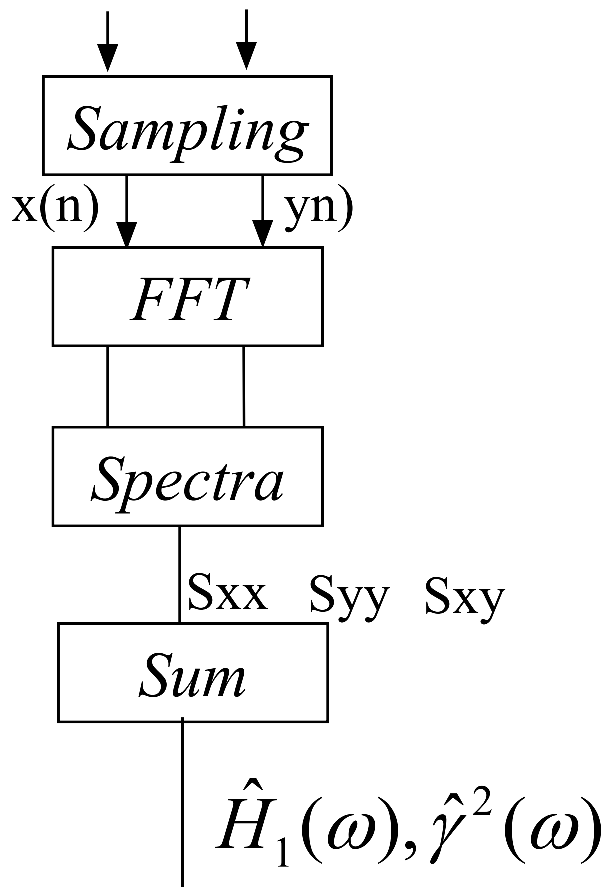
  - *Assesses the degree to which the response is linearly related to the excitation*
  - *The part of the total output energy that is attributed to linear effect of the input.*
  - *Determines the confidence in estimating the FRF*
  - *Estimated by*

$$\hat{\gamma}^2 = \frac{|\bar{S}_{xy}|^2}{\bar{S}_{xx}\bar{S}_{yy}}$$

Frequency response identification – output additive noise - coherence

- *The coherence function assesses the degree to which the response is linearly related to excitatio*
  - $\gamma^2(\omega)=1$ *indicates noiseless case*
    - *Trivia result for single pair of input/output signals*
    - *Trivia result for purely periodic excitation and response*
  - $\gamma^2(\omega)=0$ *indicates no relation*
  - $0<\gamma^2(\omega)<1$ *may result from*
    - *Additive noise*
    - *Effect of other excitations*
    - *Non-linear effects*
    - *Lack of excitation*

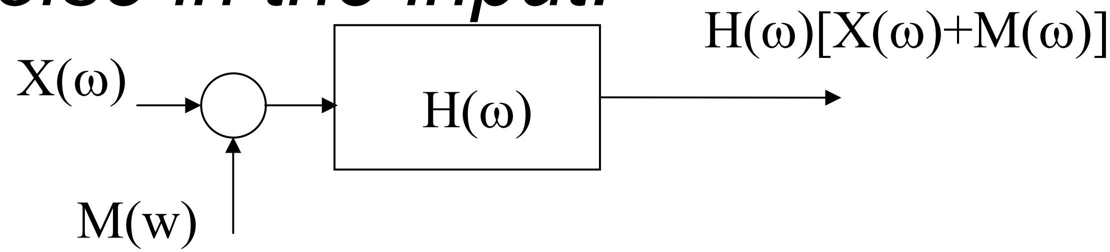
Summary of system identification for additive output noise



- *The coherence function is evaluated independently for different frequencies.*
- *The identification result is considered*
  - *meaningful for frequency ranges where the coherence is high*
  - *unacceptable for others.*

Frequency response identification – input additive noise

- *Includes noise in the input:*



- *In analogy with least-squares regression*

$$\hat{H}_{ip}(\omega)=\frac{\sum_{l=1}^{L} Y_l^*(\omega)Y_l(\omega)}{\sum_{l=1}^{L} Y_l^*(\omega)X_l(\omega)}=\frac{\sum_{l=1}^{L} S_{yy_l}(\omega)}{\sum_{l=1}^{L} S_{yx_l}(\omega)}=\frac{\bar{S}_{yy}}{\bar{S}_{xy}}=\frac{\hat{H}_{op}(\omega)}{\hat{\gamma}^2}$$

  - *Computed from the same information and general procedure as for output additive noise*
  - *Improper estimator would lead to bias errors.*

Alternative derivation: Spectral relationships

- *For the constant parameter linear system with single input:* $S_{yy}(\omega) = \left|H(\omega)\right|^2 S_{xx}(\omega)$
- *The cross spectra is:* $S_{xy}(\omega) = H(\omega) S_{xx}(\omega)$
- *So:* $S_{yy}(\omega) = H_1^*(\omega) S_{xy}(\omega)$
- *Since Sy(w) is real:* $S_{yy}(\omega) = H(\omega) S_{xy}^*(\omega) = H(\omega) S_{yx}(\omega$
- *Given estimates of the spectra at discrete freq wk, the FRF can be estimated as:*
  - *Most common (why later)*
  - *Alternatives*

$$H_{cc}(\omega_k) \equiv H_{op}(\omega_k) \equiv \frac{S_{xy}(\omega_k}{S_{xx}(\omega_k}$$

$$H_{ip}(\omega_k) \equiv \frac{S_{yy}(\omega_k)}{S_{yx}(\omega_k)}$$

$$\left|H_{ac}(\omega_k)\right|^2 \equiv \frac{S_{yy}(\omega_k)}{S_{xx}(\omega_k)}$$

Alternative derivation: coherence

- The estimated FRF can be used to calculate the output spectra: $\tilde{S}_{yy}(\omega_k) = H(\omega_k) S_{yx}(\omega_k)$
- To what extent does this calculated output spectra resemble the measure spectra:
  - Should agree well is assumptions hold (no noise, single input, constant parameters linear system, unbiased)
  - Would differ due to nonlinearities, noise and extra inputs
  - Assessed by the 'ordinary coherence function' defined b the ratio

$$\gamma^2(\omega_k) = \frac{\tilde{S}_y(\omega_k)}{S_y(\omega_k)} \qquad \gamma^2(\omega_k) = \frac{\tilde{S}_y(\omega_k)}{S_y(\omega_k)} = \frac{H(\omega_k) S_{yx}(\omega_k)}{S_y(\omega_k)} = \frac{\left|S_{xy}\right|^2}{S_x S_y}$$

misleading coherence

- *Coherence based on single record (with no averaging) is always unit, regardless of other factors (linearity, noise, inputs)*

$$S_{xx}(\omega_k) \cong \frac{T}{2\pi} X_k^* X_k \qquad S_{xy}(\omega_k) \cong \frac{T}{2\pi} X_k^* Y_k$$

$$S_{yx}(\omega_k) \cong \frac{T}{2\pi} Y_k^* X_k \qquad S_{yy}(\omega_k) \cong \frac{T}{2\pi} Y_k^* Y_k$$

- *So*

$$\gamma^2(\omega_k) = \frac{\left|S_{yx}\right|^2}{S_x S_y} \cong \frac{\left|\hat{S}_{yx}\right|^2}{\hat{S}_x \hat{S}_y} \cong \frac{X_k^* Y_k Y_k^* X_k}{X_k^* X_k Y_k^* Y_k} = 1$$

- *Hence: Averaging is critical to assess coherency*

Output noise

- Output noise n → $$S_{yy}(\omega) = S_{vv}(\omega) + S_{nn}(\omega)$$
- Where $$S_{vv}(\omega) = \left|H(\omega)\right|^2 S_{xx}(\omega)$$
- So $$S_{vv}(\omega) = \frac{\left|S_{xy}(\omega)\right|^2}{\left|S_{xx}(\omega)\right|^2} S_{xx}(\omega) = \overbrace{\gamma_{xy}^2(\omega) S_{yy}(\omega)}^{\text{Coherent Power Spectru}}$$
- Hence

$$\gamma_{xy}^2(\omega) = \frac{S_{vv}(\omega)}{S_{yy}(\omega)} \qquad \gamma_{xy}^2(\omega) = \frac{S_{yy}(\omega) - S_{nn}(\omega)}{S_{yy}(\omega)} = 1 - \frac{S_{nn}(\omega)}{S_{yy}(\omega)}$$

  - *The coherence is the fractional portion of the output spectrum of y(t) that is linearly due to x(t) and w.*

Input (measurement) noise

- *Input noise m (uncorrelated)* → $S_{xx}(\omega) = S_{uu}(\omega) + S_{mm}(\ldots$
- *Where* $S_{yy}(\omega) = \left|H(\omega)\right|^2 S_{uu}(\omega)$ $\quad$ $S_{xy}(\omega) = H(\omega) S_{uu}(\ldots$
- *So the input spectrum is*

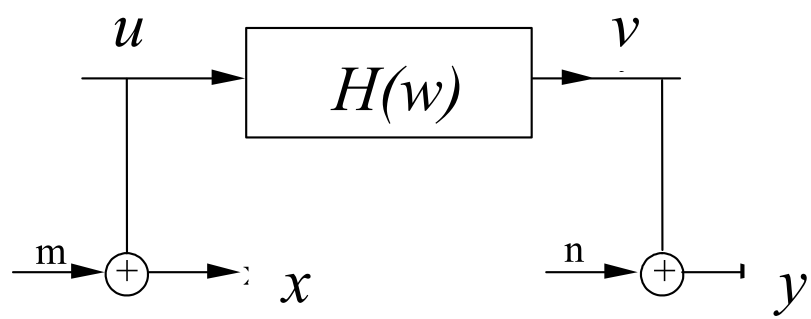
$$S_{uu}(\omega) = \frac{\left|S_{xy}(\omega)\right|^2}{S_{yy}(\omega)} = \gamma_{xy}^2(\omega) S_{xx}(\omega)$$

- *Hence*

$$\gamma_{xy}^2(\omega) = \frac{S_{uu}(\omega)}{S_{xx}(\omega)} \qquad \gamma_{xy}^2(\omega) = \frac{S_{xx}(\omega) - S_{mm}(\omega)}{S_{xx}(\omega)} = 1 - \frac{S_{mm}(\omega)}{S_{xx}(\omega)}$$

- *NOTE: the coherence is preserved under linear transformations*

Uncorrelated Input & Output measurement noise



$$S_{nm}(\omega) = S_{nx}(\omega) = S_{my}(\omega) = 0$$

(a) $$S_{xx}(\omega) = S_{uu}(\omega) + S_{mm}(\omega) \geq S_{uu}(\omega)$$

(b) $$S_{yy}(\omega) = S_{vv}(\omega) + S_{nn}(\omega) \geq S_{vv}(\omega)$$

(c) $$S_{vv}(\omega) = \left|H(\omega)\right|^2 S_{uu}(\omega)$$

(d) $$S_{xy}(\omega) = S_{uv}(\omega) = H(\omega) S_{uu}(\omega)$$

Uncorrelated Input & Output measurement noise

- *The true coherence*

$$\gamma_{uv}^2(\omega) = \frac{|S_{uv}(\omega)|^2}{S_{uu}(\omega)S_{vv}(\omega)}$$

- *The computed coherence*

$$\gamma_{xy}^2(\omega) = \frac{|S_{xy}(\omega)|^2}{S_{xx}(\omega)S_{yy}(\omega)}$$

- *From (d) and (c)*

$$|S_{xy}|^2 = |S_{uv}|^2 = |H|^2|S_{uu}|^2 = S_{uu}S_{vv}$$

- *Hence*

$$\gamma_{xy}^2(\omega) = \frac{S_{uu}S_{vv}}{S_{xx}S_{yy}} = \frac{S_{uu}S_{vv}}{(S_{uu}+S_{mm})(S_{uu}+S_{nn})} = \left(1-\frac{S_{mm}}{S_{uu}}\right)^{-1}\left(1-\frac{S_{nn}}{S_{vv}}\right)^{-1} \le 1$$

Auto- and Cross-spectrum estimates

- *Auto-spectrum method for FRF estimation*

$$S_{yy}(\omega)=\left|H(\omega)\right|^2 S_{xx}(\omega) \longrightarrow (e)\ \left|H(\omega)\right|_a^2=\frac{S_{yy}(\omega)}{S_{xx}(\omega)}$$

- *Cross-spectrum method for FRF estimation*

$$S_{xy}(\omega)=H(\omega)S_{xx}(\omega) \longrightarrow (f)\ \left|H(\omega)\right|_c^2=\frac{\left|S_{xy}(\omega)\right|^2}{S_{xx}^2(\omega)}$$

- *The ratio gives the coherence:*

$$\frac{\left|H(\omega)\right|_c^2}{\left|H(\omega)\right|_a^2}=\frac{\left|S_{xy}(\omega)\right|^2}{S_{xx}(\omega)S_{yy}(\omega)}=\gamma_{uv}^2(\omega)$$

- *Coherence less than unit implies FRF based on cross-spectra is smaller than FRF based on auto*

Auto- and Cross-spectrum estimates – effect of noise

- Auto-spectrum method (a and b in e) → biased even when Smm<<Suu

$$\left|H(\omega)\right|_a^2 = \frac{S_{vv}+S_{nn}}{S_{uu}+S_{mm}} = \left|H\right|^2\left[\frac{1+S_{nn}/S_{vv}}{1+S_{mm}/S_{uu}}\right]$$

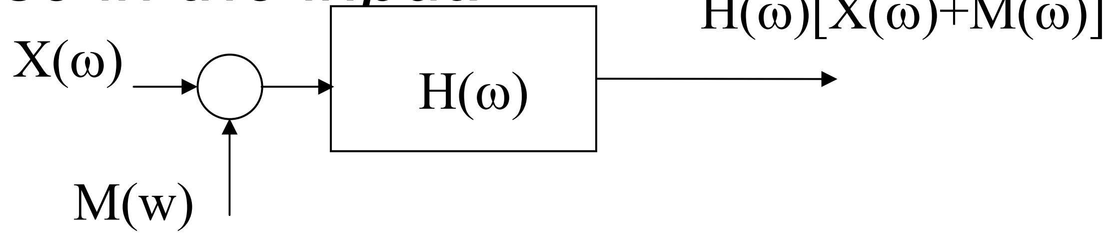
- Cross-spectrum method (a and d in f) → unbiase when Smm<<Suu regardless of Snn

$$\left|H(\omega)\right|_c = \frac{\left|S_{uv}\right|}{S_{uu}+S_{mm}} = \left|H\right|\left[\frac{1}{1+S_{mm}/S_{uu}}\right] = \left|H\right|\left[1+\frac{S_{mm}}{S_{uu}}\right]^{-1}$$

- Hence the cross-spectrum method is superior to the auto-spectrum method!!!

Input noise passing thru system – cc-estimate

- *Includes noise in the input:*



- *Special case*

Of 2 inputs

$$S_{x_1y}(\omega)=H_1S_{x_1}+H_2S_{x_1x_2}$$

$$S_{xy}(\omega)=H(\omega)\{S_{xx}(\omega)+S_{xm}(\omega)\}$$

- *In many cases the noise is correlated with the input so Sxm>0*
  - *True FRF:*
  - *Measured FRF*

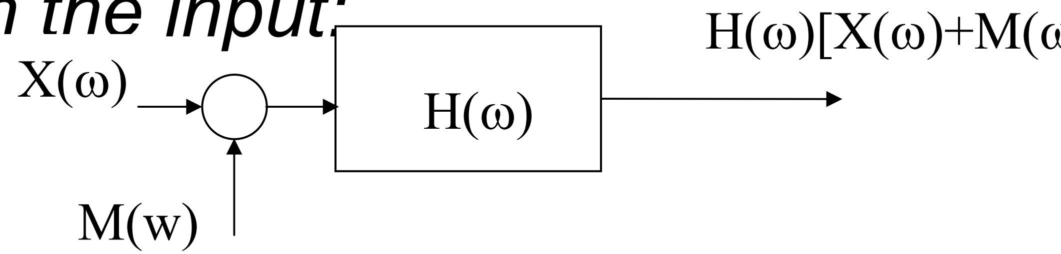
$$H(\omega)=\frac{S_{xy}(\omega)}{S_{xx}(\omega)+S_{xm}(\omega)}$$

Estimate is biased only for correlated noise

$$H_{xy}(\omega)=\frac{S_{xy}(\omega)}{S_{xx}(\omega)} \longrightarrow H_{xy}(\omega)=H(\omega)\left[1+\frac{S_{xm}(\omega)}{S_{xx}(\omega)}\right]$$

Input noise passing thru system – bias 'output noise' estimate

- *Includes noise in the input:*



$$S_y(\omega) = H_1^* H_1 S_{x_1} + H_1^* H_2 S_{x_1 x_2} + H_2^* H_1 S_{x_2 x_1} + H_2^* H_2 S_{x_2}$$

- *Special case* $S_y(\omega) = \left|H(\omega)\right|^2 \left\{S_{xx} + S_{xm} + S_{mx} + S_{mm}\right\}$

Of 2 inputs $S_{xy}(\omega) = H(\omega)\{S_{xx}(\omega) + S_{xm}(\omega)\}$

- *"Method 2"*
  - *True FRF:* $H(\omega) = \dfrac{S_{yy}(\omega)}{S_{yx}(\omega) + S_{ym}(\omega)}$
  - *Measured FRF* $H_{ip}(\omega) = \dfrac{S_{yy}(\omega)}{S_{yx}(\omega)}$

Estimate is biased even for (*) uncorrelated noise

$$H_{ip}(\omega) = H(\omega)\left[1 + \frac{S_{ym}(\omega)}{S_{yx}(\omega)}\right] = (*) H(\omega)\left[1 + \frac{S_{mm}(\omega}{S_{xx}(\omega)}\right.$$

Error mechanisms and their control

- *The general types of errors are similar to those occurring in Spectral Analysis:*
  - *Bias errors*
  - *Random errors*
  - *Leakage*
- *The coherence function is an essential quantifier of the identification → Errors have to be considered both for the FRF and the coherence*

Bias errors

- *Considered as the most severe error, especially if resonance peaks in the FRF are underestimated.*
- *Sources of bias errors:*
  - *Insufficient frequency resolution (as in Spectral analysis)*
    - *For small damping ration* ξ

$$BW = 2\xi f_n$$

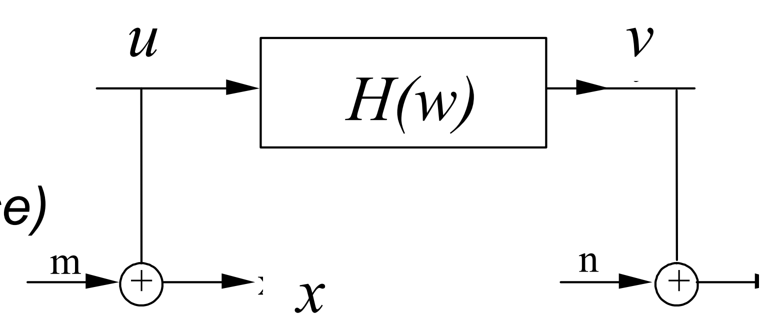
    - *Required Freq resolution*

$$\Delta f = \frac{BW}{4}$$

    - *Bias error in coherence -> underestimated, especially at resonance when using a window*

Bias errors – Con't

- Sources of bias errors:
  - *Additive noise sources*
    - *Affects magnitude (not phase)*



$$\left|H_{cc}(\omega)\right| = \left|H(\omega)\right|\left(1+\frac{S_{mm}}{S_{xx}}\right)^{-1}$$

- *Decreases coherence.*

$$\gamma^2 = \left(1+\frac{S_{mm}}{S_{xx}}\right)^{-1}\left(1+\frac{S_{nn}}{S_{yy}}\right)^{-1}$$

Bias errors

- Considered as the most severe error, especially if resonance peaks in the FRF are underestimated.
- Sources of bias errors:
  - Insufficient frequency resolution (as in Spectral analysis)
  - Additive noise sources.
  - Unmonitored additional excitations
  - delays between excitation and response

Alternative derivation: Spectra relationships

- For the constant parameter linear system with two inputs and one output:

$$S_y(\omega) = H_1^*(\omega) H_1(\omega) S_{x_1}(\omega) + H_1^*(\omega) H_2(\omega) S_{x_1 x_2}(\omega)$$

$$+ H_2^*(\omega) H_1(\omega) S_{x_2 x_1}(\omega) + H_2^*(\omega) H_2(\omega) S_{x_2}(\omega)$$

- The cross spectra are:

$$S_{x_1 y}(\omega) = H_1(\omega) S_{x_1}(\omega) + H_2(\omega) S_{x_1 x_2}(\omega)$$

$$S_{x_2 y}(\omega) = H_2(\omega) S_{x_2}(\omega) + H_1(\omega) S_{x_1 x_2}(\omega)$$

- So:

$$S_y(\omega) = H_1^*(\omega) S_{x_1 y}(\omega) + H_2^*(\omega) S_{x_2 y}(\omega)$$

- Since Sy(w) is real:

$$S_y(\omega) = H_1(\omega) S_{x_1 y}^*(\omega) + H_2(\omega) S_{x_2 y}^*(\omega) = H_1(\omega) S_{y x_1}(\omega) + H_2(\omega) S_{y x_2}(\omega$$

Alternative derivation: Spectra relationships → estimated FRF

- *Given estimates of the spectra and cross spectra at discrete freq wk, the FRF can be estimated as:*
  - *For two inputs and one output:*

$$H_1(\omega_k) = \frac{S_{x_2} S_{x_1 y} - S_{x_1 x_2} S_{x_2 y}}{S_{x_2} S_{x_1} - S_{x_1 x_2} S_{x_2 x_1}} \qquad H_2(\omega_k) = \frac{S_{x_1} S_{x_2 y} - S_{x_2 x_1} S_{x_1 y}}{S_{x_1} S_{x_2} - S_{x_2 x_1} S_{x_1 x_2}}$$

  - *For single input:*

$$H(\omega_k) = \frac{S_{xy}}{S_x}$$

Alternative derivation: Spectra relationships → coherence

- *The estimated FRF can be used to calculate the output spectra:*

$$\tilde{S}_y(\omega_k) = H_1(\omega_k) S_{yx_1}(\omega_k) + H_2(\omega_k) S_{yx_2}(\omega_k)$$

- *To what extent does this calculated output spectra resemble the measure spectra:*
  - *Should agree well is assumptions hold (no noise, only 2 inputs, constant parameters linear system)*
  - *Would differ due to nonlinearities, noise and extra inputs*
  - *Assessed by the 'multiple coherence function' defined by the ratio*

$$\gamma^2(\omega_k) = \frac{\tilde{S}_y(\omega_k)}{S_y(\omega_k)}$$

Alternative derivation: Spectra relationships → coherence

- *Multiple coherence function:*

$$\gamma^2(\omega_k) = \frac{\tilde{S}_y(\omega_k)}{S_y(\omega_k)} = \frac{H_1(\omega_k)S_{yx_1}(\omega_k) + H_2(\omega_k)S_{yx_2}(\omega_k)}{S_y(\omega_k)}$$

$$= \frac{S_{x1}\left|S_{yx_2}\right|^2 + S_{x2}\left|S_{yx_1}\right|^2 - 2\,\mathrm{Re}\left\{S_{yx1}S_{x_1x_2}S_{x_2y}\right\}}{S_y(S_{x1}S_{x_2} - S_{x1x2}S_{x_2x_1})}$$

- *For the single input case the ordinary coherence function is:*

$$\gamma^2(\omega_k) = \frac{\tilde{S}_y(\omega_k)}{S_y(\omega_k)} = \frac{H(\omega_k)S_{yx}(\omega_k)}{S_y(\omega_k)} = \frac{\left|S_{yx}\right|^2}{S_xS_y}$$

System Identification
Multiple input single output

- For the constant parameter linear system with two inputs and one output:

$$S_y(\omega) = H_1^*(\omega)H_1(\omega)S_{x_1}(\omega) + H_1^*(\omega)H_2(\omega)S_{x_1x_2}(\omega)$$
$$+ H_2^*(\omega)H_1(\omega)S_{x_2x_1}(\omega) + H_2^*(\omega)H_2(\omega)S_{x_2}(\omega)$$

- The cross spectra are:

$$S_{x_1y}(\omega) = H_1(\omega)S_{x_1}(\omega) + H_2(\omega)S_{x_1x_2}(\omega)$$

$$S_{x_2y}(\omega) = H_2(\omega)S_{x_2}(\omega) + H_1(\omega)S_{x_2x_1}(\omega)$$

- So:

$$S_y(\omega) = H_1^*(\omega)S_{x_1y}(\omega) + H_2^*(\omega)S_{x_2y}(\omega)$$

- Since Sy(w) is real:

$$S_y(\omega) = H_1(\omega)S_{x_1y}^*(\omega) + H_2(\omega)S_{x_2y}^*(\omega) = H_1(\omega)S_{yx_1}(\omega) + H_2(\omega)S_{yx_2}($$

Alternative derivation: Spectra relationships → estimated FRF

- *Given estimates of the spectra and cross spectra at discrete freq wk, the FRF can be estimated as:*
  - *For two inputs and one output:*

$$H_1(\omega_k) = \frac{S_{x_2} S_{x_1 y} - S_{x_1 x_2} S_{x_2 y}}{S_{x_2} S_{x_1} - S_{x_1 x_2} S_{x_2 x_1}} = \frac{S_{x_1 y}}{S_{x_1 x_1}} \frac{\left[1 - \dfrac{S_{x_1 x_2} S_{x_2 y}}{S_{x_2 x_2} S_{x_1 y}}\right]}{1 - \gamma^2_{x_1 x_2}}$$

  - *For single (or uncorrelated) input:* $H_{1s}(\omega_k) = \dfrac{S_{x_1 y}}{S_{x_1 x_1}}$

Alternative derivation: Spectra relationships → coherence

- The estimated FRF can be used to calculate the output spectra:

$$\tilde{S}_y(\omega_k) = H_1(\omega_k) S_{yx_1}(\omega_k) + H_2(\omega_k) S_{yx_2}(\omega_k)$$

- To what extent does this calculated output spectra resemble the measure spectra:
  - Should agree well is assumptions hold (no noise, only 2 inputs, constant parameters linear system)
  - Would differ due to nonlinearities, noise and extra inputs
  - Assessed by the 'multiple coherence function' defined by the ratio

$$\gamma^2(\omega_k) = \frac{\tilde{S}_y(\omega_k)}{S_y(\omega_k)}$$

Alternative derivation: Spectra relationships → coherence

- *Multiple coherence function:*

$$\gamma^2(\omega_k) = \frac{\widetilde{S}_y(\omega_k)}{S_y(\omega_k)} = \frac{H_1(\omega_k)S_{yx_1}(\omega_k) + H_2(\omega_k)S_{yx_2}(\omega_k)}{S_y(\omega_k)}$$

$$= \frac{S_{x1}\left|S_{yx_2}\right|^2 + S_{x2}\left|S_{yx_1}\right|^2 - 2\,\mathrm{Re}\left\{S_{yx_1}S_{x_1x_2}S_{x_2y}\right\}}{S_y(S_{x1}S_{x_2} - S_{x1x2}S_{x_2x_1})}$$

- *For the single input case the ordinary coherence function is:*

$$\gamma^2(\omega_k) = \frac{\widetilde{S}_y(\omega_k)}{S_y(\omega_k)} = \frac{H(\omega_k)S_{yx}(\omega_k)}{S_y(\omega_k)} = \frac{\left|S_{yx}\right|^2}{S_xS_y}$$

Error mechanisms and their control

- The general types of errors are similar to those occurring in Spectral Analysis:
  - Bias errors - Considered as the most severe error, especially if resonance peaks in the FRF are underestimated.
  - Random errors
  - Leakage
- The coherence function is an essential quantifier of the identification
  - Coherence function should be estimated too
  - Errors have to be considered both for the FRF and the coherence

Bias errors

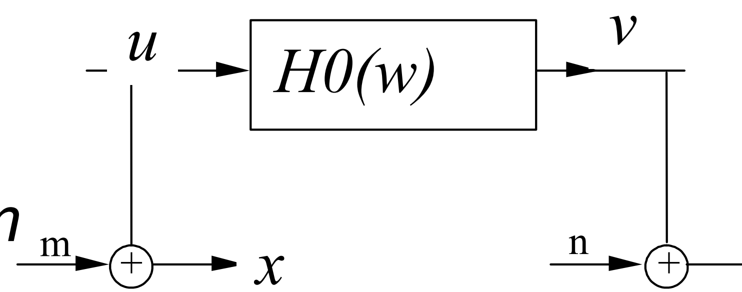
- *Considered as the most severe error, especially if resonance peaks in the FRF are underestimated.*
- *Sources of bias errors:*
  - *Insufficient frequency resolution (as in Spectral analysis)*
  - *Uncorrelated measurement input noise Not going thru system*
  - *Correlated unmonitored excitations*
  - *Nonlinearities*
  - *delays between excitation and response*

Bias errors

- Sources of bias errors:
  - Insufficient frequency resolution (as in Spectral analysis)
    - Bias Error in auto/ cross spectrum (Bendat p76) $\varepsilon_b = -\frac{1}{3}\left(\frac{B_e}{B_r}\right.$
      - Be = spectral resolution bw = 1/T (Record time!!!) $B_e = \Delta f =$
      - Br = Half point bw of spectral peak
        - For small damping ratio ξ $B_r \equiv BW = 2\xi f_n$
    - Required Freq resolution $\Delta f = \frac{BW}{4}$ $\varepsilon_b \approx 4\%$
    - Bias error in coherence -> underestimated, especially at resonance when not using a window (due to leakage)

Bias errors – Con't

- Sources of bias error
  - Uncorrelated measurement input noise Not going thru system



    - Affects magnitude (not phase)

    FRF is underestimated

$$\left|\hat{H}_1(\omega)\right| = \left|H_0(\omega)\right| \left(1 + \frac{S_{mm}}{S_{xx}}\right)^{-1}$$

    - Decreases coherence.

$$\gamma^2 = \left(1 + \frac{S_{mm}}{S_{xx}}\right)^{-1} \left(1 + \frac{S_{nn}}{S_{yy}}\right)^{-1}$$

Bias errors – Con't

- Sources of bias error
  - Correlated unmonitored excitations:
    - 2 inputs, x and z, thru diff paths:

$$\frac{H_{meas}(\omega_k)}{H_{real}(\omega_k)} = \frac{1-\gamma_{xz}^2}{\left[1-\dfrac{S_{xz}S_{zy}}{S_{zz}S_{xy}}\right]}$$

$$H_1(\omega_k) = \frac{S_{xy}}{S_{xx}} \frac{\left[1-\dfrac{S_{xz}S_{zy}}{S_{zz}S_{xy}}\right.}{1-\gamma_{xz}^2}$$

Bias only for correlated excitation!!!

- Uncorrelated excitation appear as uncorrelated op noise and cause no bias (but contribute to random err)

  - Correlated input z thru same path: $S_{xy} = H_1\{S_{xx} + S_{x}$

$$H_1 = \frac{S_{xy}}{S_{xx}} \frac{1}{\{1+S_{xz}/S_{xx}\}} \longrightarrow \frac{H_{meas}}{H_{real}} = 1 + \frac{S_{xz}}{S_{xx}}$$

Bias errors – Averaging Note

- Sources of bias error
  - *Note: The estimated FRF is based on averaging the FRF estimated from single records:*

$$E[\hat{H}_{single}(\omega_k)] = E\left[\frac{S_{xy}(\omega_k)}{S_{xx}(\omega_k)}\right] \neq \frac{E[S_{xy}(\omega_k)]}{E[S_{xx}(\omega_k)]} = \hat{H}_{avg}(\omega_k)$$

  - *So*

$$E[\hat{H}_{single}(\omega_k)] \neq \hat{H}_{avg}(\omega_k)$$

  - *But this bias is usually negligible when using many averages*

Bias errors – Con't

- Sources of bias error
  - Nonlinearities
    - The estimates produce only a linear approximation to the system response
    - HOWEVER, the result is the best possible linear approximation in the least squares sense for the system response function UNDER THE SPECIFED INPT CONDITIONS → MOTIVEATES estimating the FRF of systems with suspected non-linearities using natural inputs or simulated natural inputs rather than arbitrary lab input

Bias Errors in Coherence estimate

- *Bias error produce indicative anomalies in the measured coherence function*
- *Guidelines (based on experience and common sense)*
  - $\hat{\gamma}_{xy}^{2}(\omega)$ *falls broadly over a freq range where* $\left|\hat{H}(\omega)\right|$ *is not small and Gxx is relatively small >> input noise due to meas noise*
    - *Will also cause random error*
  - $\hat{\gamma}_{xy}^{2}(\omega)$ *notches sharply at a freq where* $\left|\hat{H}(\omega)\right|$ *displays a sharp peak or notch → inadequate spectral resolution (or nonlinearities at peaks of*
  - *Resolve by repeating with improved resolution.*
    - *Increased coherence confirms resolution issue;*
    - *otherwise, nonlinearities should be studied*

Random Errors

- Sources
  - Measurement noise
    - Output noise
    - Input noise
  - Unmonitored excitations (uncorrelated with the measured input) – seems like output meas noise
  - Nonlinearities
- Related to
  - The ip/op coherence function $\gamma_{xy}^2(\omega)$
  - Number of averages (in spectral estimates) n_d

Random Errors

- *Normalized random error in the estimated*
  - *Autospectrum*

$$\varepsilon[\hat{S}_{xx}] \approx \frac{1}{\sqrt{n_d}}$$

  - *Cross-spectrum*

$$\varepsilon[|\hat{S}_{xy}|] \approx \frac{1}{|\hat{\gamma}_{xy}(\omega)|\sqrt{n_d}}$$

  - *Gain factor*

$$\varepsilon[|\hat{H}(\omega)|] \approx \frac{\sqrt{1-\hat{\gamma}^2_{xy}(\omega)}}{|\hat{\gamma}_{xy}(\omega)|\sqrt{2n_d}}$$

  - *Phase*

$$\varepsilon[\hat{\phi}(\omega)] \approx \sin^{-1}\{\varepsilon[|\hat{H}(\omega)|]\}$$

Random Errors – con't

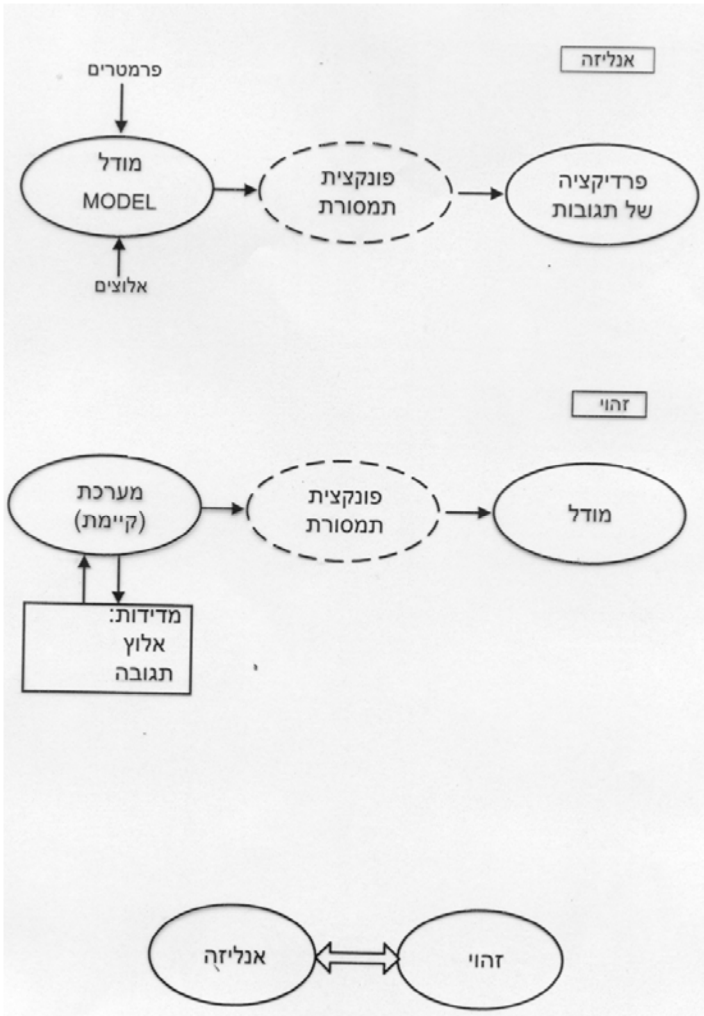
- *Normalized random error in the estimated*
  - *Coherence*

$$\varepsilon[\hat{\gamma}_{xy}^2] \approx \frac{\sqrt{2}\sqrt{1-\hat{\gamma}_{xy}^2(\omega)}}{\left|\hat{\gamma}_{xy}(\omega)\right|\sqrt{n_d}}$$

- *Low coherence → large random error*
- *Lab vs Field conditions*
  - *In lab: Experiments designed with well defined single input and low meas. noise so coherence is close to one >>modest number of averages*
  - *In field: coherence might be low especially with natural inputs*

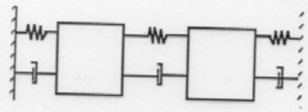
Random Errors - guidelines

- Guidelines (based on experience and common sense)
  - $\hat{\gamma}^2_{xy}(\omega)$ falls broadly over a freq range where $\left|\hat{H}(\omega)\right|$ is small >> output noise due to meas. noise or contributions of uncorrelated input
  - $\hat{\gamma}^2_{xy}(\omega)$ falls broadly over a freq range where $\left|\hat{H}(\omega)\right|$ is not small and Gxx is relatively small >> input noise due to meas noise
    - Will also cause bias error
  - Sharply peaked gain factors (at resonance), $\hat{\gamma}^2_{xy}(\omega)$ will usually peak sharply too (high SNR)
    - No peak, or even notches>>
      - Non-linearties
      - Resolution Bias error



Analytical and Experimental Modal Analysis

Modal description of vibrating system

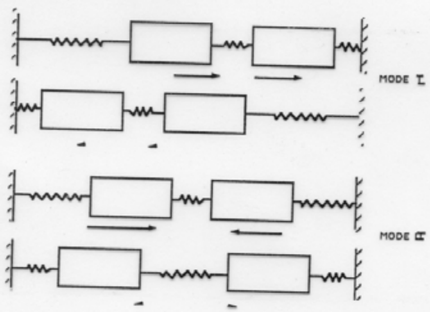


Two deflection shapes exist for each eigenfrequency.

They are described by the modal matrix

Deflection shapes are eigenvectors of modal matrix

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} = \left[(\Phi)_1 \; (\Phi)_2 \right]$$

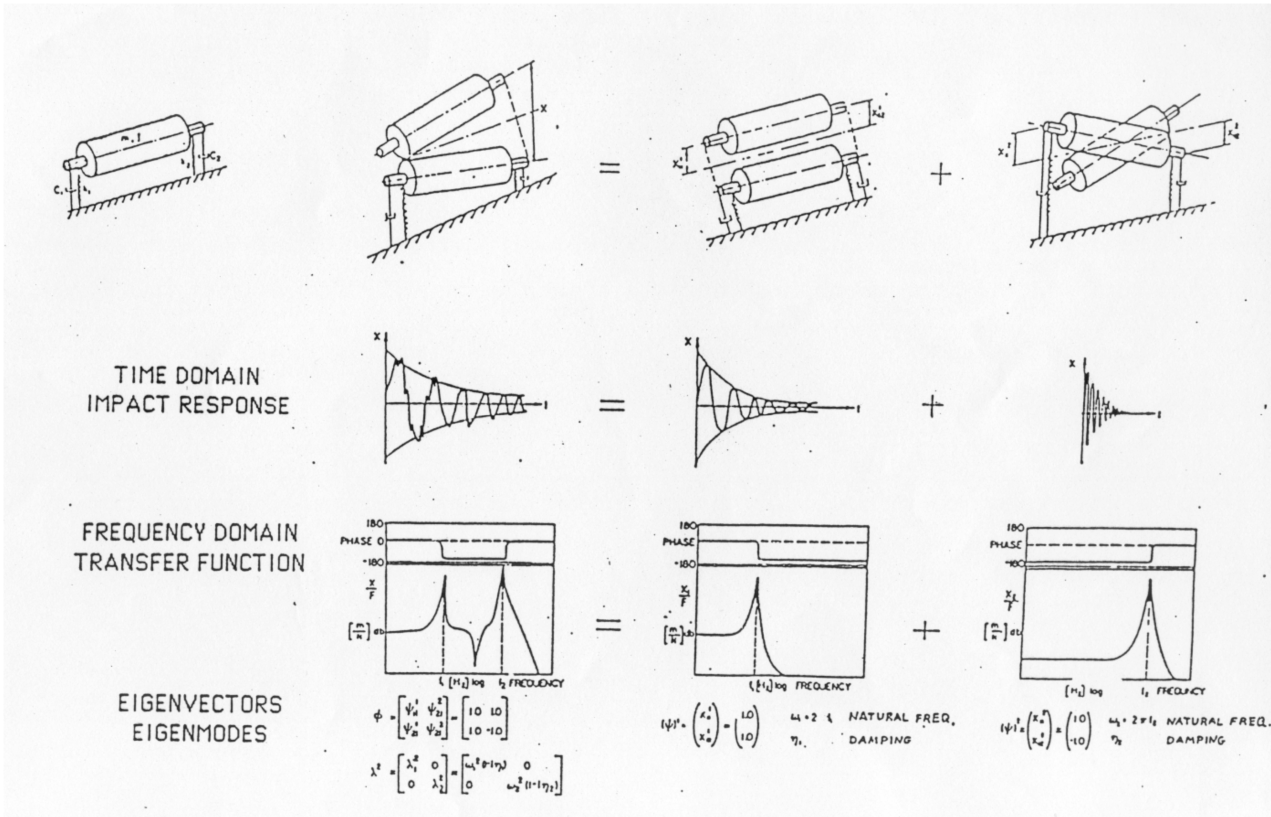


$$(\Phi)_r = \begin{bmatrix} \Phi_{1r} \\ \Phi_{2r} \end{bmatrix}$$

The eigenvalue matrix describes natural frequencies

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\lambda_r = \omega_r^2$$



Signal Processing: Modal Analysis

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + c_2(\dot{x}_1 - \dot{x}_2) + k_1x_1 + k_2(x_1 - x_2) = f_1$$

$$m_2\ddot{x}_2 + (c_2 + c_3)\dot{x}_1 + (k_2 + k_3)x_2 - c_2\dot{x}_1 - k_2x_1 = f_2$$

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 + (k_1 + k_2)x_1 - c_2\dot{x}_2 - k_2x_2 = f_1$$

$$m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_3x_2 + k_2(x_2 - x_1) = f_2$$

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad [c] = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \quad [k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

$$(x) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (\dot{x}) = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} \quad (\ddot{x}) = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} \quad (f) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

General: $[m](\ddot{x}) + [c](\dot{x}) + [k](x) = (f)$

$$m(N \times N) \quad c(N \times N) \quad x(N \times 1) \quad f(N \times 1)$$

Undamped system:

$$[c] = 0$$

$$(\ddot{x}) = -\omega^2 (\psi) \, exp(j\omega t)$$

$$-\omega^2 [m](\psi) + [k](\psi) = 0$$

$$-\lambda [m](\psi) + [k](\psi) = 0$$

$$\left\{ [m]^{-1} [k] - \lambda [I] \right\} (\psi) = 0$$

λ_i *eigenvalue*

$(\psi)_i$ *eigenvector*

$m_1 = 5\ kg\ m_2 = 10\ kg \qquad k_1 = k_2 = 2N/m \qquad k_3 = 4N/m$

$$\left| \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0 \qquad (4/5-\lambda)(3/5-\lambda)-(-1/5)(-2/5)=0$$

$$\lambda_1 = 2/5 \qquad \omega_1 = (2/5)^{1/2} \qquad [rad/sec]$$

$$\lambda_2 = 1 \qquad \omega_2 = 1 \qquad [rad/sec]$$

$$\lambda_1 = \omega_1^2 = 2/5 \qquad \begin{bmatrix} 4/5-2/5 & -2/5 \\ -1/5 & 3/5-2.5 \end{bmatrix} \begin{pmatrix} \psi_{11} \\ \psi_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2/5\ \psi_{11} - 2/5\psi_{21} = 0$$

$$\psi_{11} = \psi_{21}$$

$$(\psi)_1 = \begin{pmatrix} \psi_{11} \\ \psi_{21} \end{pmatrix} = \begin{pmatrix} \psi_{11} \\ \psi_{11} \end{pmatrix}$$

$$\lambda_2 = \omega_2^2 = 1 \qquad (\psi)_2 = \begin{pmatrix} \psi_{12} \\ -\frac{1}{2}\psi_{12} \end{pmatrix}$$

Multi Degree of Freedom Models

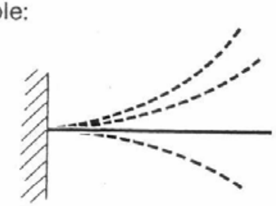
• Mode Shape

$$[B(p_r)]\,\{X\}_r = \{0\} \qquad (1)$$

$\{X\}_r = \{\psi\}_r$ mode shape for mode # r

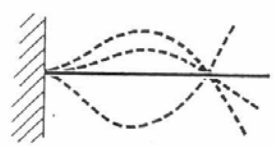
Example:

$\{\psi\}_1$



$$\begin{Bmatrix} 0{,}00 \\ ,07 \\ ,17 \\ ,33 \\ ,50 \\ ,67 \\ 1{,}00 \end{Bmatrix}$$

$\{\psi\}_2$



$$\begin{Bmatrix} 0{,}00 \\ ,28 \\ ,67 \\ 1{,}00 \\ ,67 \\ 0{,}00 \\ -,50 \end{Bmatrix}$$

$\{\psi\}_r$ is solution for homogenous equation (1), i.e. only the relative deflections are found

841806/1

$$\lambda_i \neq \lambda_j \quad (\psi)_i^T \; [m](\psi)_j = 0$$

$$(\psi)_i^T [k](\psi)_j = 0$$

Weighted Orthogonality relations

$$\lambda_i = \lambda_j \quad (\psi)_i^r [m](\psi)_i = M_i$$

$$(\psi)_i^T [k](\psi)_i = K_i$$

$$[\psi]^T [m][\psi] = \begin{bmatrix} \ddots & & \\ & M_i & \\ & & \ddots \end{bmatrix}$$

Generalized mass and stiffness

$$[\psi]^T [k][\psi] = \begin{bmatrix} \ddots & & \\ & K_i & \\ & & \ddots \end{bmatrix}$$

$$K_i/M_i = \omega_i^2$$

$$\omega_1 = (2/5)^{1/2} \quad set \;\; \psi_{11} = 1 \quad (\psi)_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = M_1 = 15$$

$$K_1 = \omega_1^2 M_1 = 6 \quad \omega_2 = 1 \quad Set \;\; \psi_{12} = 1 \quad (\psi)_2 = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}^T \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} = M_2 = 15/2$$

$$K_2 = \omega_2^2 M_2 = 15/2$$

$$[M] = \begin{bmatrix} 15 & 0 \\ 0 & 15/2 \end{bmatrix} \quad [K] = \begin{bmatrix} 6 & 0 \\ 0 & 15/2 \end{bmatrix}$$

Mass normalized eigenvectors

$$M_i = 1$$

$$K_i = \omega_i^2 M_i = \omega_i^2$$

$$\omega_1 = (2/5)^{1/2} \qquad \begin{pmatrix} \phi_{11} \\ \phi_{21} \end{pmatrix}^T \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{pmatrix} \phi_{11} \\ \phi_{21} \end{pmatrix} = M_1 = 1$$

$$\phi_{11} = \phi_{21}$$

$$(\phi)_1 = \begin{pmatrix} \sqrt{1/15} \\ \sqrt{1/15} \end{pmatrix}$$

$$\omega_2 = 1 \qquad \begin{pmatrix} \phi_{12} \\ \phi_{22} \end{pmatrix}^T \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{pmatrix} \Phi_{12} \\ \Phi_{22} \end{pmatrix} = M_2 = 1$$

$$(\phi)_2 = \begin{pmatrix} \sqrt{2/15} \\ -0.5\sqrt{2/15} \end{pmatrix}$$

$$(\phi)_i = \frac{1}{\sqrt{M_i}} (\psi)_i$$

Generalized coordinates

$$[\psi] = [(\psi)_1 \ldots (\psi_2) \ldots (\psi)_n]$$

$$(x) = [\psi] q \quad q = [\psi]^{-1}(x)$$

$$[m](\ddot{x}) + [k](x) = (f)$$

$$[\psi]^T [m][\psi](\ddot{q}) + [\psi]^T [k][\psi](q) = [\psi]^T (f)$$

$$[M](\ddot{q}) + [K](q) = [\psi]^T (f)$$

$$M_i \ddot{q}_i + K_i q_i = (\psi)_i (f) = f_i$$

$$\ddot{q}_i + \omega_i^2 q_i = f_i / M_i = \frac{(\psi)_i^T (f)}{(\psi)_i^T [m](\psi)_i}$$

$$\ddot{q}_i + \omega_i^2 q_i = f_i = (\phi)_i^T (f)$$

$$\psi] = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}$$

$$(x) = [\psi] q = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

$$[\psi]^T [m][\psi] = \begin{bmatrix} 15 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

$$[\psi]^T [k]\ [\psi] = \begin{bmatrix} 6 & 0 \\ 0 & \frac{15}{0} \end{bmatrix}$$

$$\begin{bmatrix} 15 & 0 \\ 0 & 15/2 \end{bmatrix} + \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & \frac{15}{2} \end{bmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$15\ddot{q}_1 + 6q_1 = f_1 + f_2$$

$$\frac{15}{2}\ddot{q}_1 + \frac{15}{2} q_2 = f_1 - \frac{f_2}{2}$$

Uncoupling of equations

$$m\ddot{x} + kx = f\,exp(\,j\omega t\,)$$

$$(\,k - \omega^2 m\,)x = f$$

Transfer Function approach

$$\left\{[\,k\,] - \omega^2[\,m\,]\right\}(\,x\,) = (\,f\,)$$

$$[\,H\,]^{-1}(\,x\,) = (\,f\,) \qquad (\,x\,) = [\,H\,](\,f\,)$$

$$[\,H\,]^{-1} = [\,k\,] - \omega^2[\,m\,]$$

$$[\,\phi\,]^T[\,H\,]^{-1}[\,\phi\,] = [\,\phi\,]^T[\,k\,][\,\phi\,] = \omega^2[\,\phi\,]^T[\,m\,][\,\phi\,] = \begin{bmatrix} \ddots & & \\ & \omega_r^2 - \omega^2 & \\ & & \ddots \end{bmatrix}$$

$$[\,H\,] = [\,\phi\,]\begin{bmatrix} \ddots & & \\ & \omega_r^2 - \omega^2 & \\ & & \ddots \end{bmatrix}^{-1}[\,\phi\,]^T$$

$$f = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f_k \\ 0 \end{pmatrix} \qquad (x) = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \phi_{21} & & & \\ \vdots & & & \\ \phi_{N1} & & & \phi_{NN} \end{bmatrix} \begin{bmatrix} \ddots & & \\ & (\omega_r^2 - \omega^2) & \\ & & \ddots \end{bmatrix}^{-1} \begin{bmatrix} \phi_{11} & \phi_{21} & \cdots & \phi_{N1} \\ \phi_{12} & & & \\ \vdots & & & \\ \phi_{1N} & & & \phi_{NN} \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f_2 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \vdots & & & \\ \vdots & & & \\ \phi_{N1} & & & \phi_{NN} \end{bmatrix} \begin{bmatrix} \ddots & & \\ & (\omega_r^2 - \omega^2) & \\ & & \ddots \end{bmatrix}^{-1} \begin{pmatrix} \phi_{k1} \\ \phi_{k2} \\ \vdots \\ \phi_{kN} \end{pmatrix} f_k$$

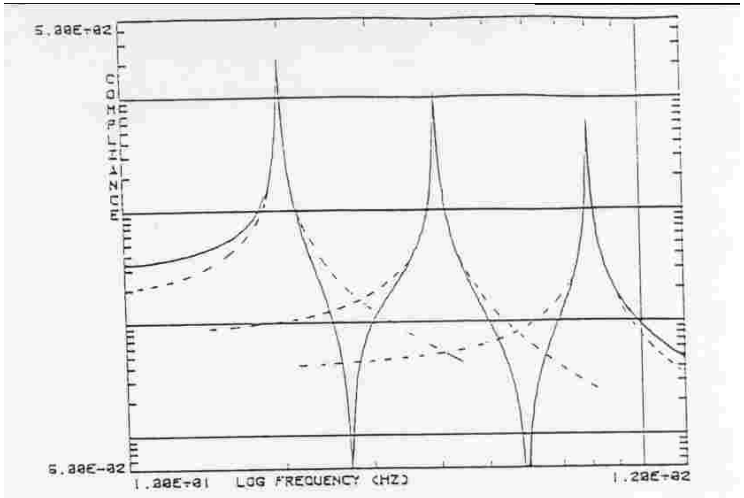
$$x_i = (\phi_{i1}\phi_{i2}\ldots\phi_{iN} \begin{bmatrix} \ddots & & \\ & (\omega_r^2 - \omega^2) & \\ & & \ddots \end{bmatrix}^{-1} \begin{pmatrix} \phi_{k1} \\ \vdots \\ \phi_{kN} \end{pmatrix} f_k$$

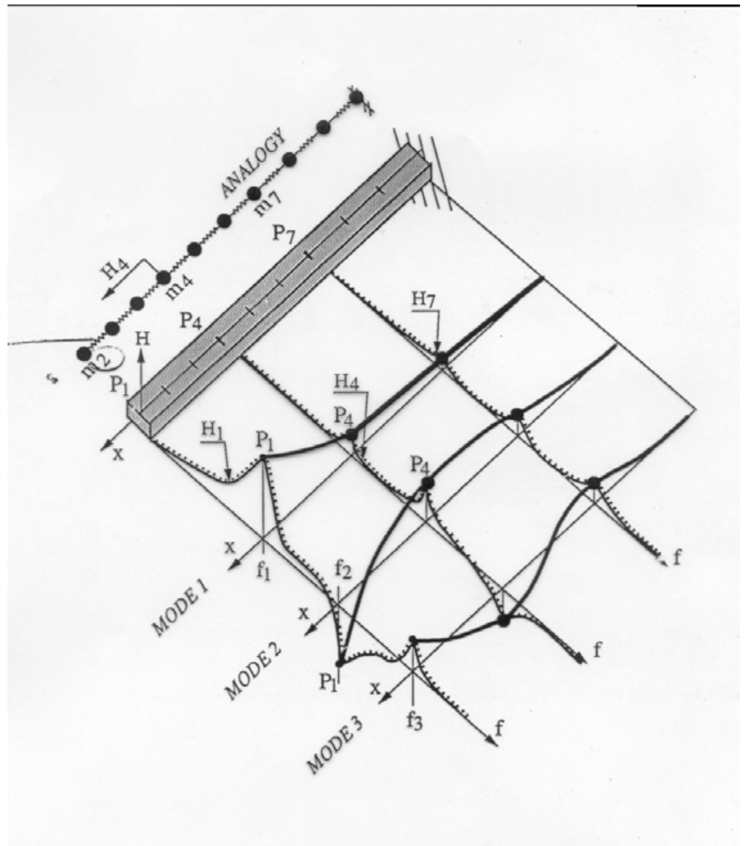
Summary:

$$H_{ik} = \sum_{r=1}^{N} \frac{\phi_{ir}\phi_{kr}}{\omega_r^2 - \omega^2}$$

$$H_{ik} = \sum_{r=1}^{N} \frac{\phi_{ir}\phi_{kr}}{\omega_r^2 - \omega^2 + j2\varepsilon_r\omega_r\omega} = \sum_{r} \frac{{}_{v}A_{ik}}{\omega_r^2 - \omega^2 + j^2\varepsilon_r\omega_r\omega}$$

$${}_{r}A_{ik} = \phi_{ir}\phi_{kr}$$

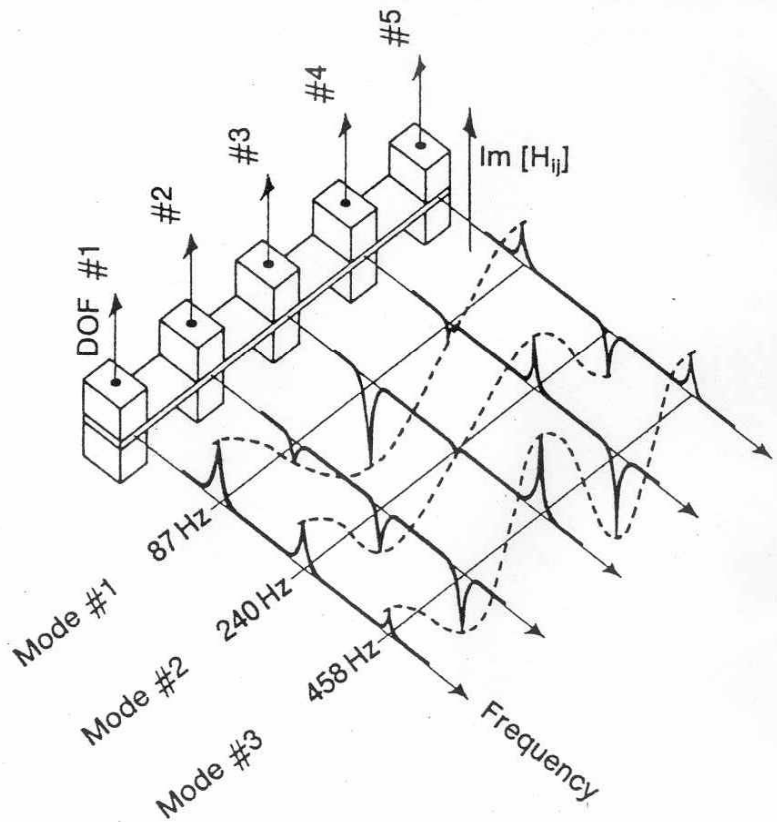




Signal Processing: Modal Analysis

Frequency Response Function and Modal Parameters

- **Mode Shapes from Quadrature Picking**

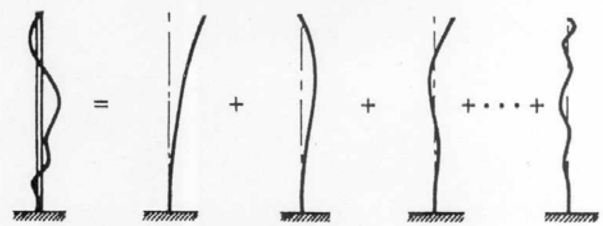


Signal Processing: Modal Analysis

Modal Analysis Introduction

Modal Behaviour

- The dynamic response of a structure can be decomposed into a discrete set of independent particular motions
- These motions are called Modes of Vibration
- A Mode is described by the Modal Parameters:
  * Natural Frequency & Damping
  * Mode Shape



$$x(y,t) = q_1(t)\{\phi\}_1 + q_2(t)\{\phi\}_2 + q_3(t)\{\phi\}_3 + \cdots + q_n(t)\{\phi\}$$

- Modal Analysis is the process of determining the modal parameters

  and ultimately to

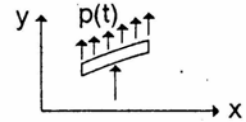
- Create a mathematical Model

83117C

Modal Analysis Introduction

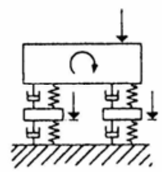
• Mathematical Models

- Wave Equation



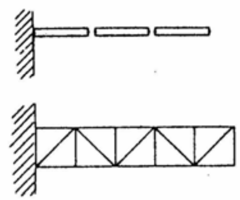
$$\frac{\partial^2}{\partial x^2}\left[IE(x)\frac{\partial^2 y}{\partial x^2}\right] + m(x)\frac{\partial^2 y}{\partial t^2} =$$

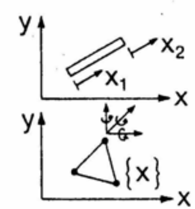
- Lumped Parameters



$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} =$$

- Finite Elements



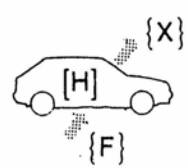


$[k]_{2\times 2}$

$[k]_{18\times 1}$

$$[m]\{\ddot{x}\} + [k]\{x\} = \{f\}$$

- Experimental Modal Model



$$\{X\} = [H]\{F\}$$

$$[H] = \sum_{r}^{2m} \frac{\{\psi\}_r \ \{\psi\}_r^T}{j\omega + \sigma_r - j\omega_{dr}}$$

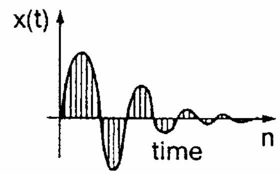
841

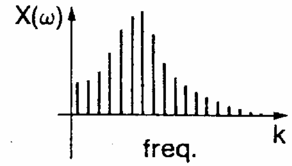
Signal Processing: Modal Analysis

The Modal Transformation

• Fourier – vs. Modal Transformation

Fourier

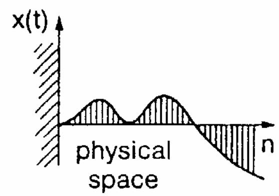


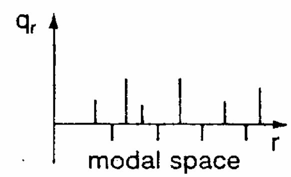


$$X(n) = \sum_{k}^{N/2} X(k)\, e^{j\frac{nk}{N}}$$

or $$x(t) = [e^{j\omega t}]\,\{X(\omega)\} = \sum_{n=1}^{N/2} \sin(\omega_n t) \times X_n$$

Modal





$$X(n) = \sum_{r=1}^{m} \phi_{nr} \cdot q_r$$

or $$\{x\} = [\Phi]\,\{q\} = \sum_{r=1}^{m} \{\phi\}_r \cdot q_r$$

Imagine:

$$[e^{j\omega t}] = [\;\cdots\;] \quad \text{and} \quad \Phi = [\;\cdots\;]$$

850237/2

Signal Processing: Modal Analysis

Experimental Modal Analysis

Conduct test excitation/response

Compute transfer functions FRFs

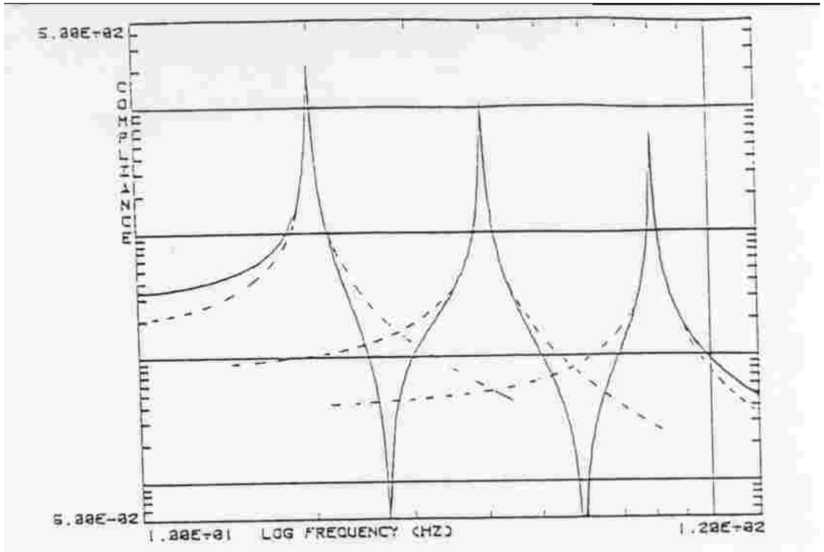
Extract modal parameters from FRFs

$$H_{ik} = \sum_{r=1}^{N} \frac{\phi_{ir}\phi_{kr}}{\omega_r^2 - \omega^2 + j2\varepsilon_r\omega_r\omega} = \sum_{r} \frac{{}_{v}A_{ik}}{\omega_r^2 - \omega^2 + j^2\varepsilon_r\omega_r\omega}$$

$${}_{r}A_{ik} = \phi_{ir}\phi_{kr}$$

$$H_{ik} = \sum_{r=1}^{N} \frac{\phi_{ir}\phi_{kr}}{\omega_r^2 - \omega^2 + j2\varepsilon_r\omega_r\omega} = \sum_{r} \frac{{}_{v}A_{ik}}{\omega_r^2 - \omega^2 + j^2\varepsilon_r\omega_r\omega}$$

$${}_{r}A_{ik} = \phi_{ir}\phi_{kr}$$



SDOF - Peak method

$$H_{ik} = \frac{{}_{r}A_{ik}}{\omega_r^2[1-(\omega/\omega_r)^2+2j\xi_r\omega/\omega_r]}$$

ω_r where H peaks

ξ_r 3 db frequencies ω_2, ω_1

$$\xi_r = \frac{\omega_2 - \omega_1}{2\omega_r}$$

Assuming $\xi_r << 1, \quad \omega \approx \omega_r$

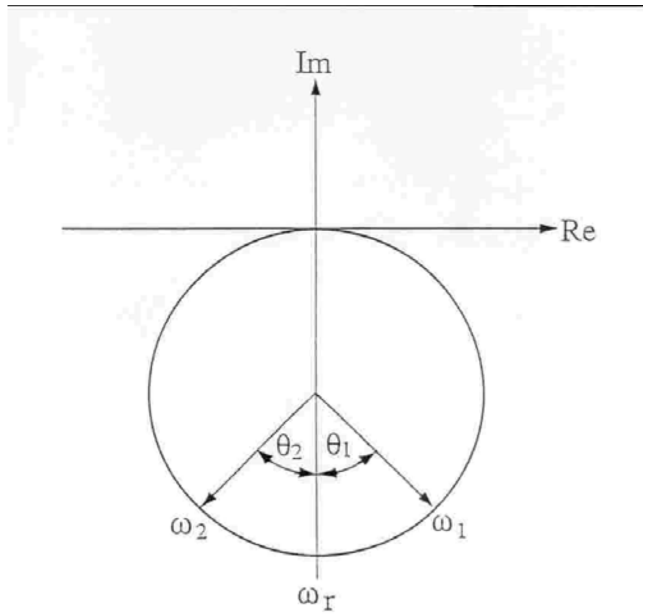
$$|H(\omega_r)| = \frac{{}_{r}A_{ik}}{2\xi_r\omega_r^2}$$

SDOF - circle method

ω_r maximum angular spacing

$$\xi_r = \frac{\omega_2 - \omega_1}{\omega_r\left[\tan\left(\frac{\theta_1}{2}\right) + \tan\left(\frac{\theta_2}{2}\right)\right]}$$

$$Dr = \frac{{}_rA_{ik}}{2\xi_r\omega_r^2}$$



Note: Frequencies, damping – global

Shapes - local

MDOF - complex exponentail fit

$$h_{ik}(t) = \sum_r {}_r h_{ik}(t)$$

$${}_r h_{ik}(t) = \frac{{}_r A_{ik}}{\omega_r[1-\xi_r^2]^{1/2}} \mathrm{Sin}[\omega_r(1-\xi_r^2)^{1/2} t]$$

Finite Element Model of Disc Head

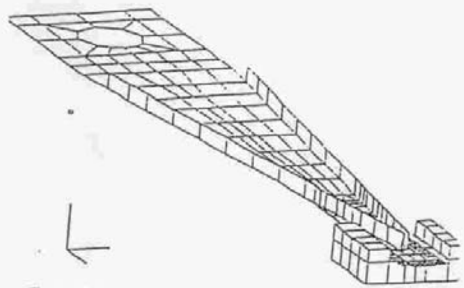


Fig. 8 Finite element model of read/write head suspension

In free head vibration test

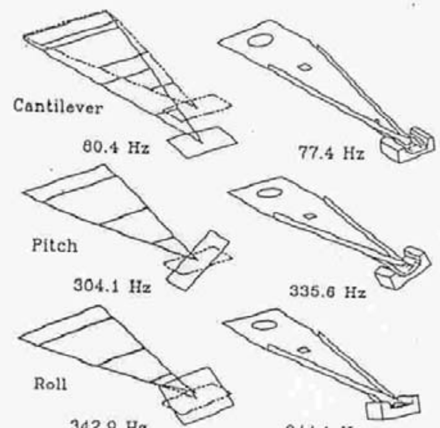
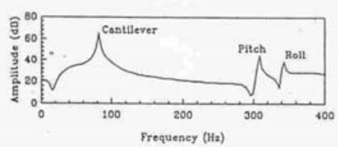
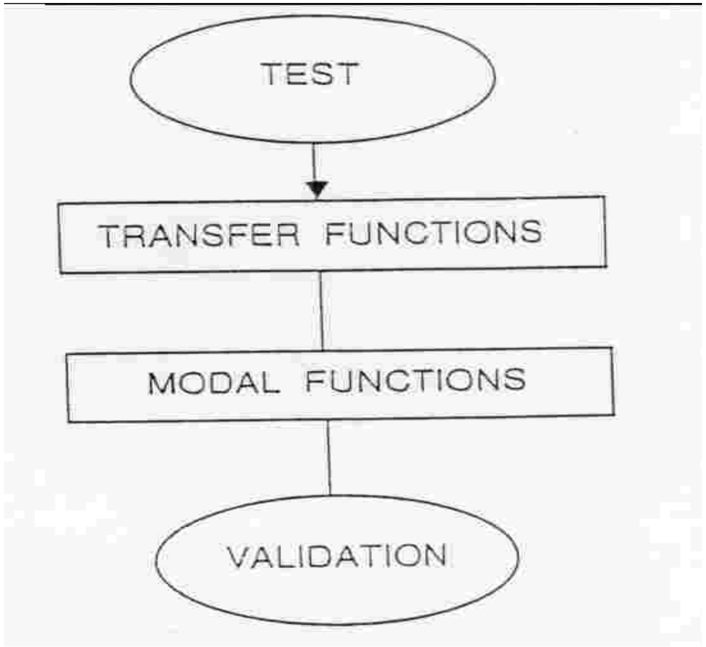


Fig. 7 Experimental (left) and numerical (right) mode shapes for free head vibration test (Compliance Modes)

Frequency Response Function (FRF) of head



Bode Plot



$$ {}_rA_{ik} = \frac{{}_rA_{ij}\ {}_rA_{k1}}{{}_rA_{jj}} $$

$$ H_{ik} = \frac{H_{ij}\ H_{kj}}{H_{jj}} $$

$\therefore$ synthesize H_{ij} from H_{1j}, H_{2j} and H_{jj}

(response moved)

Signal Processing: Modal Analysis

$$[H] = \begin{bmatrix} H_{11} & \cdots & H_{1k} & & \\ & & H_{2k} & & \\ H_{i1} & \cdots & H_{2k} & \cdots & H_{iN} \\ & & \vdots & & \\ & & H_{nk} & & H_{NN} \end{bmatrix} \leftarrow$$

moving excitation

$\uparrow$

moving response

Moving excitation: Impact testing

Moving response: Shaker

Mathematical equivalence does not imply technology equivalence!

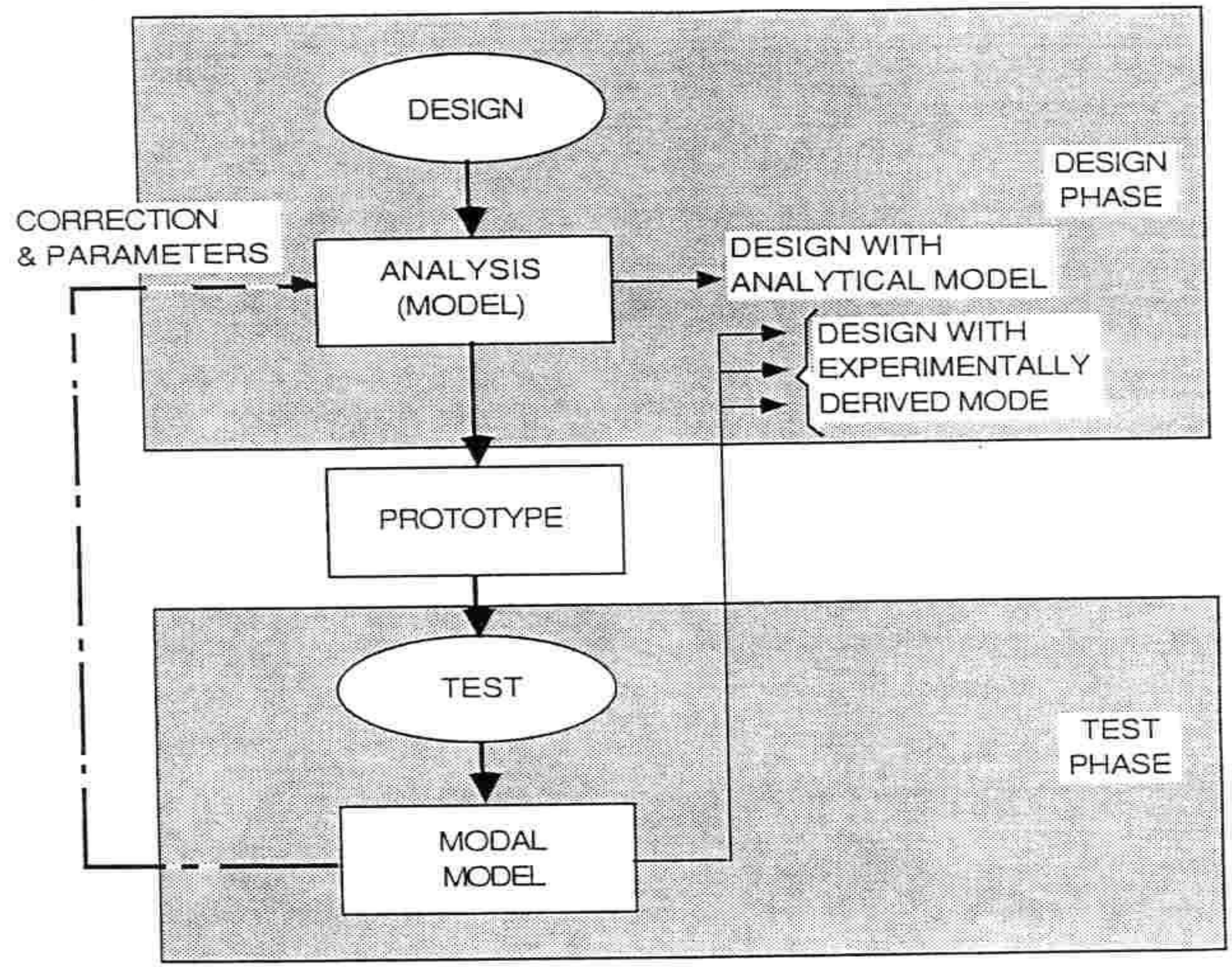
Modal Analysis applications

Model updating

Structural modifications

Diagnostics

Force evaluation via response



Signal Processing: Modal Analysis

STRUCTURAL MODIFICATIONS

$$\{[m]+[\Delta m]\}(\ddot{x})+\{[c]+[\Delta c]\}(\ddot{x})+\{[k]+[\Delta k]\}(x)=(f)$$

$$[\overline{M}]=\begin{bmatrix} \ddots & & \\ & I & \\ & & \ddots \end{bmatrix}+[\phi]^{r}[\Delta m][\phi]$$

$$[\overline{K}]=\begin{bmatrix} \ddots & & \\ & \omega_r^2 & \\ & & \ddots \end{bmatrix}+[\phi)^{T}[\Delta k][\phi]$$

$$[\overline{C}]=\begin{bmatrix} \ddots & & \\ & 2\xi\omega & \\ & & \ddots \end{bmatrix}+[\phi^{T}]\,[\Delta c][\phi]$$

$$\left.\begin{array}{l}\Delta m\\ \Delta c\\ \Delta k\end{array}\right\}\ \text{modification matrices}$$

Find eigensolution for $[\overline{M}](\ddot{q})+[\overline{c}](\dot{q})+[\overline{K}](q)=[\phi]^{T}(f)$

Diagnostics

Change in Modal parameters

Frequencies (cracks etc)

Type (and changes) of deflection shapes

Force evaluation via response

$$[X]\exp(j\omega t) = [H](f)\exp(j\omega t)$$

$$(f) = [H]^{-1}(X)\exp(j\omega t)$$

Problem often ill conditioned

Appropriate numerical approaches needed

Model Based Signal Processing

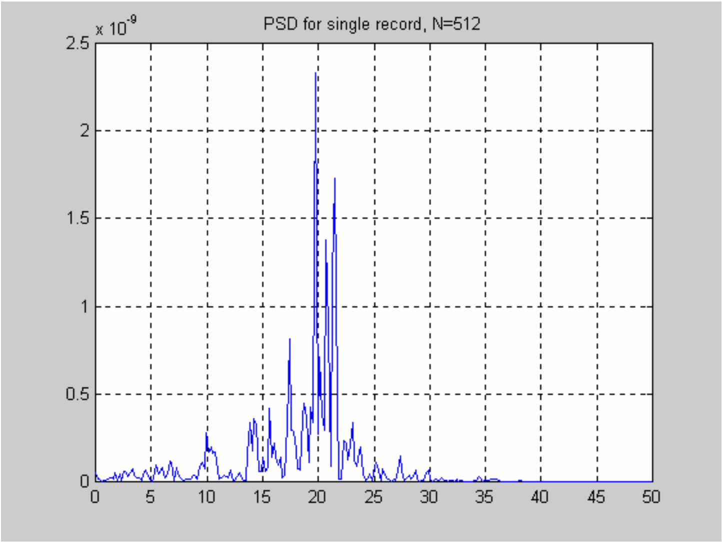
Vz Fourier (which is non parametric)

Signal modeling:

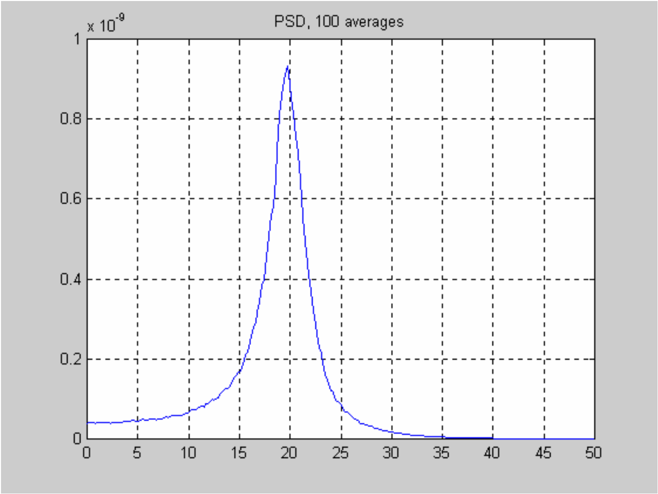
Consider a signal $x(i\Delta T)$, with $i = 0, 1 \ldots n$.
It may be possible to model the signal with a number of parameters p, where $p < n$.

Signal modeling is thus concerned with the representation of signals in an efficient manner.

Applications of such signal models include data compression, signal prediction, classification , diagnostics and spectral analysis techniques.

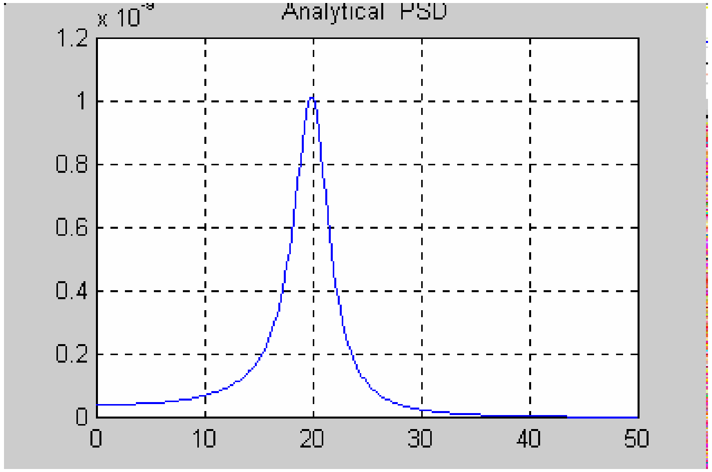


512 data points



512*1000 data points

3 parameters and 1 data descriptor



<u>Data reduction problem</u>

Modeling of Signals

Measured signal $\longrightarrow x$

Random

Whitenoise $e \longrightarrow \boxed{\text{Model}} \longrightarrow \hat{x}$ $\qquad S_{\hat{x}\hat{x}} = S_{xx}$

Deterministic

Im pulse $\delta(t) \longrightarrow \boxed{\text{Model}} \longrightarrow \hat{x}$

Signal Models

1. MA – Moving Average

All zero

$$x_i = e_i + b_1 e_{i=1} + b_2 e_{i-2}$$

$$H(z) = B(z)$$

$$H(z) = B(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots = \sum_{i=0}^{q} b_0 z^{-i}$$

2. AR – Autoregressive

All Pole

$$x_i = a_1 x_{i-1} + a_i x_{i-2} + \ldots e_i \qquad (a_0 = -1)$$

$$H(z) = \frac{1}{A(z)} = \frac{1}{a_0 - a_1 z^{-1} - a_2 z^{-2} \ldots} = \frac{1}{\sum_{i=0}^{p} -a_i z^{-i}}$$

3. ARMA

Autoregressive Moving Average

Pole/zero

$$x_i = a_1 x_{i-1} + a_2 x_{i-2} + \ldots + e_i + b_1 e_{i+1} + b_2 e_{i-2} \qquad (b_0 = 1)$$

$$H(z) = \frac{B(z)}{A(z)}$$

Correlation Functions

$$R(\tau) = E\left[x(t)x(t+\tau)\right]$$

$$R(0) = E[x^2(t)] = MS$$

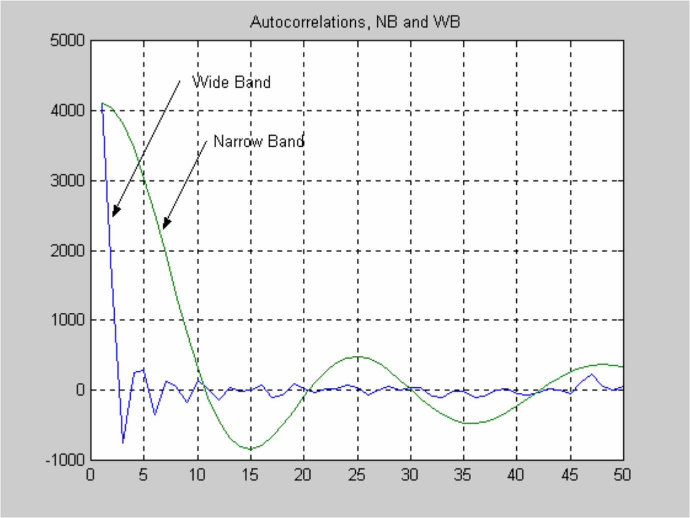
$$R(-\tau) = R(\tau)$$

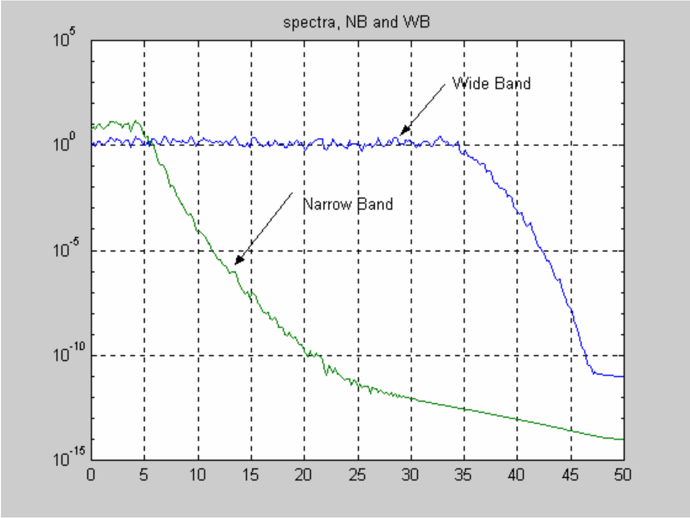
$$R(k) = E[x_i \; x_{i+k}]$$

$$\mathrm{S_{xx}(f) = F[R_{xx}(\tau)]} = \int_{-\infty}^{\infty} \mathrm{R_{xx}(\tau)\exp(-j\omega\tau)d\tau}$$

$$\mathrm{R_{xx}(\tau) = F^{-1}[S_{xx}(f)]}$$

Autocorrelation $$R_{xx}(\tau) = E[x(t)x(t+\tau)]$$





Autocorrelations and PSD

Narrow and wideband signals

Computation of AR parameters

$$x_i = -a_1 x_{i-1} - a_2 x_{i-2} + e_i$$

$$E[x_{i-1} x_i] + a_1 E[x_{i-1} x_{i-1}] + a_2 E[x_{i-1} x_{i-2}] = 0$$

$$E[x_{i-2} x_i] = a_1 E[x_{i-2} x_{i-1}] + a_2 E[x_{i-1} x_{i-2}] = 0$$

$$R(1) = -a_1 \; R(0) - a_2 \; R(1)$$
$$R(2) = -a_1 \; R(i) - a_2 R(0)$$

$$E(x_i x_i] = -a_1 \; E[x_i x_{i-1}] - a_2 E[x_i x_{i-2}] + E[x_i e_i]$$

$$\begin{aligned} R(0) &= -a_1 R(1) \quad - a_2 R(2) + \; E[-a_1 x_{i+1} - a_2 x_{i-2} + e_i) e_i] \\ &= -a_1 R(1) - a_2 R(2) + \sigma_e^2 \end{aligned}$$

$$\begin{bmatrix} R(0) & R(1) & R(2) \\ R(1) & R(0) & R(1) \\ R(2) & R(1) & R(0) \end{bmatrix} \begin{pmatrix} 1 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \sigma^2 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} \boldsymbol{R}(0) & \boldsymbol{R}(1) & \boldsymbol{R}(\boldsymbol{p}) \\ \boldsymbol{R}(1) & \boldsymbol{R}(0) & \boldsymbol{R}(\boldsymbol{p}-1) \\ \vdots & & \vdots \\ \boldsymbol{R}(\boldsymbol{p}) & \boldsymbol{R}(\boldsymbol{p}-1) & \boldsymbol{R}(0) \end{bmatrix} \begin{pmatrix} 1 \\ \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_p \end{pmatrix} = \begin{pmatrix} \sigma^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\mathbf{Ra} = \sigma_e^2\, \mathbf{I} \qquad \mathbf{I} = [\ 1\ 0\ 0\ 0\ \ldots..0\ ]^T$$

Model based Spectral Analysis

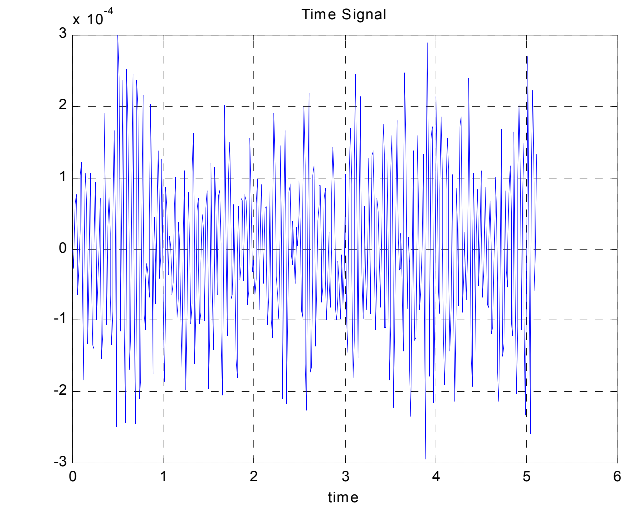
$$S_{ee} = \sigma^2 s\Gamma$$

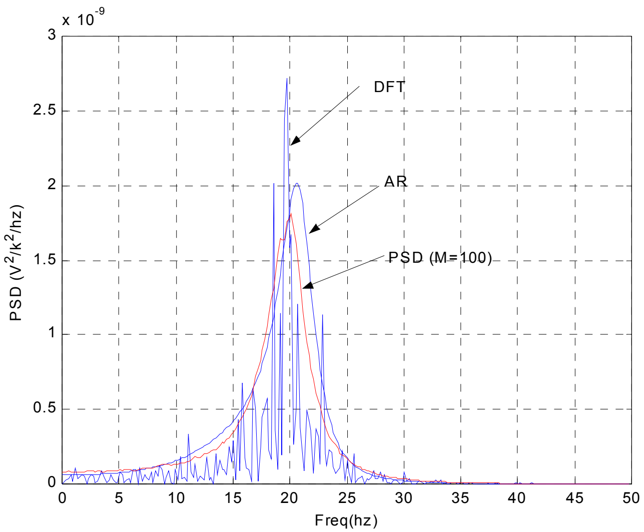
$$S_{xx} = S_{ee} \left| H(j\omega) \right|^2$$

$$H(z) = \frac{1}{1 + \sum_{i+1}^{p} a_i z^{-i}}$$

$$H(j\omega) = H(z)\Big|_{z=\exp(j\omega\Delta\Gamma)}$$

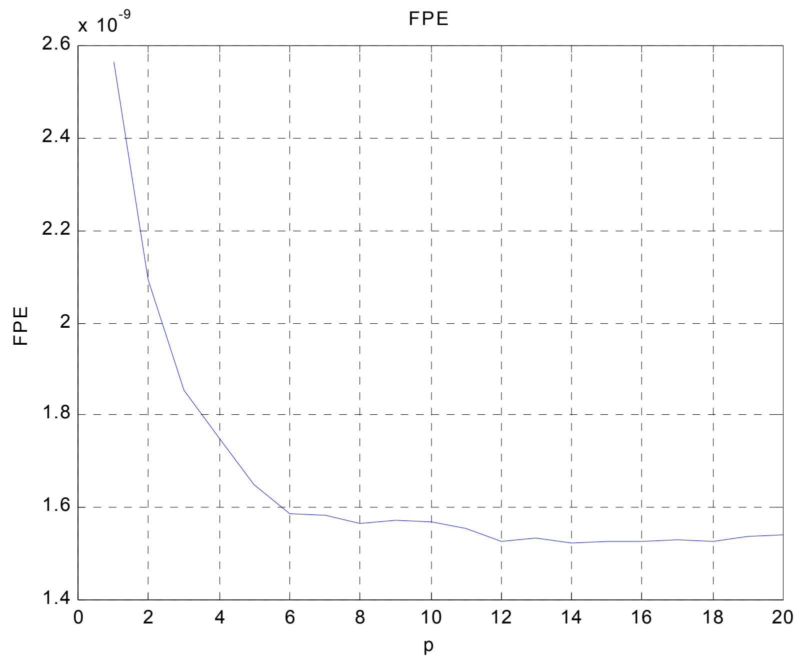
$$FPE(p) = \frac{N + (p+1)}{N - (p-1)} \sigma_e^2(p)$$



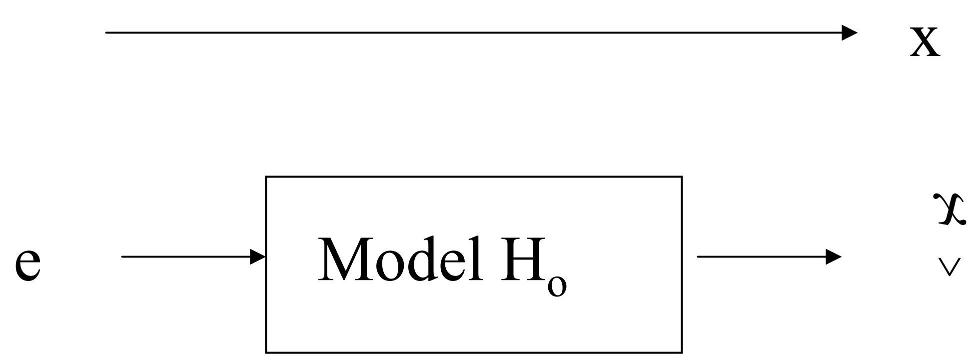


AR Spectrum

(p=4)


FPE
x 10^-9
2.6
2.4
2.2
2
1.8
1.6
1.4
FPE
0
2
4
6
8
10
12
14
16
18
20
p

Model Based Diagnostics



The signal x , monitored from a no-fault system is first modeled:

The result is H_o (parameters) and white noise variance σ_e^2

1. **Apply and inverse filter**

2. **Check properties of resulting output**

Example: AR model

Model: $x_i = - a_1 x_{i-1} - a_2 x_{i-2} + e_i$

Inverse: $e_i = x_i + a_1 x_{i-1} + a_2 x_{i-2}$

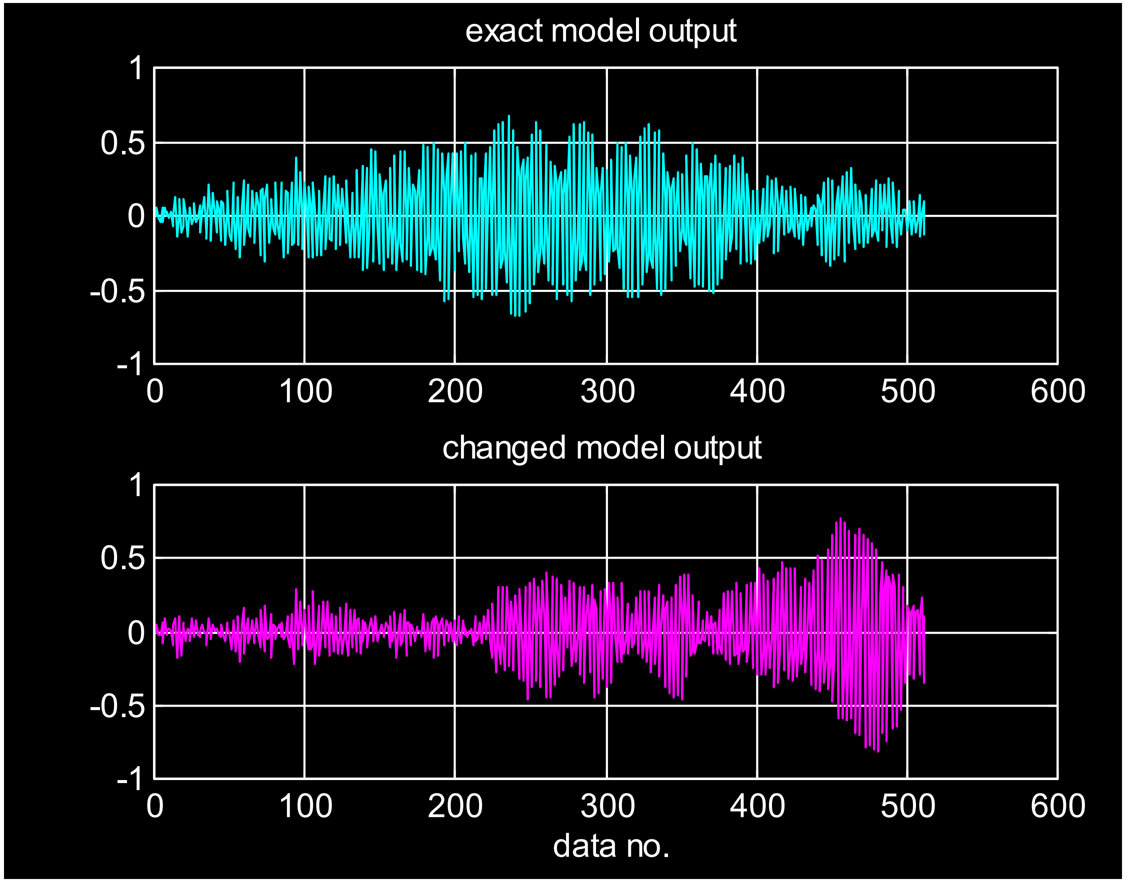
Inverse filter output

If $H=Ho$, e has white noise properties

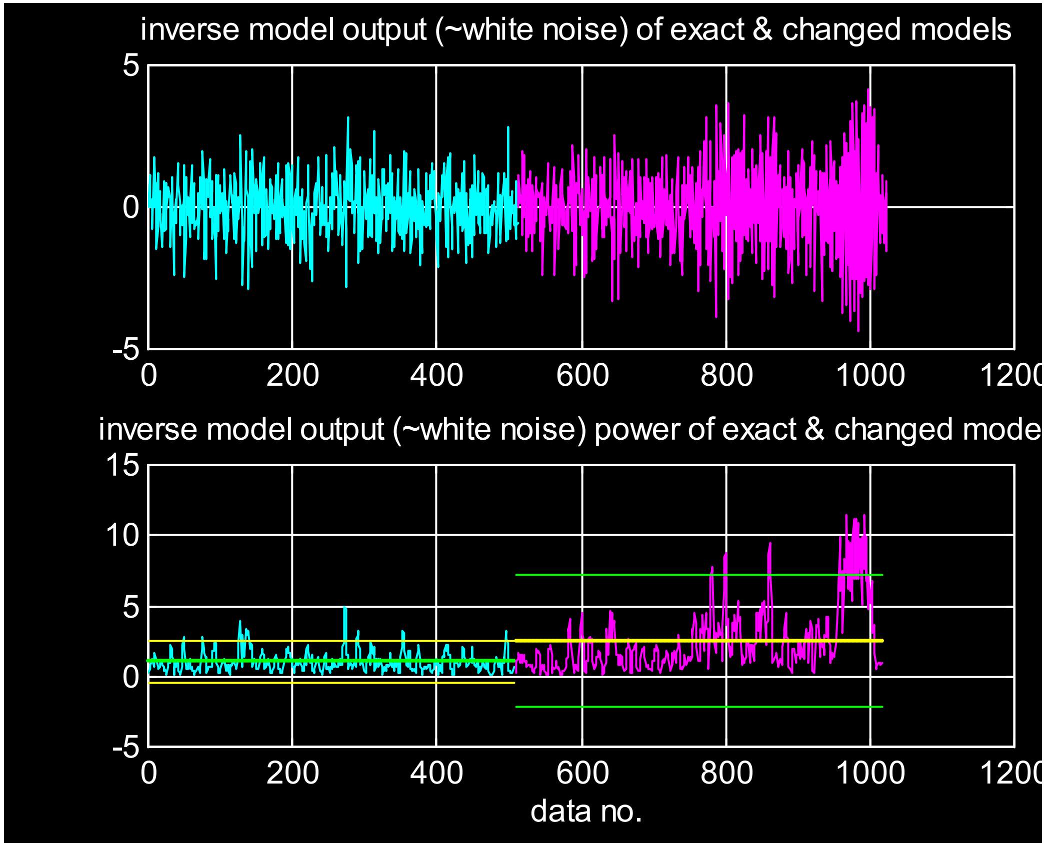
Else $H=H_1$, i.e a change occured

Residual is investigated via inverse filter. Tests for whiteness of e can be based on Spectrum or autocorrelation of residual

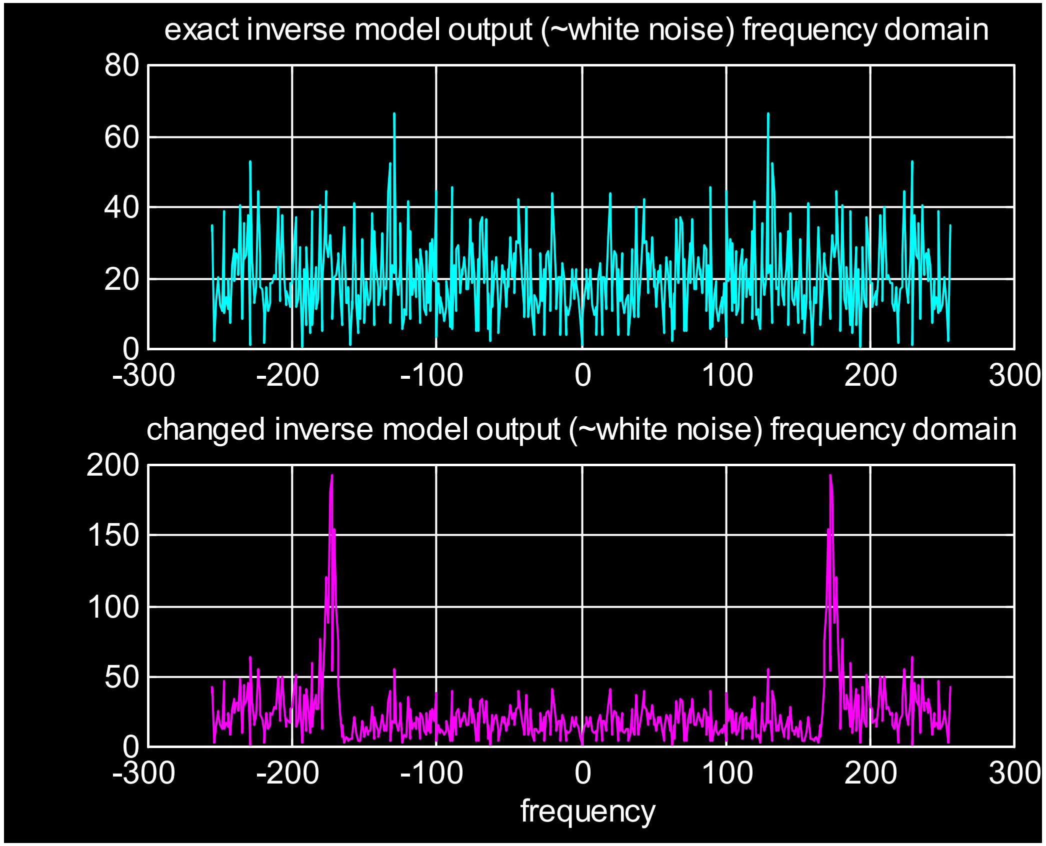
The method can detect changes, not identify cause !



Signal Processing: Model based



Signal Processing: Model based


exact inverse model output (~white noise) frequency domain
80
60
40
20
0
-300
-200
-100
0
100
200
300
changed inverse model output (~white noise) frequency domain
200
150
100
50
0
-300
-200
-100
0
100
200
300
frequency