

An analytical introduction to Descriptive
Geometry

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Preface

This is a book for people who want to understand Descriptive Geometry while they learn its methods and it is a book for those who have difficulties in directly visualizing spatial relationships, but can grasp them by analytical reasoning. The book is also meant for those who want to know on what Engineering Drawing is based. In addition, the book can serve as a bridge between traditional, manual methods and computer graphics. Therefore, the audience can include students of Engineering, Architecture, Design and Graphics.

We live in a three-dimensional world and we often want to represent this world in a drawing, a painting or a photography. Now, drawings, paintings and photos are two-dimensional. So are, indeed, the images formed in each of our eyes. Seen from two different points, those images are different. It is our brain that restores a three-dimensional view from the two images formed in our eyes. In technical drawing we must use another procedure. The aim of Descriptive Geometry is to describe the three-dimensional objects by two-dimensional drawings so as to allow to reconstitute their original forms and

their correct measures.

Let us explain in plain words what we mean above by the term *dimension*. We may consider, for example, a parallelepipedic box. We can measure its length, its breadth and its height, in total three dimensions. We can draw this object on a sheet of paper that has only two dimensions, a breadth and a height. So do photos, and TV and computer screens.

When representing a three-dimensional object on the two-dimensional surface of our retina, of a camera film or array of sensors, of a paper sheet, or of a TV or computer screen, the number of dimensions is reduced from three to two. The general process of reducing the number of dimensions of a given object is called **projection**. The type of projection that produces the image in our eye, on the film of a conventional camera or on the array of electronic sensors of a digital camera is called **central projection**. The images produced by central projection convey the sensation of depth, or, in other words, of the actual space. This is the only projection that produces the real **perspective**. However, drawing in perspective is tedious. Moreover, reading the correct measures in a central projection requires auxiliary constructions or calculations. These matters are taught in courses on *Perspective*, such as given in schools of Art or Architecture. The understanding of perspective is indispensable in Computer Vision and, as such, important in Robotics. Descriptive Geometry is based on another type of projection, specifically **parallel projection**, in most cases **parallel, orthogonal projection**. The result is realistic for small objects placed at large distances

from the viewer. Even if the result is not realistic for other objects, the processes of projection and of measuring in the resulting drawing are simplified to such an extent that the method imposed itself in Engineering. In fact, Descriptive Geometry, as developed by Monge, requires no calculations, but only graphical constructions. This advantage may not be appreciated today by students who learned early how to use hand-held calculators and desk computers and do not realize that this technology is very recent. Although we can use today computers and software for drawing, the theoretical basis of technical drawing still derives from Descriptive Geometry. In practical engineering there are instances in which hand drawings and Descriptive Geometry can be used to check a solution much faster than by digitizing the data and reverting to a computer. Not to talk about the situations in which we are not close to a computer. Moreover, research shows that Descriptive Geometry helps in developing **spatial abilities**, qualities essential in many branches of engineering, and especially in **Design**. But not only engineers have to understand spatial relationships and be able to interpret ‘flat’ images. Orthopedists have learned a long time ago how to locate fractures in two x-ray images perpendicular one to the other. Dentists too use x-ray pictures in their work. And what is Computer Tomography if not a very advanced application of several geometric techniques? Descriptive geometry teaches when it is possible to measure a length or an angle in a drawing, or how to recover that information when direct measuring is not possible. As another example, descriptive geometry teaches how to find out if a given object lies

on a surface, or on which side of that surface. An engineer who does not know how to do this can place a piece of equipment outside a vehicle, and an Interior Designer who lacks this skill can block a door by wrongly placing an object. During their practice, two of the authors encountered more than one example of such cases.

Descriptive geometry continues to live even when the use of it is not openly acknowledged. Thus, the excellent book of Chirone and Tornincasa (2002) does not mention the term *Descriptive Geometry*, but pages 89–105 and 137–153 in it do belong to this discipline. Similarly, Simmons and Maguire (2003) do not use the explicit term, but chapter 11–13 of their book treat, in fact, subjects of descriptive geometry. As a counterexample, Bertoline and Wiebe (2003) have no problem to give the title *Descriptive Geometry to Part Three* of their monumental book. This part runs from page 571 to 636 and it begins with the sentence, ‘Descriptive Geometry is the mathematical foundation of engineering graphics.’

Many, probably most books on Descriptive Geometry are, in fact, ‘cook-books’. The various procedures are given as ‘recipes’ without any serious justification. Descriptive Geometry, however, is a branch of geometry, in particular, and of mathematics, in general. There ‘is reason behind’ any procedure and it is possible to give proofs for all procedures. Obviously, the basis is elementary geometry, plane and solid. For many procedures one can find a one-to-one relationship to Linear Algebra. Also, not a few procedures can be reduced to calculations in Analytic Geometry and, sometimes, in Dif-

ferential Geometry. More advanced results belong to Projective Geometry, that branch of Mathematics that evolved from the study of perspective.

Historically, Geometry is the first branch of science that used deductive logic and rigorous formalism. More than one philosopher used the qualifier ‘geometrical’ or one derived from it to show that his reasoning is logical, rigorous. To cite only one famous example, Spinoza’s most important work bears the title *Ethica, more geometrico demonstrata* (Ethics, demonstrated in a geometrical manner). A report issued in 1997 by the US Department of Education, and known as the ‘Riley report’, shows that high-school students who took a course in geometry succeeded better in college than their colleagues who did not take such a course. ‘The report did not say anything about getting a good grade in those mathematics courses or even about passing. Simply taking the courses led to benefits. What’s more, students got the same benefits regardless of the subjects that they pursued at college.’ (Devlin, 2000, p. 262). Descriptive Geometry is a branch of Geometry and as such, if properly taught, it can help in improving the thinking skills of the students. Therefore, we consider that it’s wrong to write DG textbooks styled as cookbooks and we try to follow quite another course in our book. Truly said, there are a few German DG textbooks that try to explain the procedures and, to do so, they use also other branches of mathematics. See, for example, Reutter (1972).

It is possible to establish a one-to-one correspondence between chapters of geometry, among them descriptive geometry, and linear algebra. In Italy, for

example, university courses that cover Linear Algebra are called *Geometria* and treat geometry and linear algebra as one subject. For example, the book of Abate (1996) describes the main concept of descriptive geometry, **projection**, in pure terms of linear algebra. The links between linear algebra and geometry are also shown by Banchoff and Wermer (1983) who give their book the significant title *Linear algebra through geometry*.

In this book we try to take into account the above considerations. The book does not cover all topics found in extended treaties on the subject. We preferred to restrict ourselves to basics. Students interested in more complex problems can continue by reading traditional textbooks. Alternatively, having understood the basics, they can solve their problems by using one of the available CAD packages. We cannot refrain from pointing out that the indiscriminate use of a CAD program can produce wrong results. One must know how to prepare the input and how to interpret the output. While the acronym ‘CAD’ is meant to stay for ‘Computer-Aided Design’, it may easily become ‘Computer-Aided Disasters’.

Further, in this book we try to justify most procedures. To do so we use Analytic Geometry, Linear Algebra, Vector Calculus and Differential Geometry, and we exemplify the relationships between these disciplines. As some of our students testified, by proceeding in this way the course helped them to better understand courses in Calculus and possibly also others. By explaining the procedures, we are also trying to help students in approaching books and courses in advanced graphics, especially computer graphics. Sim-

ply said, a course based on this book may have an *Added Value*. On the other hand, to fully profit of this book, the reader must have some knowledge in the mathematical disciplines mentioned above. The more advanced graphic programming is, the more mathematics it needs.

In our book we added a few complements not found in traditional DG textbooks. Those additions provide the link between descriptive geometry and other disciplines such as geometric modelling, computer graphics, maps, or stereo vision.

To enhance readability we illustrate the book by coloured figures. Almost all figures are produced in MATLAB. The companion software of the book can be retrieved from the site ? provided by the editors, and from the site ? provided by The MathWorks. Figures called in the MATLAB environment can be rotated and this may contribute to their visualization. Other files found in the companion software can be run to see animations of some procedures described in the book. These ‘movies’ can greatly contribute to the understanding of the text.

Throughout this book we use the **first angle projection**, as usual in most European countries. In addition to traditional terminology, we are trying to introduce in this book the newest ISO terms.

Besides having its own value, the book can be used as an auxiliary text to books on Engineering Graphics and help in understanding them better and extend the material exposed in them. An example of such a book is the excellent manual of Simmons and Maguire (2003).

Contents

Notations

We summarize here the notations used in this book. Alternative notations may be found in the literature of specialty. Where we think it is important, we are trying to explain our choice as compared to other notations.

Object	Notation	Examples
Point	upper-case Latin letters	$A, B, \dots P$
Straight line	lower-case Latin letters	l, g
Plane	upper-case Greek letter	Δ, Γ
Projection planes	Greek letter π with subscripts	π_1 for the horizontal plane π_2 for the frontal plane π_3 for the side plane π_4 for an additional plane
Projected object	superscript	P' horizontal projection of P P'' frontal projection of P P''' side projection of P P^4 projection on π^4 similarly for lines
Lines connecting two projections of the same point	dashed, blue line	see Figure 1
Auxiliary lines	continuous, red line	see Figure 1

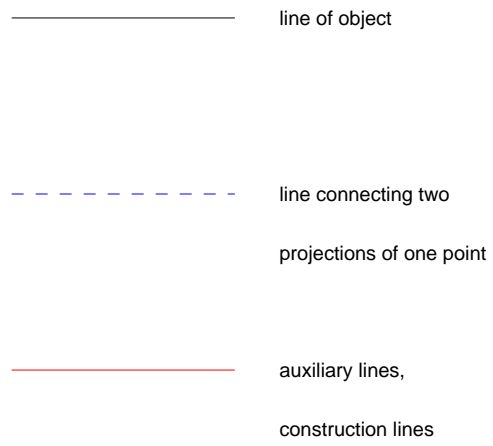


Figure 1: Notations - lines

Chapter 1

Geometric constructions

1.1 Introduction

1.2 Drawing instruments

1.3 A few geometric constructions

1.3.1 Drawing parallels

1.3.2 Dividing a segment into two

1.3.3 Bisecting an angle

1.3.4 Raising a perpendicular on a given segment

1.3.5 Drawing a triangle given its three sides

1.4 The intersection of two lines

1.4.1 Introduction

1.4.2 Examples from practice

1.4.3 Situations to avoid

Chapter 2

Introduction

2.1 How we see an object

Before learning how to represent an object, we must understand how we see it. In Figure 2.1 the object is the letter *V*; it lies in the **object plane**.

Rays of light coming from the object are concentrated by the **lens** of the eye and projected on the **retina**, forming there the **image** of the object.

This image is *inverted*. The retina is not plane, but, for the small angles of ray bundles involved, we can consider it to be planar.

Images in photographic cameras are formed in the same way as in the eye, but in this case the image is, indeed, plane. In traditional photographic cameras the image is formed on films. In digital cameras the image is formed on arrays of electronic sensors that may be of the CCD (charged-coupled devices) or the CMOS (complementary metal oxide

semiconductor) type. In photographic cameras too the image is inverted, and the image plane is known as the **focal plane**. In *camera obscura*, a device known for at least 2000 years, light is admitted through a narrow hole into a dark chamber. An inverted image is formed on the opposing wall.

To simplify the geometrical reasoning, it is convenient to consider an image plane situated between the lens and the object, at the same distance from the lens as the true image plane. In Figure 2.2 we see both the true image plane and the one introduced here. It is obvious that the size of the image is the same, but the position on the new image plane is the same as on the object plane and not inverted. Therefore, from here on we shall consider only such situations (see Figure 2.3).

2.2 Central projection

2.2.1 Definition

In the preceding section we considered an object and its image. We say that the image is the **projection** of the object on the plane of the retina, film or array of electronic sensors. All the lines that connect one point of the object with its projection (that is its image) pass through the same point that we call **projection centre**. This kind of projection is called **central projection**. In French we find the term *perspective centrale* (see, for example, Flocon and Taton, 1984), and in German *Zentralprojektion*

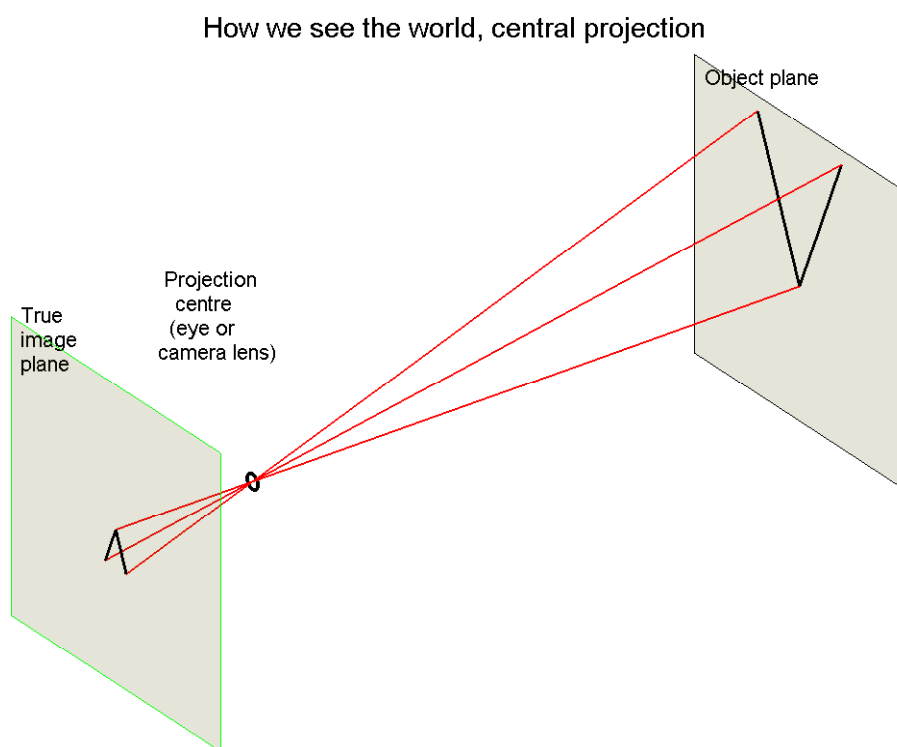


Figure 2.1: How we see the world

(see, for example, Reutter, 1972).

Definitions regarding the central projection can be read in ISO 5456-4:1996(E).

2.2.2 Properties

Figure 2.3 allows us to discover a first property of the central projection. If the object and the image planes are parallel, the image is **geometrically similar** to the object.

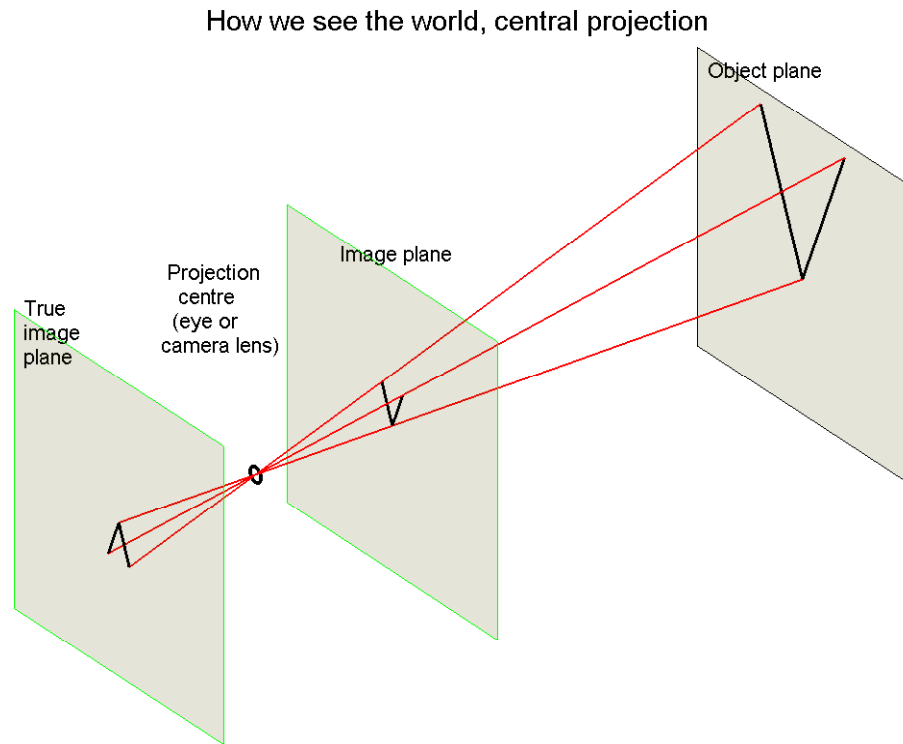


Figure 2.2: How we see the world

We examine now other situations. In Figure 2.4 the point P is the object and E the eye. We consider a system of coordinates in which the xy -plane is horizontal and contains the point P , the yz -plane is the image plane, and the eye lies in the zx -plane. The image, or projection of the point P is the point P' in which the *line of sight* PE pierces the image plane.

Theoretically, we can consider object points situated before or behind the image plane.

In Figure 2.4 we can see a graphic method for finding the projection P' . If the method is not obvious at this stage, we invite the reader to accept it

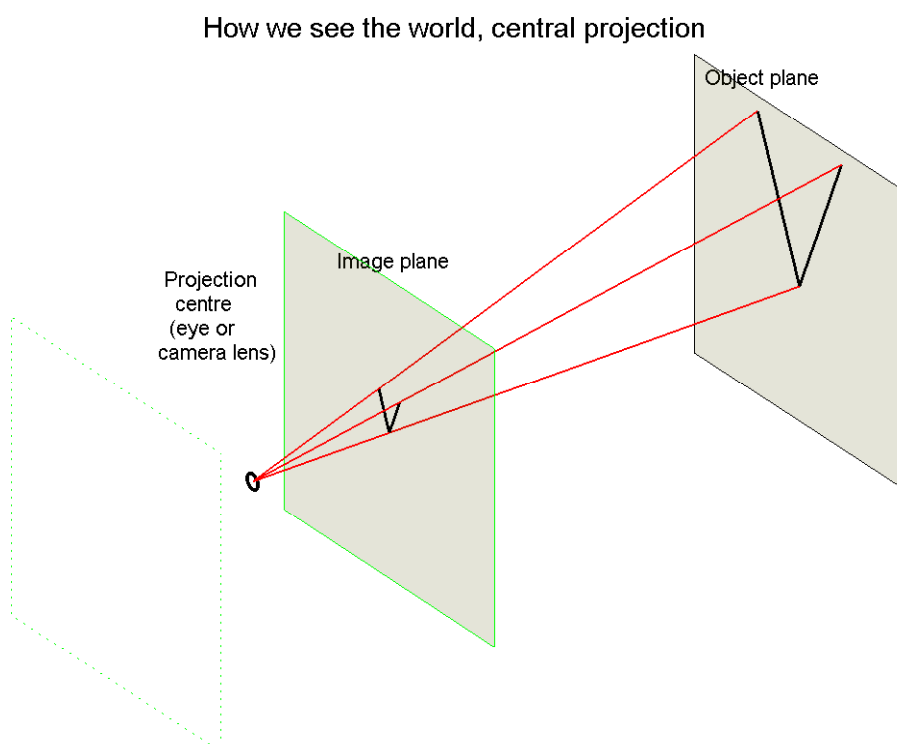


Figure 2.3: How we see the world

now as such and to return to it after reading this chapter and the following one. We draw through E a perpendicular on the horizontal, xy -plane and find the point F in which it intersects the x -axis. The line FP intersects the y -axis in P_h . We raise the vertical from P_h and intersect the line EP in P' . By construction the point P' belongs to the image plane; it is the central projection of the point P .

The central projection of the projection centre, E , is not defined. Neither can we find the projection of the point F . In general, we cannot construct the projection of a point that lies in the plane that contains the projection

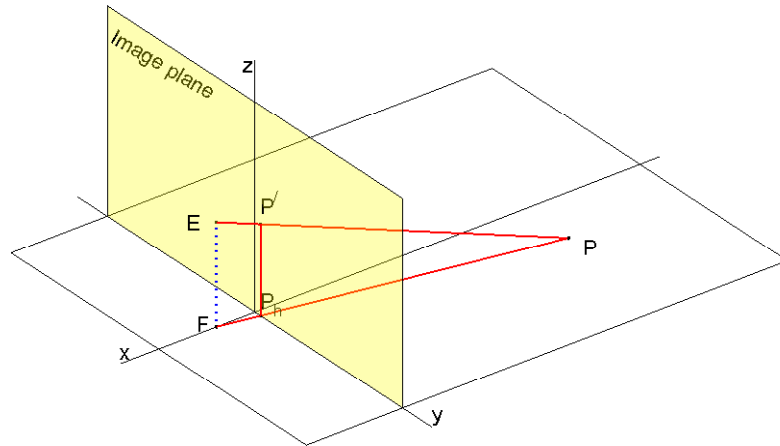


Figure 2.4: A point in central projection

centre and is parallel to the image plane. These images are sent to infinity. In practice, this fact may not appear as a disadvantage, but it is a nuisance for mathematicians. To build a consistent theory it is necessary to include the points at infinity into the *space* under consideration. These additional points are called **ideal**.

Let us turn now to Figure 2.5. The segment $\overline{EE_p}$ is perpendicular to the image plane and E_p belongs to this plane. We cannot find in the plane xy a point whose image is E_p ; however, we can think that if we consider a point P on the axis x and its image, when the point P moves to infinity (in either

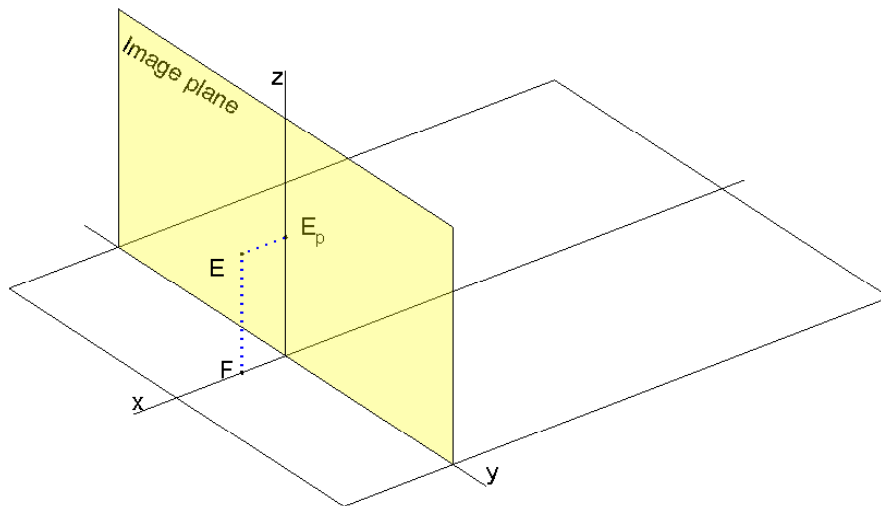


Figure 2.5: A point that is not the image of any point

direction) its image approaches E_p . To understand the geometrical significance of the point E_p we must see how straight lines are projected under a central projection.

In Figure 2.6 we consider the line d belonging to the horizontal plane xy , and perpendicular to the image plane. The point P_1 belongs both to the line d and to the image plane. Therefore, this point coincides with its projection. We can choose two other points, P_2 and P_3 , of the line d and look for their projections. The line d and the point E define a plane. Let us

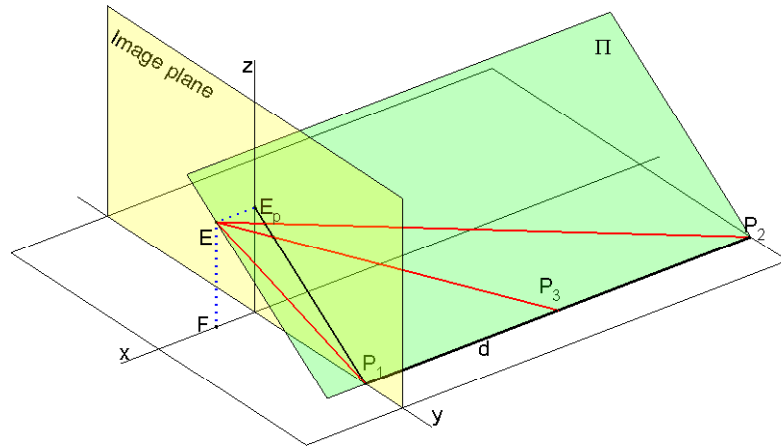


Figure 2.6: Central projection of line perpendicular to the image plane

call it Π . The plane Π contains all the projection rays of points on d ; it is the plane that **projects** d on the image plane, with E as the projection centre. It follows that the projection of d is the intersection of the plane Π with the image plane. Then, the projections of the points P_1 , P_2 and P_3 , and, in general, the projections of all points of d lie on the line that passes through the points P_1 and E_p . Remember, the point E_p was defined in Figure 2.5. As $\overline{EE_p}$ is perpendicular to the image plane, it is parallel to d and belongs to the plane Π . Let us think now that the point P_2 moves towards infinity. Then, its projection moves towards E_p ; this is the image of

the ideal point of the line d .

In the above reasoning we made no assumption as to the position of the line d ; all we said is that it is perpendicular to the image plane. We can conclude that the projections of all lines perpendicular to the image plane end in the point E_p . In other words, E_p is the projection of the ideal points of all lines perpendicular to the image plane. In fact, in *Projective Geometry* — the branch of geometry that evolved from the study of the central projection — all lines parallel to a given direction have the same ideal point. An important consequence, illustrated in Figure 2.7, is that the projections of parallel lines (not parallel to the image plane) converge in one point, the image of their ideal point.

The reader may try the above considerations on a line that is not perpendicular to the image plane, but makes any angle with it. We are going to examine the special case in which the line is parallel to the image plane. Thus, in Figure 2.8 we consider two lines, d_1 and d_2 , parallel to the image plane. Their projections, d'_1 and d'_2 , are parallel to the y -axis and, thus, are parallel one to the other.

The analysis of Figures 2.7 and 2.8 leads us to the conclusion that *in central projection parallel lines are projected as parallel if and only if they are parallel to the image plane*. In mathematical terms we like to say that the parallelism of lines parallel to the image plane is an **invariant** of central projection. In general, however, parallelism is not conserved under central projection.

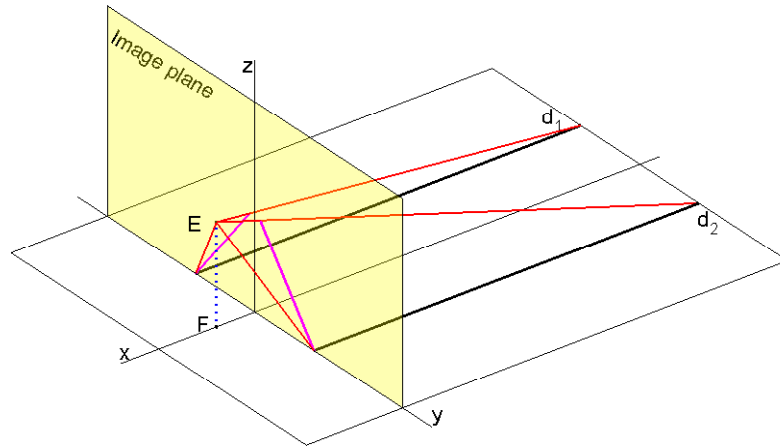


Figure 2.7: Central projection, lines not parallel to the image plane

Figure 2.8 reveals us another property of central projection. The segments of the lines d_1 and d_2 shown there are equal, but the projection of the segment of d_2 is shorter than the projection of the segment of d_1 . The farther a segment is from the image plane, the shorter is its projection. This phenomenon is known as **foreshortening** and it contributes to the realistic sensation of depth conveyed by central projection. We shall talk more about it in the section on vanishing points.

There is another property that is not invariant under central projection. In Figure 2.9 the point O is the centre of projection, \overline{AB} is a line segment in

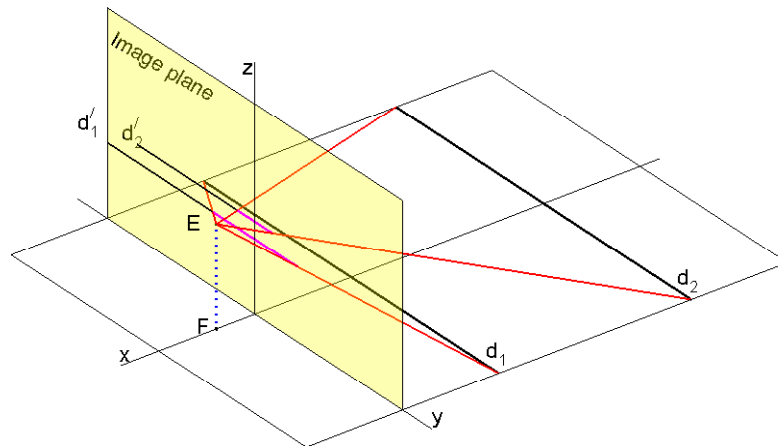


Figure 2.8: Central projection, lines parallel to the image plane

the object plane, and $\overline{A'B'}$, its central projection on the image plane. As one can easily check, M' is the midpoint of the segment $\overline{A'B'}$, while M is not the midpoint of the segment \overline{AB} . This simple experiment shows that central projection does not conserve the **simple ratio** of three collinear points. Formerly we can write that, in general,

$$\frac{\overline{A'M'}}{\overline{M'B'}} \neq \frac{\overline{AM}}{\overline{MB}}$$

An exception is the case in which the object and the image pane are parallel.

Central projection, however, conserves a more complex quantity, the

Central projection, variation of the simple ratio

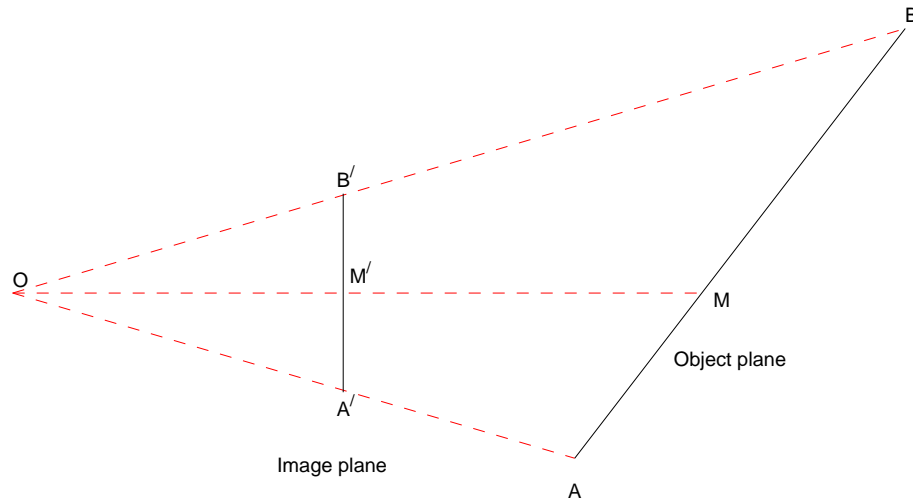


Figure 2.9: Central projection, the simple ratio is not invariant

cross-ratio of four points. In Figure 2.10 we consider the projection centre, O , and three collinear points, A , B , C , of the object plane. The three points lie on the line d that together with the projection centre defines a projecting plane. Its intersection with the image plane is the line d' . The projection of the point A is A' , that of B , B' , and that of C is C' . Following Reutter (1972) we draw through O a line parallel to the object line, d . It intersects d' in F . We also draw through C' a line parallel to d . Its intersections with the projection rays are A_1 , B_1 and C_1 , the latter being identical, by construction, with C' .

Applying the theorem of Thales in the triangle ACO we write

$$\frac{\overline{AC}}{\overline{BC}} = \frac{\overline{A_1C_1}}{\overline{B_1C_1}} \quad (2.1)$$

The triangles $A_1A'C_1$ and $OA'F$ are similar. Therefore, we can write

$$\frac{\overline{A'C'}}{\overline{A'F}} = \frac{\overline{A_1C_1}}{\overline{OF}} \quad (2.2)$$

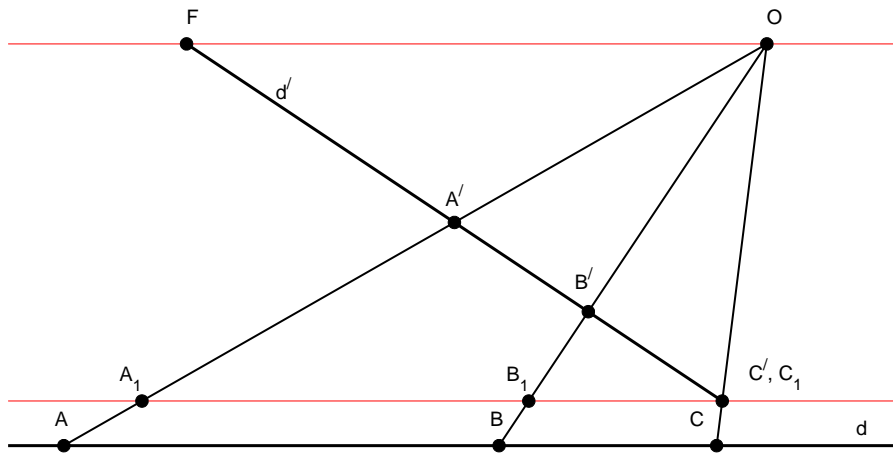


Figure 2.10: The invariance of the cross-ratio

The triangles $B_1B'C_1$ and $OB'F$ are also similar. Therefore, we can write

$$\frac{\overline{B'C'}}{\overline{B'F}} = \frac{\overline{B_1C_1}}{\overline{OF}} \quad (2.3)$$

Dividing Equation 2.2 by Equation 2.3 and using Equation 2.1 we obtain

$$\begin{aligned} \frac{\overline{A'C'}}{\overline{A'F}} \cdot \frac{\overline{B'F}}{\overline{B'C'}} &= \frac{\overline{A_1C_1}}{\overline{OF}} \cdot \frac{\overline{OF}}{\overline{B_1C_1}} \\ &= \frac{\overline{A_1C_1}}{\overline{B_1C_1}} = \frac{\overline{AC}}{\overline{BC}} \end{aligned} \quad (2.4)$$

Summing up, we obtained

$$\frac{\overline{AC}}{\overline{BC}} = \frac{\overline{A'C'}}{\overline{B'C'}} \cdot \frac{\overline{B'F}}{\overline{A'F}} \quad (2.5)$$

Taking instead of C a point D , we can write

$$\frac{\overline{AD}}{\overline{BD}} = \frac{\overline{A'D'}}{\overline{B'D'}} \cdot \frac{\overline{B'F}}{\overline{A'F}} \quad (2.6)$$

It remains to divide Equation 2.5 by Equation 2.6 and write

$$\frac{\overline{AC}}{\overline{BD}} : \frac{\overline{AD}}{\overline{BD}} = \frac{\overline{A'C'}}{\overline{B'C'}} : \frac{\overline{A'D'}}{\overline{B'D'}} \quad (2.7)$$

The quantity at the left of the equal sign is the **cross-ratio** of the points A, B, C, D and is noted by $(ABCD)$. The quantity at the right of the equal sign is the cross-ratio of the points A', B', C', D' and is noted by $(A'B'C'D')$. We have proved that the *cross-ratio of four collinear points is an invariant of central projection*. This property has important theoretical applications, but it is also used in practice, for instance in the interpretation of aerial photos. In French, the term cross-ratio is translated as *birapport*, and in German as *Doppeltverhältnis*. More about the subject can be read, for example, in Brannan, Esplen and Gray (1999), or in Delachet (1964). In Figure 2.11 we discover another property of the central projection. The object segments \overline{AB} and \overline{CD} are of different lengths, but their images, $\overline{A'B'}$ and $\overline{C'D'}$ have the same length. In concise notation,

$$\overline{AB} \neq \overline{CD}, \overline{A'B'} = \overline{C'D'}$$

Ambiguity of central projection

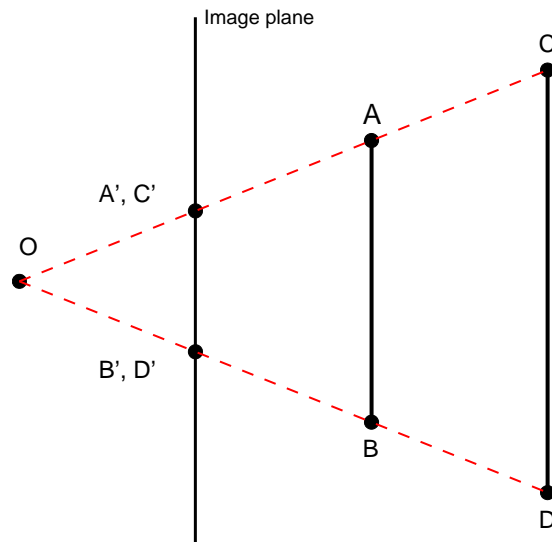


Figure 2.11: The ambiguity of the central projection

This **ambiguity** of the central projections is used with success in cinematography, but only complicates the interpretation of engineering images.

2.2.3 Vanishing points

Let us list the properties revealed by studying the projections of various lines.

- Parallel lines parallel to the image plane are projected as parallels.
- Parallel lines not parallel to the image plane are projected as

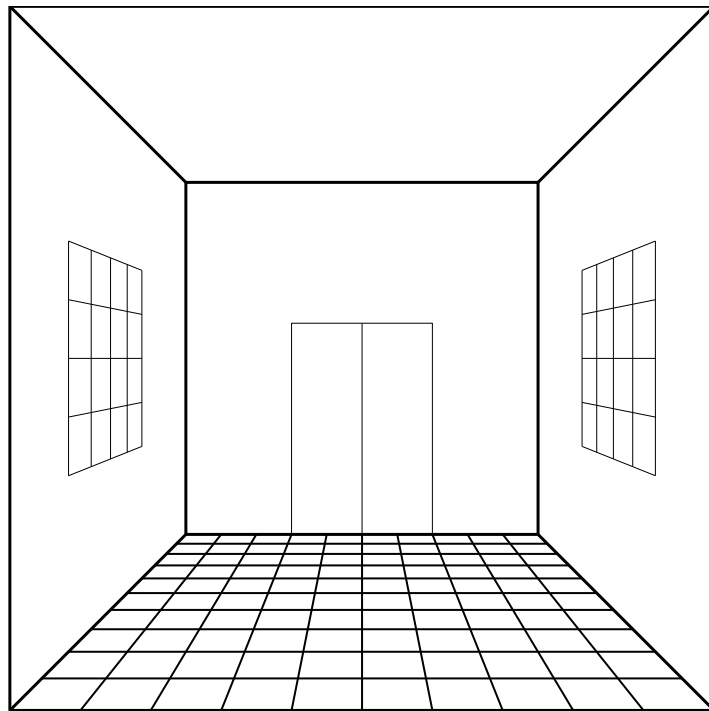


Figure 2.12: One-point perspective

converging lines.

- All lines parallel to a given direction are projected as lines converging to the same point.

The above properties determine the character of images formed under central projection. Thus, Figure 2.12 shows a room defined by lines parallel or perpendicular to the image plane. While the former are projected as parallel lines, the latter appear as lines converging to one point, as seen in the centre of Figure 2.13. The point of convergence is called **vanishing point**. In Figures 2.12 and 2.13 there is only one vanishing point.

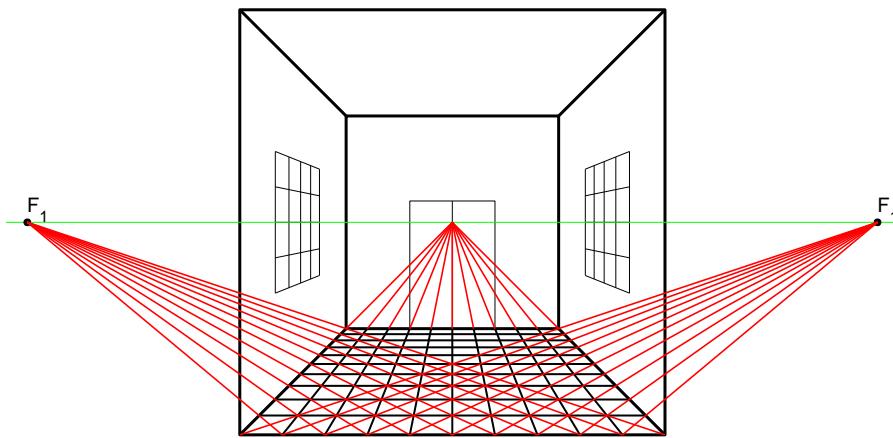


Figure 2.13: The construction of a one-point perspective

Therefore, such a projection is known as **one-point perspective**.

In a more general situation, the object can be defined by a set of lines parallel to the image plane, namely vertical, and sets of lines parallel to two directions oblique to the image plane. We obtain then a **two-point perspective**, as in Figure 2.14.

In the most general situations there are no lines parallel to the image plane. If we can identify sets of lines parallel to three directions, we can construct a **three-point perspective**, as in Figures 2.15 and 2.16. Such perspectives are used, for example, to obtain *bird views* of tall buildings.

When Renaissance artists discovered the laws of perspective, they liked to exercise with it and draw spectacular images. One particularly popular item was the *pavement* (in Italian *pavimento*) which, as exemplified in Figures 2.12 and 2.13, easily contributed to the perception of space. To cite only two names, more recently the Dutch graphic artist Maurits Cornelis

Two-point perspective

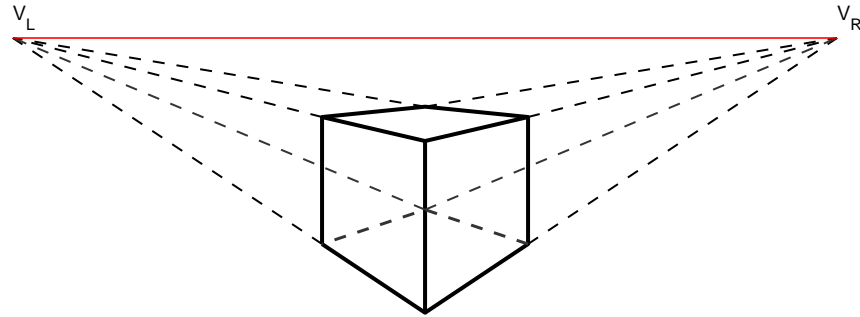


Figure 2.14: Two-point perspective

Escher (1898-1972) and the Spanish Surrealist painter Salvador Dali (1904-1989) exploited in remarkable manners the possibilities of drawing perspectives.

2.2.4 Conclusions

The central projection produces the true perspective, that is images as we see them, and as photographic cameras (film or digital) and TV cameras see.

As we saw, central projection conserves the parallelism of lines parallel to the image plane, the geometric similarity of objects in planes parallel to the image plane, and the cross-ratio. On the other hand, the central projection does not conserve the parallelism of lines that have any direction oblique to

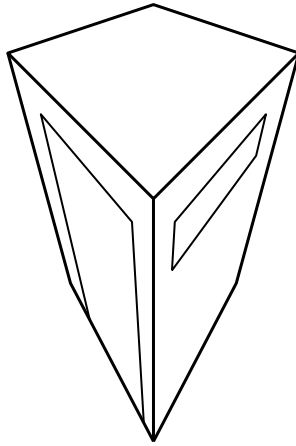


Figure 2.15: Three-point perspective

the image plane, does not conserve the simple ratio, and the foreshortening effect can lead to ambiguity of size. As a result, to draw in central projection one must use elaborate techniques, and Figure 2.13 is just a very simple example. Still worse, to pick up measures from a central-projection drawing one has to use special calculations. Architects and graphic artist have to draw in central projection in order to obtain realistic effects. The interpretation of aerial photos, and that of images seen by a robot require the use of elaborate techniques and not-so-simple mathematics. In engineering drawing, however, we are interested in using the simplest techniques, both for drawing and for reading a drawing. Therefore, we must accept an assumption that makes our life simpler, even if we have to pay some cost. These matters are described in the next sections. We think, however, that the preceding introduction was worth reading because

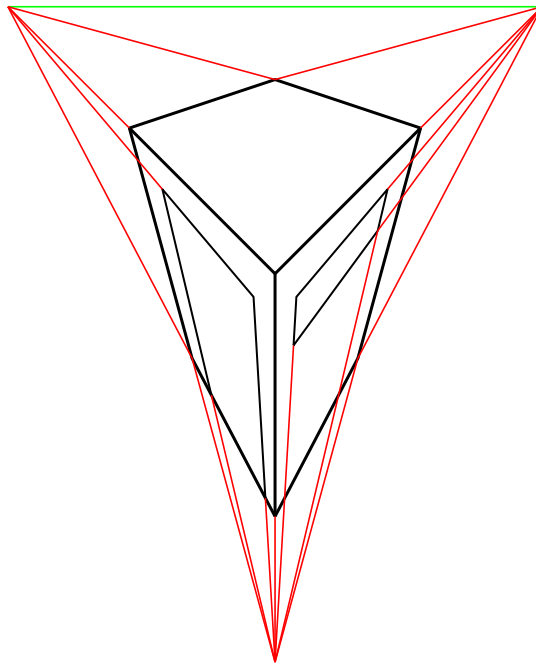


Figure 2.16: The construction of a three-point perspective

1. the user of projections described in continuation must know to what extent his work departs from reality and why he has to accept this fact;
2. the student who wants to reach higher levels or continue to the study of computer vision, must know in which direction to aim.

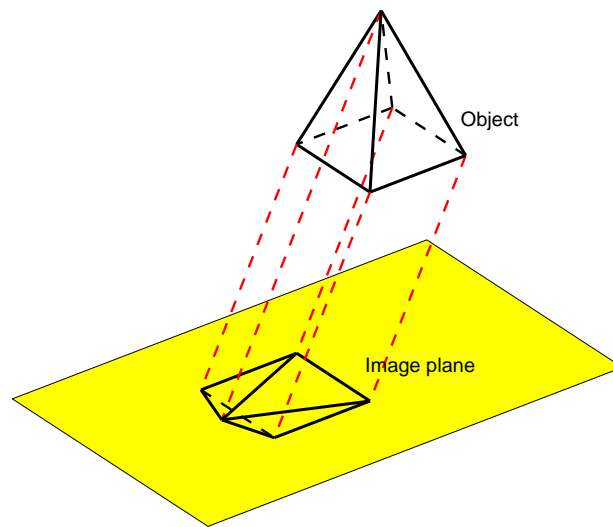


Figure 2.17: Parallel projection of a pyramid

2.3 Parallel projection

2.3.1 Definition

We make now a new, strong assumption: *the projection centre is sent to infinity*. Obviously, all projection lines become parallel. What we obtain is called **parallel projection**. An example is shown in Figure 2.17. As we shall see in the next subsection, by letting the projection rays be parallel we add a few properties that greatly simplify the execution and the reading of drawings.

In real life we cannot think about looking at objects from infinity. The farther we go from an object, the smaller its image is. From a certain distance we practically see nothing. We ignore this fact and, as said, we

assume that we do see the world from infinity. In fact, in Engineering Drawing we assume many other conventions that do not correspond to reality, but greatly simplify the work.

2.3.2 A few properties

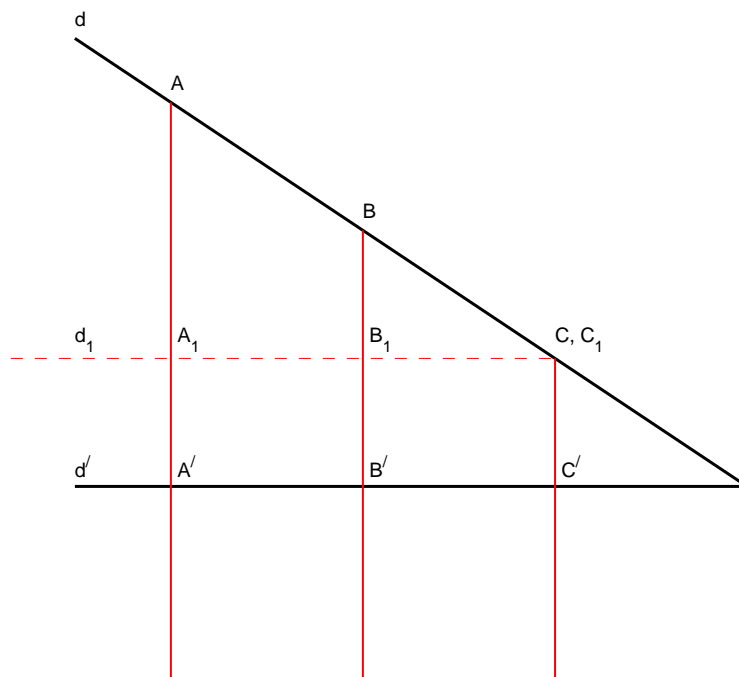


Figure 2.18: Parallel projection — The invariance of the simple ratio

In Figure 2.18 the line d belongs to the object plane and its parallel projection is d' . The projection of the point A is A' , that of B is B' , and that of C is C' . The projection lines, AA' , BB' , CC' , are parallel. They can make any angle with the image plane, except being parallel to it. Let

us draw through the point C the line d_1 parallel to d' . By construction

$$\overline{A_1C_1} = \overline{A'C'}, \overline{B_1C_1} = \overline{B'C'} \quad (2.8)$$

Applying the theorem of Thales in the triangle ACA_1 yields

$$\frac{\overline{BC}}{\overline{AC}} = \frac{\overline{B_1C_1}}{\overline{A_1C_1}} \quad (2.9)$$

Substituting the values given by Equations 2.8 in Equations 2.9 we obtain

$$\frac{\overline{BC}}{\overline{AC}} = \frac{\overline{B'C'}}{\overline{A'C'}} \quad (2.10)$$

We conclude that *the simple ratio of three collinear points is an invariant of parallel projection.*

Another property of parallel projection can be deduced from Figure 2.17:

parallel lines are projected as parallel.

Finally, Figure 2.19 shows that *when the object plane is parallel to the image plane, the lengths of segments and the sizes of angles are invariant.*

2.3.3 The concept of scale

We know now that in parallel projection the length of the projection of a segment parallel to the image plane is equal to the real length of that segment. This property greatly simplifies the process of drawing.

Apparently it would be sufficient to arrange the object so as to have as many segments parallel to the image plane and then draw their projections in real size. In practice, however, this is not always possible. Many objects

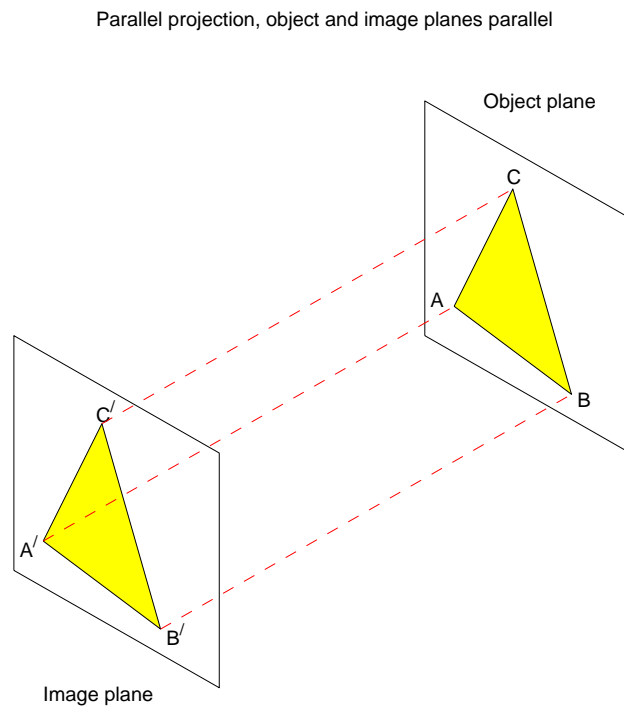


Figure 2.19: Parallel projection — The invariance of lengths and angles

are larger than the paper sheets we use. Other objects are too small and drawing them in real size would result in an unintelligible image. To solve the problem we must multiply all dimensions by a number such that the resulting image can be well accommodated in the drawing sheet and can be easily understood.

The international standard ISO 5455-1979 defines **scale** as the *‘Ratio of the linear dimension of an element of an object as represented in the original drawing to the real linear dimension of the same element of the object itself.*

Drawing at **full size** means drawing at the scale 1:1, **enlargement** means a

drawing made at a scale larger than 1:1, and **reduction** one made at a scale smaller than 1:1. For example, if a length is 5 m, that is 5000 mm, when we use the scale 1:20 we draw it as 250 mm. On the other hand, a rectangle 2 mm by 5 mm drawn at the scale 2:1 appears as a rectangle with the sides 4 and 10 mm. Table 2.1 shows the scales recommended in ISO 5455-1979.

Table 2.1: The scales recommended by ISO 545-1979

Enlargement	2:1 5:1 10:1 20:1 50:1
Reduction	1:2 1:5 1:10 1:20 1:50 1:100 1:200 1:500 1:1000 1:2000 1:5000 1:10000

2.4 Orthographic projection

2.4.1 Definition

Parallel projections are used for special applications in Engineering Graphics and, to a greater extent, in general graphics. In Descriptive Geometry, however, we go one step farther and we assume that *the projection rays are perpendicular to the image plane*. One simple example is shown in Figure 2.20. We obtain thus the **orthogonal projection**. This is a particular case of the parallel projection; therefore, it inherits all the

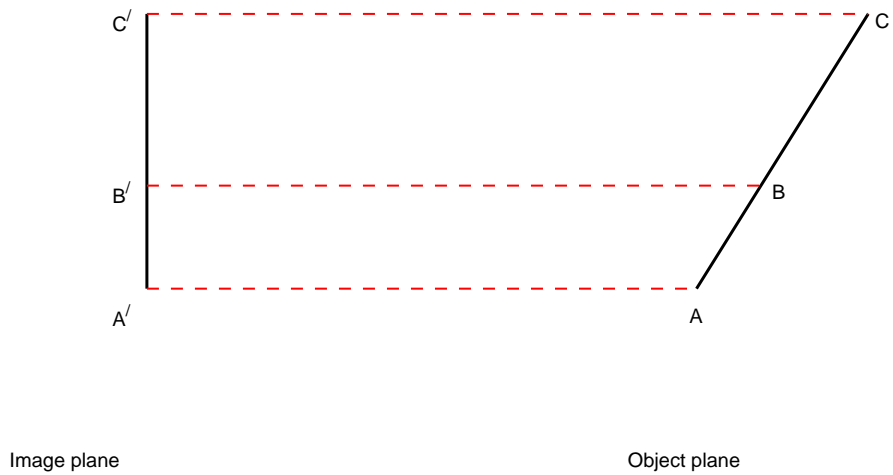


Figure 2.20: Orthographic projection

properties of the parallel projection. In addition, the assumption of projection rays perpendicular to the image plane yields a new property that is described in the next subsection.

Figure 2.21 shows a more interesting example of orthographic projection; it can be compared to Figure 2.17.

In technical drawing, when using several projections to define one object, we talk about **orthographic projections**. Some standard definitions related to orthographic representations can be read in ISO 5456-2:1996(E).

2.4.2 The projection of a right angle

Figure 2.22 shows two line segments, \overline{BA} and \overline{BC} , and two image planes, π_1 and π_2 , perpendicular one to another. As shown in Section 2.5, these are the *projection planes* adopted in Descriptive Geometry.

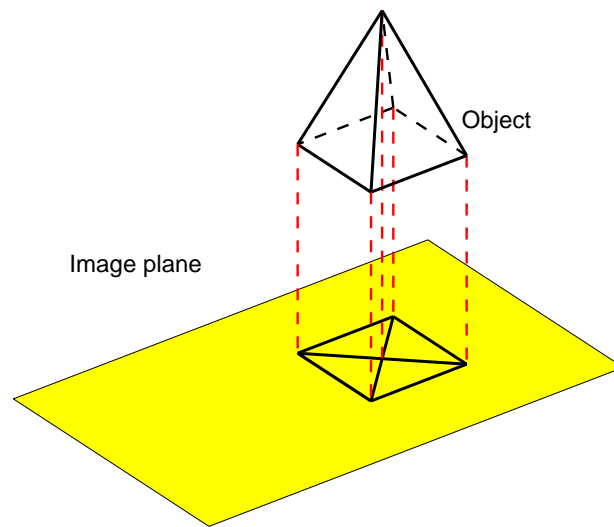


Figure 2.21: The orthographic projection of a pyramid

In Figure 2.22 the angle \widehat{ABC} is right and the side \overline{BC} is horizontal, that is parallel to the projection plane π_1 . The figure shows also the *horizontal* and the *frontal* projections of this angle. By **horizontal projection** we mean the projection on the horizontal plane π_1 , and by **frontal projection**, the projection on the frontal plane π_2 . The following theorem states an important property of orthogonal projections.

If one of the sides of a right angle is parallel to one of the projection planes, the (orthogonal) projection of the angle on that plane is also a right angle

To prove this theorem, let the coordinates of the points A , B , C , be

$$A = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}, \quad B = \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}, \quad C = \begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix}$$

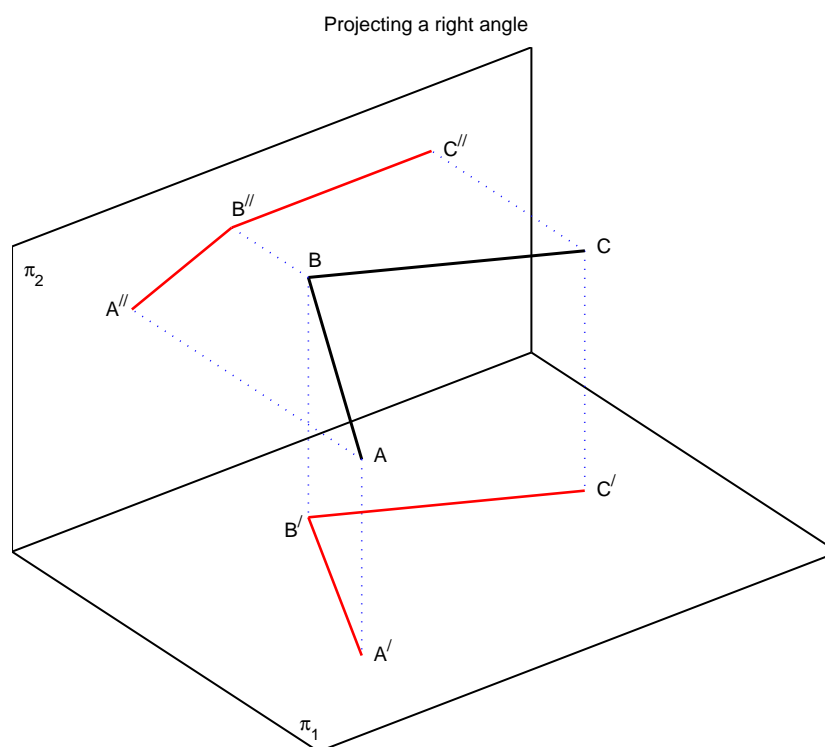


Figure 2.22: Projecting a right angle

The horizontal projections of the three points have the coordinates

$$A' = \begin{bmatrix} A_x \\ A_y \\ 0 \end{bmatrix}, \quad B' = \begin{bmatrix} B_x \\ B_y \\ 0 \end{bmatrix}, \quad C' = \begin{bmatrix} C_x \\ C_y \\ 0 \end{bmatrix}$$

The angle \widehat{ABC} has as sides the vectors

$$\begin{aligned}\overrightarrow{BA} &= (A_x - B_x)\mathbf{i} + (A_y - B_y)\mathbf{j} + (A_z - B_z)\mathbf{k} \\ \overrightarrow{BC} &= (C_x - B_x)\mathbf{i} + (C_y - B_y)\mathbf{j} + (C_z - B_z)\mathbf{k}\end{aligned}\quad (2.11)$$

If the angle \widehat{ABC} is right, the scalar product of the above vectors is zero

$$(A_x - B_x)(C_x - B_x) + (A_y - B_y)(C_y - B_y) + (A_z - B_z)(C_z - B_z) = 0 \quad (2.12)$$

The scalar product of the horizontal projections of the vectors \overrightarrow{BA} , \overrightarrow{BC} is

$$\overrightarrow{B'A'} \cdot \overrightarrow{B'C'} = (A_x - B_x)(C_x - B_x) + (A_y - B_y)(C_y - B_y) \quad (2.13)$$

If the projected angle is right, the scalar product $\overrightarrow{B'A'} \cdot \overrightarrow{B'C'}$ must be zero and this happens only if

$$(A_z - B_z)(C_z - B_z) = 0$$

that is, if either $A_z = B_z$, or $C_z = B_z$. In other words, the right angle \widehat{ABC} is projected as a right angle, $\widehat{A'B'C'}$, if \overline{BA} or \overline{BC} is parallel to the horizontal projection plane, π_1 . This proves that the theorem states a necessary condition. We can see immediately that this condition is also sufficient. Indeed, if in Equation 2.12 either $A_z = B_z$, or $C_z = B_z$, then the scalar product in Equation 2.13 is zero. In Figure 2.22 the side \overline{BC} is parallel to the horizontal projection plane, π_1 . The extension of the proof to

To the theorem of the right angle

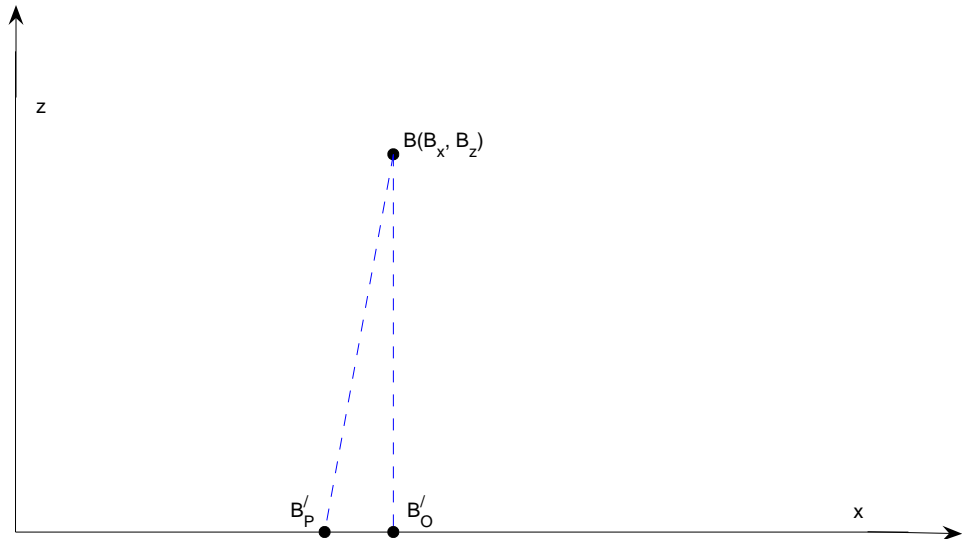


Figure 2.23: To the theorem of the projections of a right angle

cases in which one side of the right angle is parallel to another projection plane is straightforward.

Is this theorem valid only for orthogonal projections? Doesn't it hold also for an oblique parallel projection? Without loss of generality, let us check the case of a parallel, oblique projection in which the projecting rays are parallel to the frontal projection plane, π_2 , but make the angle α with the horizontal projection plane, π_1 . In Figure 2.23 we consider the point B of Figure 2.22. The orthogonal projection of the point B would be B'_O with the x -coordinate equal to B_x . We consider, however, the oblique projection with $\widehat{B'_O B'_P B} = \alpha$. The projection of B is now the point B'_P with the

x -coordinate

$$B_x - \frac{B_z}{\tan \alpha}$$

Then, the coordinates of the oblique projections would be

$$A'_P = \begin{bmatrix} A_x - \frac{A_z}{\tan \alpha} \\ A_y \\ 0 \end{bmatrix}, \quad B'_P = \begin{bmatrix} B_x - \frac{B_z}{\tan \alpha} \\ B_y \\ 0 \end{bmatrix}, \quad C'_P = \begin{bmatrix} C_x - \frac{C_z}{\tan \alpha} \\ C_y \\ 0 \end{bmatrix}$$

and the scalar product of the projections is

$$\begin{aligned} \overrightarrow{B'_P A'_P} \cdot \overrightarrow{B'_P C'_P} &= \left(A_x - B_x - \frac{A_z - B_z}{\tan \alpha} \right) \left(C_x - B_x - \frac{C_z - B_z}{\tan \alpha} \right) \\ &\quad + (A_y - B_y)(C_y - B_y) \\ &= (A_x - B_x)(C_x - B_x) + (A_y - B_y)(C_y - B_y) - \\ &\quad (A_x - B_x) \frac{C_z - B_z}{\tan \alpha} - (C_x - B_x) \frac{A_z - B_z}{\tan \alpha} \quad (2.14) \end{aligned}$$

As we assumed that the angle \widehat{ABC} is right and that one of its sides is parallel to the horizontal projection plane, the sum of the first two terms in Equation 2.14 is zero. The sum of the other two terms is zero if, and only if we have both $C_z = B_z$ and $A_z = B_z$, that is if both sides of the angle \widehat{ABC} are parallel to the horizontal projection plane. But, in this case the angle \widehat{ABC} lies in a horizontal plane, that is in a plane parallel to one of the projection planes and we already know that in parallel projections the size of angles is invariant if the object and image planes are parallel.

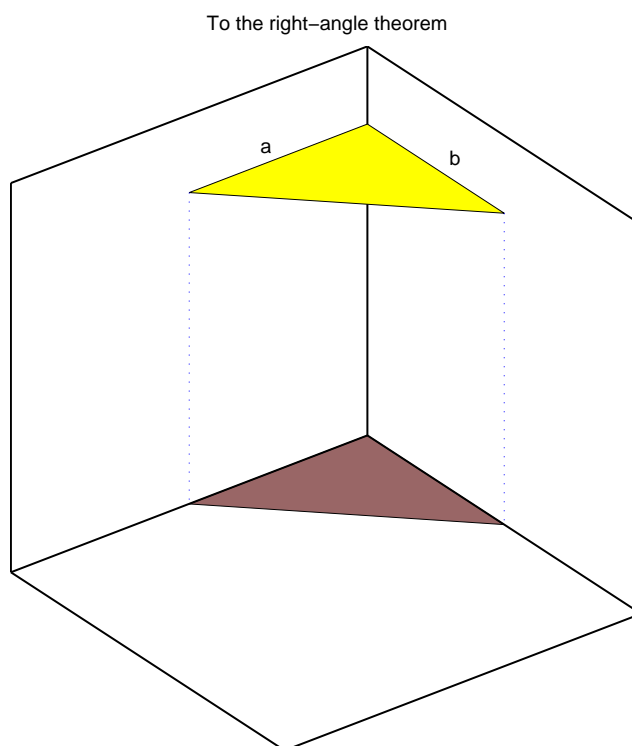


Figure 2.24: Experimental proof of the right-angle theorem

There is an experimental way of proving the right-angle theorem. We invite the reader to take a cardboard box and a square set. First, locate the square set in the corner of the box, and parallel to the bottom, as in Figure 2.24. The square set is shown in yellow, and its horizontal projection in a darker, brown hue. This projection is right-angled.

Let now the side b of the square set remain horizontal, while we rotate the side a as in Figure 2.25. The square set still fits well in the corner of the box and the horizontal projection remains right-angled. Alternatively, keep the side a horizontal and rotate the side b as in Figure 2.26. The square set

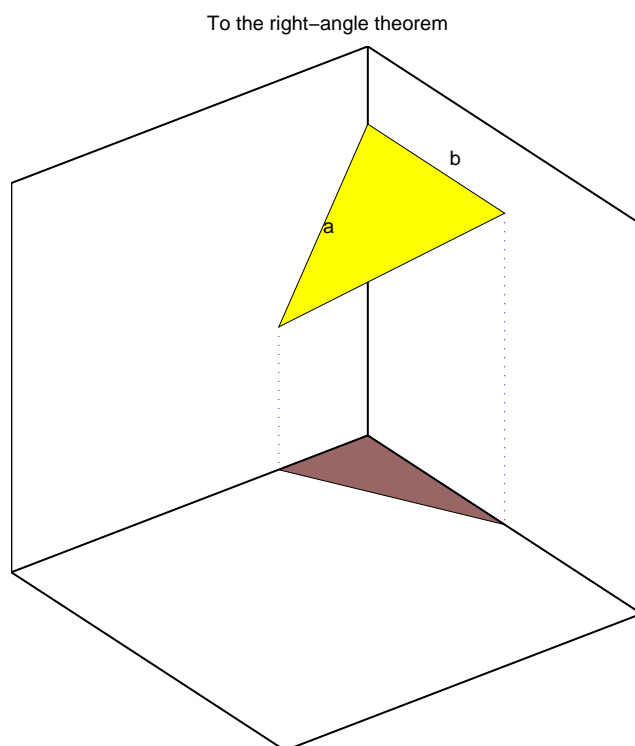


Figure 2.25: Experimental proof of the right-angle theorem

again fits well in the corner of the box and the horizontal projection remains right-angled.

Finally, try to rotate both sides a and b of the square set so that none of them is horizontal, or, in other words, none of them is parallel to the bottom. It is impossible to do this without changing the angle of the box corner. In fact it must be increased. This means that the horizontal projection is no more right-angled.

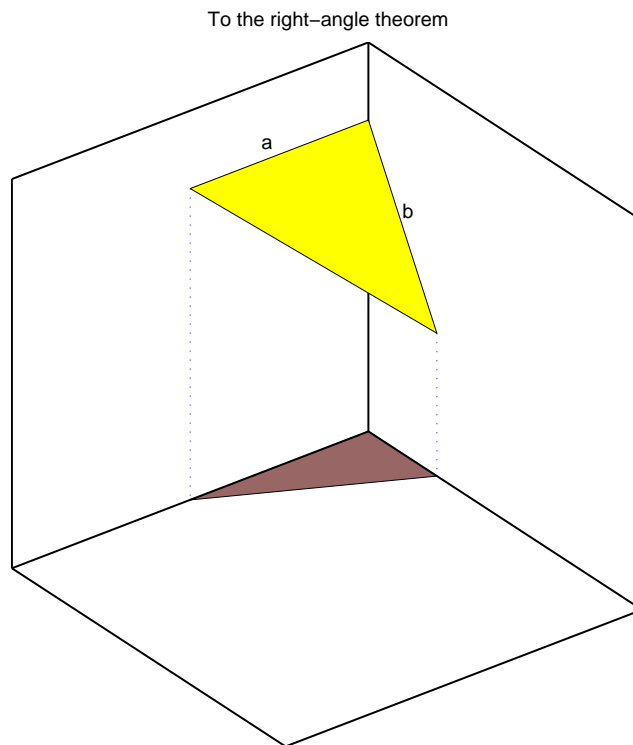


Figure 2.26: Experimental proof of the right-angle theorem

2.5 The two-sheet method of Monge

It seems that the great German painter and printmaker Albrecht Dürer (1471-1528) was the first to show how three-dimensional objects can be represented by projections on two or more planes (Coolidge, 1963, pp. 109–12, Bertoline and Wiebe, 2003, pp. 394–5). However, the first to organize the methods for doing so in a coherent system was the French mathematician Gaspard Monge (1746-1818). Therefore, he is considered to be the founder of Descriptive Geometry. Monge was one of the scientists

who accompanied Bonaparte in the Egypt campaign and was one of the founders of the École Polytechnique in Paris.

a

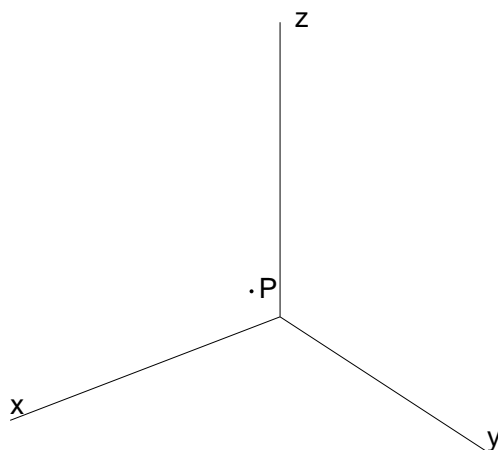


Figure 2.27: Preparing the two-sheet projection of Monge

Figure 2.27 shows a point P and a system of cartesian coordinates. The basic idea of Descriptive Geometry is to use two *projection planes*, a *horizontal* one, defined by the axes x and y , and a *frontal* one defined by the axes x and z . We note the horizontal plane by π_1 , and the frontal one by π_2 . These projection planes are shown in Figure 2.28.

The next step is to project orthogonally the point P on π_1 and π_2 . The horizontal projection, P' , is the point in which a ray emitted from P , perpendicularly to π_1 , pierces the plane π_1 . Similarly, the frontal projection, P'' , is the point in which a ray emitted from P , perpendicularly to π_2 ,

b

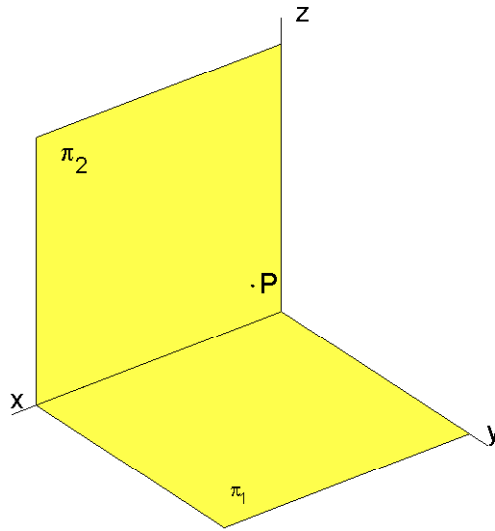


Figure 2.28: Preparing the two-sheet projection of Monge

pierces the plane π_2 . These projections are shown in Figure 2.29.

Finally, as shown in Figure 2.30, we rotate the plane π_2 until it is coplanar with π_1 . We obtain thus what we call the **sketch** of the point P . It is shown in Figure 2.31. This figure shows that the two projections contain all the information about all the coordinates of a given point.

As an example of projection on two sheets we show in Figure 2.32 the sketch of the right angle \widehat{ABC} introduced in Figure 2.22.

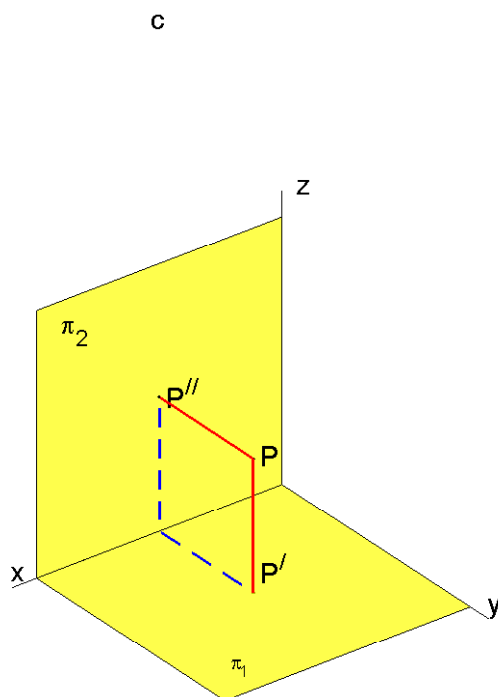


Figure 2.29: Preparing the two-sheet projection of Monge

2.6 Summary

To define the *central projection* we assumed that all projection rays pass through one point, the *projection centre*. There is no condition on the position of the projection centre. We saw then that the only properties that remain invariant between an object and its image are:

1. the parallelism of line parallel to the image plane;
2. the cross ratio of four collinear points.

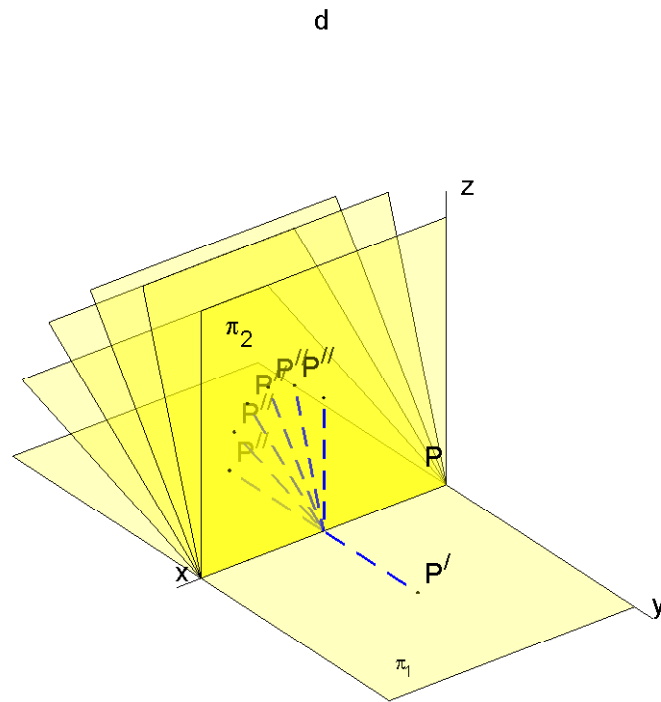


Figure 2.30: Preparing the two-sheet projection of Monge

To define the *parallel projection* we accept an additional assumption: the projection centre is sent to infinity. Then, the projection rays become parallel and we obtain three new invariants:

1. sizes of objects (segment lengths and angles) parallel to the image plane;
2. parallelism of lines;
3. the simple ratio of three collinear points.

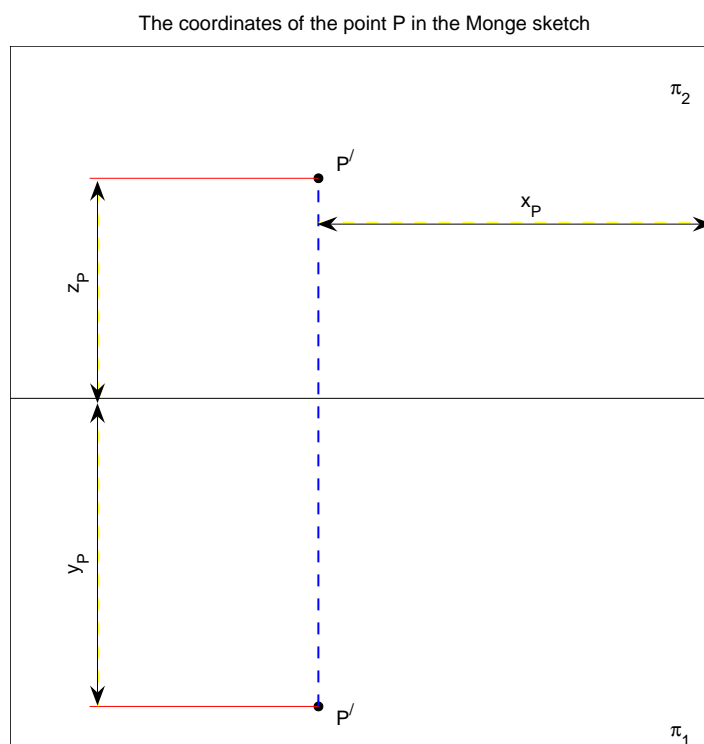


Figure 2.31: The Monge sketch contains all coordinates

The parallel projection inherits the invariants of the central projection. By adding a new assumption we restricted the number of possible projections. The parallel projection is a particular case of the central projection. On the other hand, the parallel projection has more useful properties than the central projection; it is easier to draw an object in parallel projection than in central projection and it is easier to measure lengths in parallel projection than in central projection.

Let us further assume that the projection rays are perpendicular to the image plane. We are dealing now with the *orthogonal* or *orthographic*

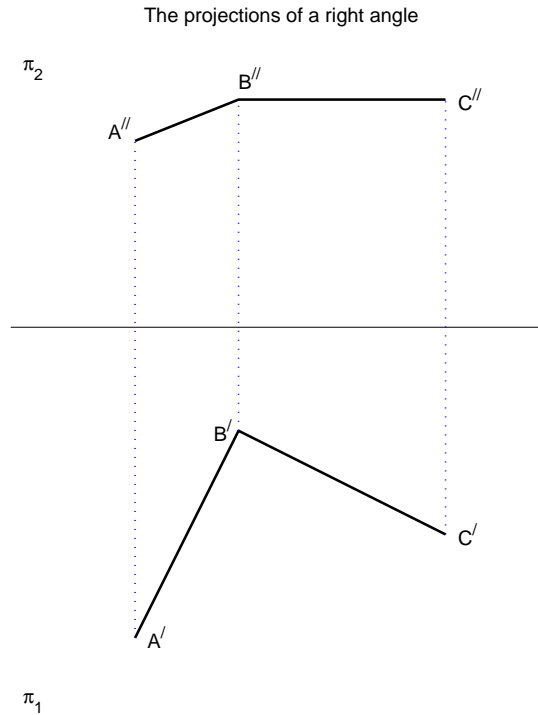


Figure 2.32: The sketch of the projections of a right angle

projection and obtain a further invariant:

1. the size of a right angle with a side parallel to the image plane.

The orthographic projection inherits the invariants of the parallel projection. The orthographic projection is a particular case of the parallel projection. By restricting further the number of possible projections we obtained an additional, useful property; it is easier to draw and to measure in orthographic projection than in an oblique parallel projection.

Figure 2.33 summarizes the hypotheses and invariants of the various projection methods and their relations of inclusion. In ISO 5456-1:1996,

Part 1, we find a summary of the properties of different projection methods. A technical vocabulary of terms relating to projection methods in English, French, German, Italian and Swedish can be found in ISO 10209-2:1993(E/F).

We said that when we draw in parallel projection we accept the seemingly unnatural assumption that we look from infinity. Nature, however, provides us with a real example of parallel projection: shadows cast by objects illuminated by the sun. The distance between us and the sun is so large that its rays are practically parallel. And if we are talking about shadows, let us say that shadows of objects illuminated by a small source of light are approximations of central projections.

2.7 Examples

Example 2.1 — Dividing a segment

In this example we learn a widely used application of one of the properties of parallel projection, specifically the invariance of the simple ratio. In Figure 2.34 we see a segment $\overline{P'_0P'_7}$ defined on the x -axis. Let us suppose that we are asked to divide it into seven equal segments. The trivial solution seems to consist of:

1. measuring the length $\overline{P'_0P'_7}$;
2. dividing the above length by 7;

3. applying the resulting length, $\overline{P'_0P'_7}/7$, seven times along the x -axis.

This procedure may be convenient in particular cases. In general however, the calculated length, $\overline{P'_0P'_7}/7$, does not fit exactly the gradations of our rulers or square sets. Then, measuring seven times the calculated length may result in unequal segments and an accumulated error towards the end. A good solution is to draw an arbitrary segment, inclined with respect to the given segment. In Figure 2.34 we draw thus the segment $\overline{P_0P_7}$, through a point outside the given segment. In practice however, we can draw the arbitrary segment through one of the ends of the given segment, for example from the point P'_0 . On the oblique segment we measure an exact multiple of 7, for example 70 mm. Let the measured segment be $\overline{P_0P_7}$. We mark on it seven equal segments

$$\overline{P_0P_1} = \overline{P_1P_2} = \dots = \overline{P_6P_7}$$

We connect P_7 to P'_0 and we draw through the divisions of $\overline{P_0P_7}$ 'projection rays' parallel to $\overline{P_7P'_0}$:

$$\overline{P_0P'_0} \parallel \overline{P_1P'_1} \parallel \dots \parallel \overline{P_7P'_7}$$

To prove that we divided the given segment into equal parts, we draw through P_0 a line parallel to the x -axis and note by Q_1, Q_2, \dots, Q_7 its intersections with the parallel lines $\overline{P_0P'_0}, \dots, \overline{P_7P'_7}$. By Thales' theorem

$$\frac{\overline{P_0P_1}}{\overline{P_0P_7}} = \frac{\overline{P_0Q_1}}{\overline{P_0Q_7}} = \dots = \frac{\overline{P_6P_7}}{\overline{P_0P_7}} = \frac{\overline{Q_6Q_7}}{\overline{P_0Q_7}}$$

As

$$\overline{P_0Q_1} = \overline{P'_0P'_1}, \dots, \overline{Q_6Q_7} = \overline{P'_6P'_7}$$

by construction, we conclude that

$$\frac{\overline{P'_0P'_1}}{\overline{P'_0P'_7}} = \frac{\overline{P_0P_1}}{\overline{P_0P_7}}$$

and so on.

Example 2.2 — Decomposing a vector

The term *projection* appears also in other contexts than Descriptive Geometry. The reader may be familiar with this use, for instance in Vector Geometry or Mechanics. For example, figure 2.35 shows a vector, \mathbf{F} , and a system of cartesian coordinates. The horizontal axis is Ox , and the vertical axis is Oy . We decompose the vector \mathbf{F} into a horizontal component, \mathbf{F}_x and a vertical component, \mathbf{F}_y . Using an alternative terminology we can say that \mathbf{F}_x is the *horizontal projection* and \mathbf{F}_y is the *vertical projection* of the given vector.

If α is the angle made by the vector \mathbf{F} with the x -axis, then

$$|\mathbf{F}_x| = |\mathbf{F}| \cos \alpha, \quad |\mathbf{F}_y| = |\mathbf{F}| \sin \alpha$$

Using the concept of *scalar product* we can, however, write a more interesting expression for these projections. Let F_x be the length of \mathbf{F}_x and F_y that of \mathbf{F}_y . As usual, we note by regular letters scalar quantities, and by bold-face letters, vectors. Then

$$\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j}$$

where \mathbf{i} is the unit vector along the x -axis, and \mathbf{j} the unit vector along the y -axis. Using the dot, ' \cdot ', to mark the scalar product (therefore known also as *dot product*), the horizontal projection of \mathbf{F} is given by

$$\mathbf{F}_x = ((\mathbf{F}) \cdot \mathbf{i})\mathbf{i} = F_x\mathbf{i}$$

Similarly, the vertical projection of \mathbf{F} is given by

$$\mathbf{F}_y = ((\mathbf{F}) \cdot \mathbf{j})\mathbf{j} = F_y\mathbf{j}$$

2.8 Exercises

In Figures 2.22 and 2.32 the coordinates of the three points A , B , C are

$$A = \begin{bmatrix} 3 \\ -7.5 \\ 4.5 \end{bmatrix}, \quad B = \begin{bmatrix} 5.5 \\ -2.5 \\ 5.5 \end{bmatrix}, \quad C = \begin{bmatrix} 10.5 \\ -5 \\ 5.5 \end{bmatrix}$$

Use the scalar product to show that the angles \widehat{ABC} and $\widehat{A'B'C'}$ are right.

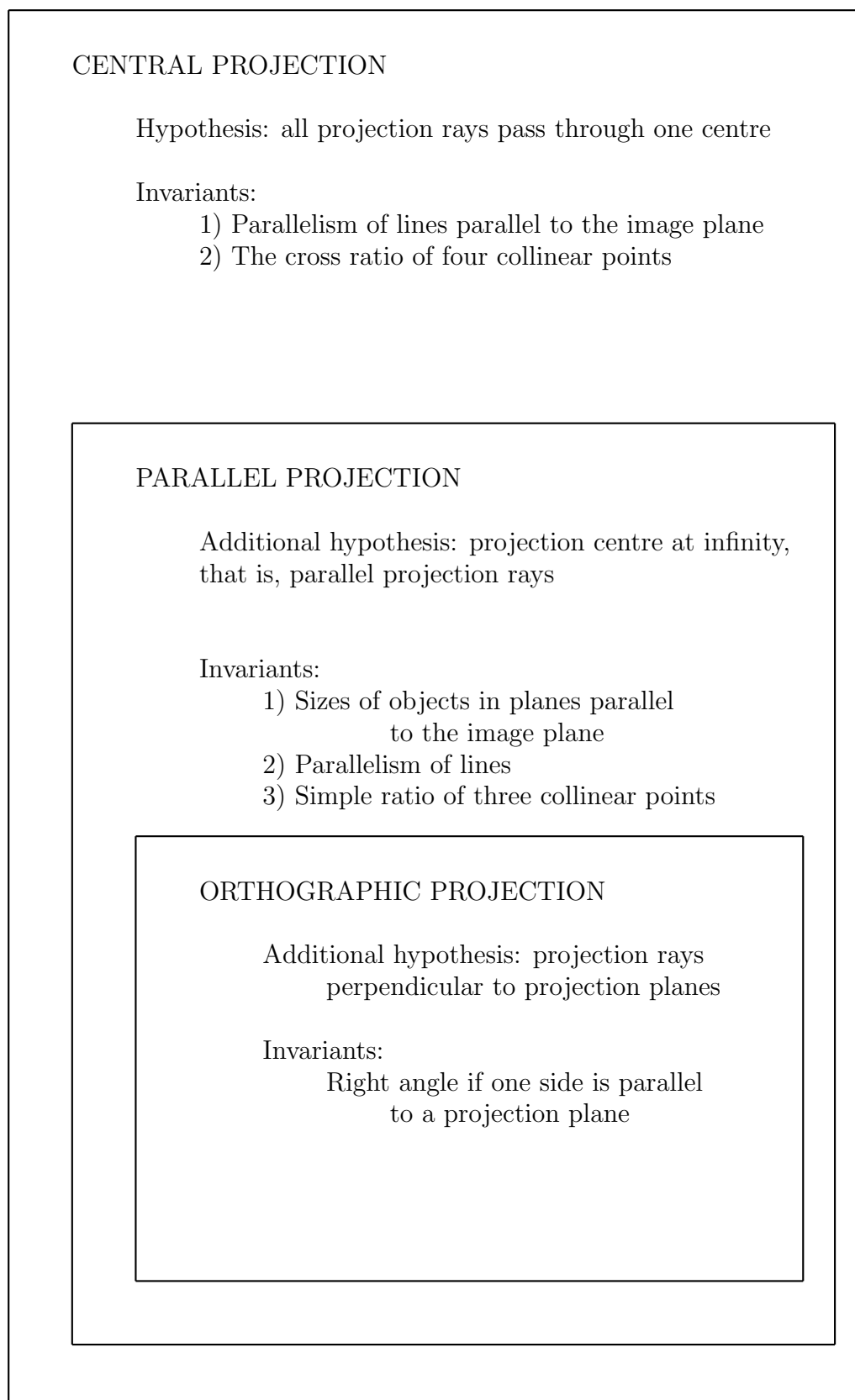


Figure 2.33: Projection methods and their invariants

Parallel projection – Applying the invariance of the simple ratio

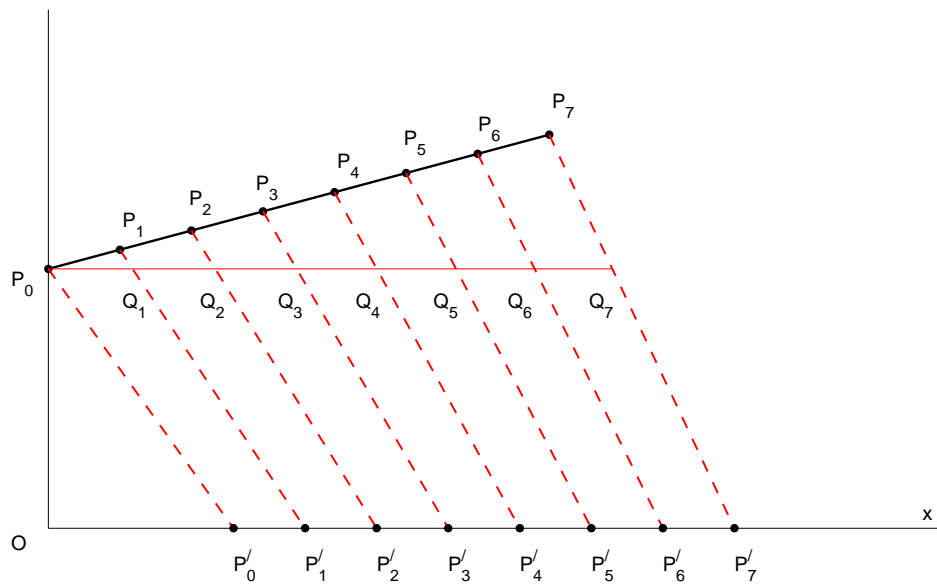


Figure 2.34: An application of the simple-ratio invariance

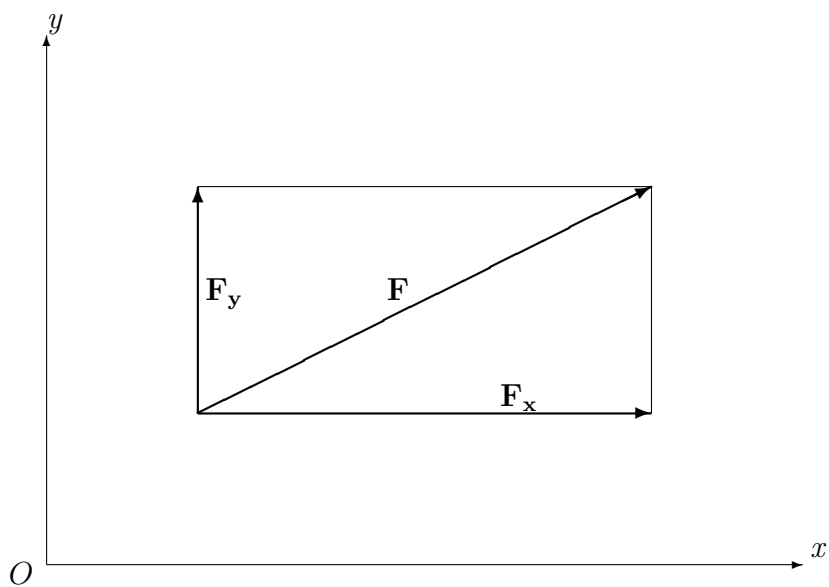


Figure 2.35: The components, or projections of a vector

Chapter 3

The point

3.1 The projections of a point

3.2 Points in four quadrants

3.3 The first angle view

3.4 The third angle view

3.5 Exercises

Chapter 4

The straight line

4.1 The projections of a straight line

4.2 Special straight lines

4.2.1 Parallel lines

4.2.2 Horizontal lines

4.2.3 Frontal lines

4.2.4 Profile lines

4.2.5 Lines perpendicular on π_2

4.2.6 Vertical lines

4.2.7 Lines perpendicular on π_3

4.3 When a third projection can help

4.4 Intersecting lines

4.5 The true length of a straight-line

Chapter 5

The plane

5.1 Defining a plane

5.2 The traces of a given plane

5.3 The traces of special planes

5.3.1 Definitions

5.3.2 How to visualize a plane

5.3.3 Horizontal planes

5.3.4 Frontal planes

5.3.5 Planes parallel to π_3

5.3.6 Planes perpendicular on π_1

5.3.7 Planes perpendicular on π_3

5.3.8 Parallel planes

5.4 Special straight lines of a given plane

Chapter 6

Axonometry I

6.1 The need for axonometry

In Figure 6.1 we see the orthographic projections of a simplified house model. In Descriptive Geometry we call the projection planes π_1 , π_2 and π_3 . In Engineering Drawing it is usual to consider the projection on π_2 as the main projection and call it **front elevation**. The projection on π_1 is called **top view** and the projection on π_3 , **side elevation**.

It is obvious that none of the three projections fully defines the shape of the house. Moreover, while the overall breadth, B_1 , can be measured in the front elevation and the top view, the breadth B_2 can be measured only in the front elevation. Similarly, while the overall length, L_1 , can be measured in the top view and the side elevation, the length L_2 can be measured only in the side elevation. Finally, the overall height, H_1 , and the height H_2 can

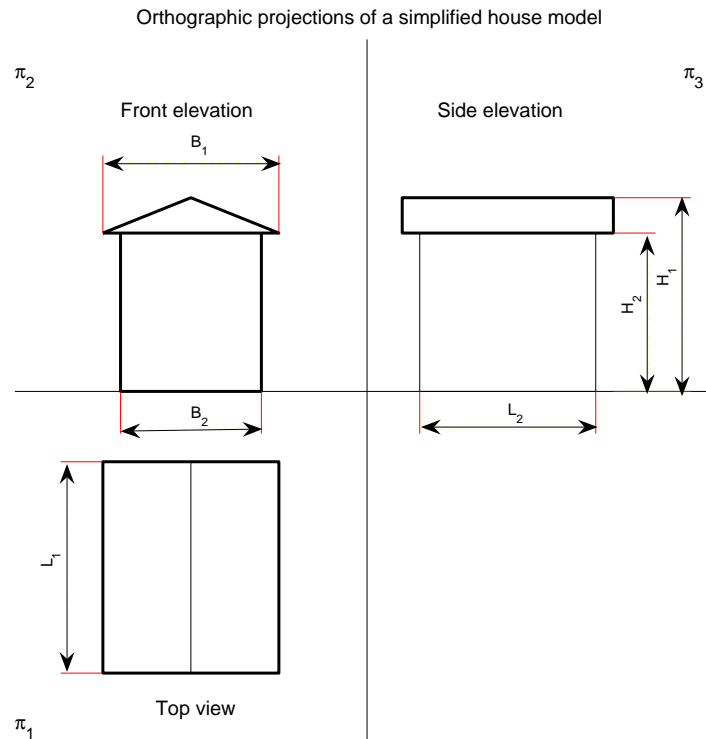


Figure 6.1: The orthographic projections of a simplified house model

be measured in the front and side elevations, but not in the top view. We conclude that to fully define this simple house form and to measure all its dimensions we need three projections. One question may naturally arise, 'Isn't it really possible to find a single projection that fully defines the object and allows us to dimension it completely?' In some simple cases the answer is, 'Yes', and one example of a corresponding drawing is shown in Figure 6.2. We call this representation of the house **axonometric projection**. Webster's Ninth New Collegiate Dictionary shows that the first appearance of the term *axonometric* dates from 1908 and that it is

composed of the Greek word *axon*, meaning ‘axis’, and the English word *metric*.

Axonometry of simple house model

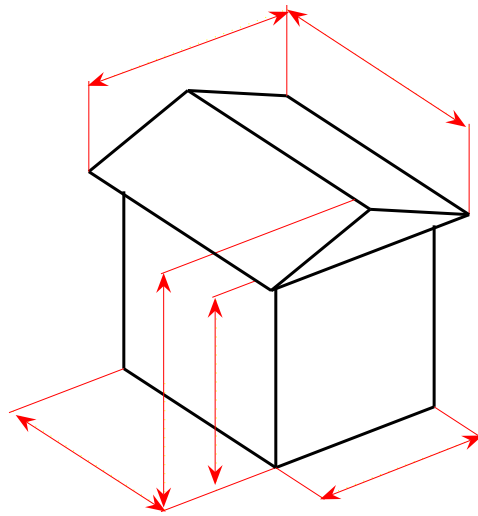


Figure 6.2: The axonometric projection of a simplified house model

6.2 Definition of orthographic axonometry

We assume that the object to be drawn is placed in a cartesian coordinate system. For orthographic projections we place the object so as to have as many plane faces as possible parallel to the planes xy , xz , yz . We noted these planes by π_1 , π_2 , π_3 , respectively. Next, we project the object on those three planes. For axonometric projections we still consider the object

as placed in the same coordinate system, but we proceed differently. There are two ways to define axonometric projections. The first possibility is to add a new plane, π , that is oblique with respect with the three planes π_1, π_2, π_3 . We project now our object on the plane π . We shall follow in continuation this definition because it allows us to draw an important conclusion. The other possibility is to rotate the object so that its plane faces are oblique with respect to the projection planes and project now the object on one of the planes π_1, π_2, π_3 .

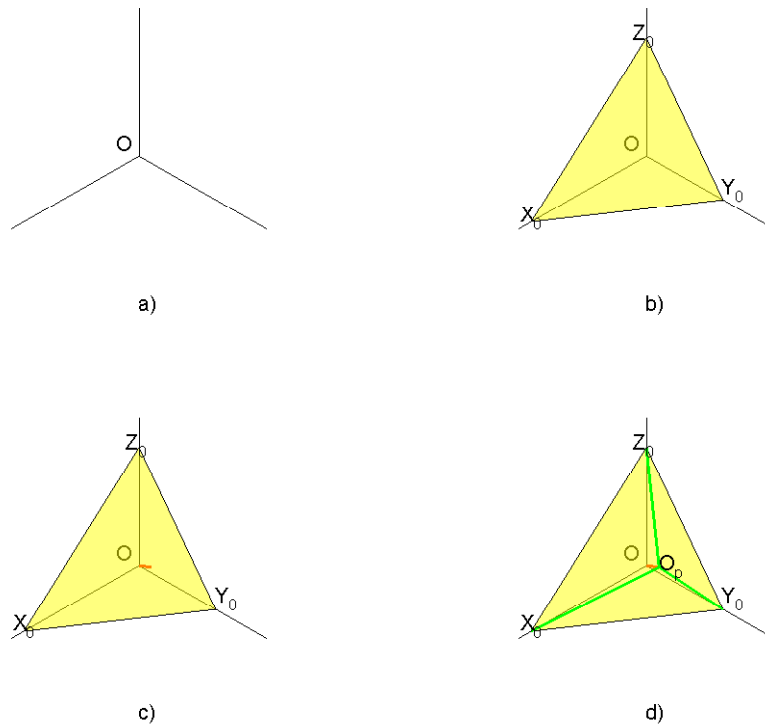


Figure 6.3: The axonometric projections of the coordinate axes

6.3 The law of scales

In Figure 6.3 a) we consider a system of cartesian coordinates with the origin in O and the axes O_x , O_y , O_z . In part b) of the figure we choose a projection plane, π , whose traces are

$$X_0Y_0 \text{ on } \pi_1$$

$$X_0Z_0 \text{ on } \pi_2$$

$$Y_0Z_0 \text{ on } \pi_3$$

Let us project orthogonally the three coordinate axes on the projection plane π , as shown in part c) of the figure. In part d) the projection of the origin is O_p , that of the x -axis is O_px_0 , and so on.

In the right-angled triangle OO_pX_0 let the coordinate along the side $\overline{OX_0}$ be x , and the coordinate along the side $\overline{O_pX_0}$ be x' . In other words, while x is the coordinate in the actual world system, x' is the coordinate in the system that we see in the axonometric projection. We can write

$$x' = x \cos \widehat{O_pX_0O} = x \sin \widehat{O_pOX_0} \quad (6.1)$$

The ratio x'/x is the scale at which lengths parallel to the x -axis are projected on the plane Δ . Let us note it by S_x . Similarly to Equation 6.1 we can write from the right-angled triangles OO_pY_0 and OO_pZ_0

$$y' = y \cos \widehat{O_pY_0O} = y \sin \widehat{O_pOY_0}$$

$$z' = z \cos \widehat{O_pZ_0O} = z \sin \widehat{O_pOZ_0}$$

We also define the scales at which lengths parallel to the y -axis and the z -axis are projected on the Δ plane as

$$\frac{y'}{y} = S_y, \quad \frac{z'}{z} = S_z$$

The sum of the squares of the three scales is

$$\begin{aligned} S_x^2 + S_y^2 + S_z^2 &= \sin^2 O_p \widehat{OX}_0 + \sin^2 O_p \widehat{OY}_0 + \sin^2 O_p \widehat{OZ}_0 \\ &= 3 - (\cos^2 O_p \widehat{OX}_0 + \cos^2 O_p \widehat{OY}_0 + \cos^2 O_p \widehat{OZ}_0) \end{aligned} \quad (6.2)$$

Now, $O_p \widehat{OX}$, $O_p \widehat{OY}$ and $O_p \widehat{OZ}$ are the *direction angles* of OO_p , and their cosines are the *direction cosines* of the line that contains OO_p (see, for example, Fuller and Tarwater, 1992, pp. 330-1). The sum of the squares of the direction cosines of a given line equals 1. This result is easy to prove.

For example, in our particular case, if we consider the projections $\overline{OO_x}$, $\overline{OO_y}$ and $\overline{OO_z}$ of the segment OO_p on the coordinate planes

$$\begin{aligned} \overline{OO_x} &= \overline{OO_p} \cos O_p \widehat{OX}_0 \\ \overline{OO_y} &= \overline{OO_p} \cos O_p \widehat{OY}_0 \\ \overline{OO_z} &= \overline{OO_p} \cos O_p \widehat{OZ}_0 \end{aligned} \quad (6.3)$$

Squaring both sides of the above equations and adding them side by side we obtain

$$\overline{OO_x}^2 + \overline{OO_y}^2 + \overline{OO_z}^2 = \overline{OO_p}^2 (\cos^2 O_p \widehat{OX}_0 + \cos^2 O_p \widehat{OY}_0 + \cos^2 O_p \widehat{OZ}_0) \quad (6.4)$$

On the other hand, by Pitagora's theorem in three dimensions

$$\overline{OO_x}^2 + \overline{OO_y}^2 + \overline{OO_z}^2 = \overline{OO_p}^2 \quad (6.5)$$

By combining Equations 6.3, 6.4 and 6.5 we obtain

$$S_x^2 + S_y^2 + S_z^2 = 2 \quad (6.6)$$

6.4 Isometry

Equation 6.6 allows us an infinity of choices. The simplest one is

$$S_x = S_y = S_z = \sqrt{2/3} \quad (6.7)$$

We call this case **isometric projection**.

6.4.1 More about the definition

6.4.2 Practical rules

6.5 Dimetry

6.5.1 Definition

In the **dimetric projection** two scales are equal. In the usual choices one of these two scales is S_z .

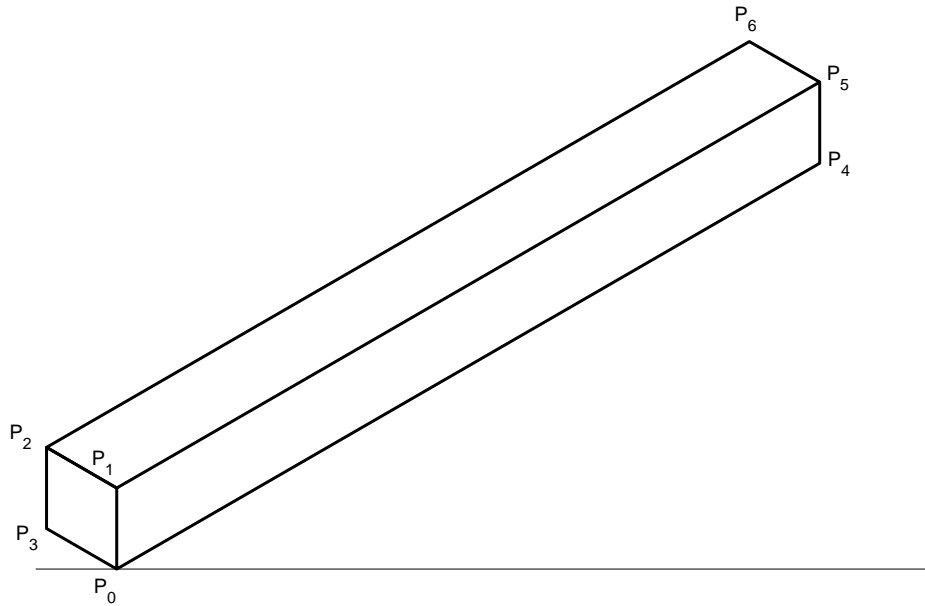


Figure 6.4: The isometry of a slender body

6.5.2 When is dimetry preferable to isometry

Figure 6.4 shows an isometric view of a parallelepiped whose breadth and height are equal, but the length is ten times larger. Looking at this figure we can have the sensation that the lines that run in the direction of the larger dimension are diverging towards right. Checking the figure with more attention reveals that those lines are parallel. The cause of our illusion resides in our visual experience. As explained in Chapter 2, the images we see are produced by central projection. In this projection, although the edges $\overline{P_0P_1}$, $\overline{P_1P_2}$, $\overline{P_2P_3}$, $\overline{P_3P_0}$, $\overline{P_4P_5}$ and $\overline{P_5P_6}$ are all equal, the projections of the edges $\overline{P_4P_5}$ and $\overline{P_5P_6}$ are shorter. This is a

consequence of the *foreshortening effect*.

As compared to the central projection, in isometry all the above-mentioned edges appear as equal. This contradicts our expectations and we have the feeling that the edge $\overline{P_4P_5}$ is longer than the edge $\overline{P_0P_1}$, and the edge $\overline{P_5P_6}$ is longer than the edge $\overline{P_1P_2}$. We may generalize and conclude that the isometric projection of objects having one dimension much larger than the others can produce an unpleasant optical illusion. To correct this effect we must use another axonometric projection and the dimetric projection is an adequate alternative.

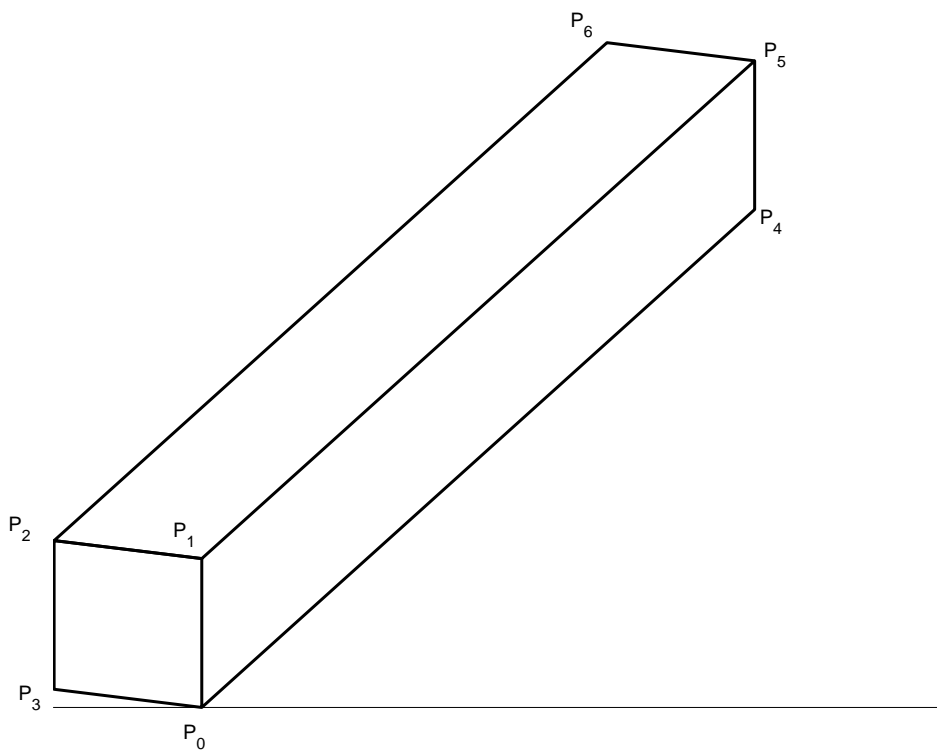


Figure 6.5: The dimetry of a slender body

6.6 Trimetry

In **trimetric projections** all three scales are different, that is

$$S_x \neq S_y \neq S_z \quad (6.8)$$

6.7 Other axonometries

6.8 Examples

Example 1 - Sketch corresponding to an axonometric projection

Figure 6.6 shows the traces of the projection plane, Δ , that appears in Figure 6.3. Thus, YX is the horizontal projection of the trace of Δ on π_1 , and XZ is the frontal projection of the trace of Δ on π_2 . Obviously, the frontal projection of the trace on π_1 and the horizontal projection of the trace on π_2 coincide with the ground line that contains the segment OX . In Figure 6.6 O_p is the projection of the origin, O , on Δ . Let us find the projections of this point in the sketch shown in Figure 6.6. To do so, we first find the projections of the perpendicular on Δ that passes through the origin, O . Next, we must find the point in which this perpendicular pierces the projection plane Δ .

To draw the perpendicular from O on Δ we use the theorem that says that a right angle is projected on a plane as a right angle if one of its sides is parallel to that plane. The trace XY is parallel to the projection plane π_1

because it belongs to it. Thus, the horizontal projection $\overline{OH'}$ of the perpendicular from O on Δ is perpendicular to the trace XY . Similarly, the trace XZ is parallel to the projection plane π_2 because it belongs to it. Then, the frontal projection $\overline{OF''}$ of the perpendicular from O on Δ is perpendicular to the trace XZ .

It remains now to find the point O_p in which the perpendicular from O on Δ intersects this plane. Let us consider the plane Γ that projects O_p on π_1 . This plane intersects the trace XY in the point H' , whose frontal projection is H'' . As the line XY belongs to Δ , the point H also belongs to Δ . But, by construction, the point H also belongs to Γ . The point Z belongs to Δ because it lies on a trace of its plane. But the point Z belongs also to Γ because the line OZ is perpendicular to π_1 , while the plane Γ is also perpendicular to π_1 and contains the origin O . We conclude that the line HZ is the intersection of the planes Δ and Γ and, therefore, any point of it belongs to Δ . On the other hand, as both lines OO_p and OH belong to the same plane Γ they intersect. Thus we find the projection O_p'' as intersection of $\overline{H''Z}$ and $\overline{OF''}$, and afterwards the projection O_p' by drawing through O_p'' the perpendicular on the ground line and intersecting it with $\overline{OH'}$.

6.9 Summary

Definitions regarding axonometric representations can be found in ISO 5456-3:1996(E). A splendidly illustrated book on axonometry is that of

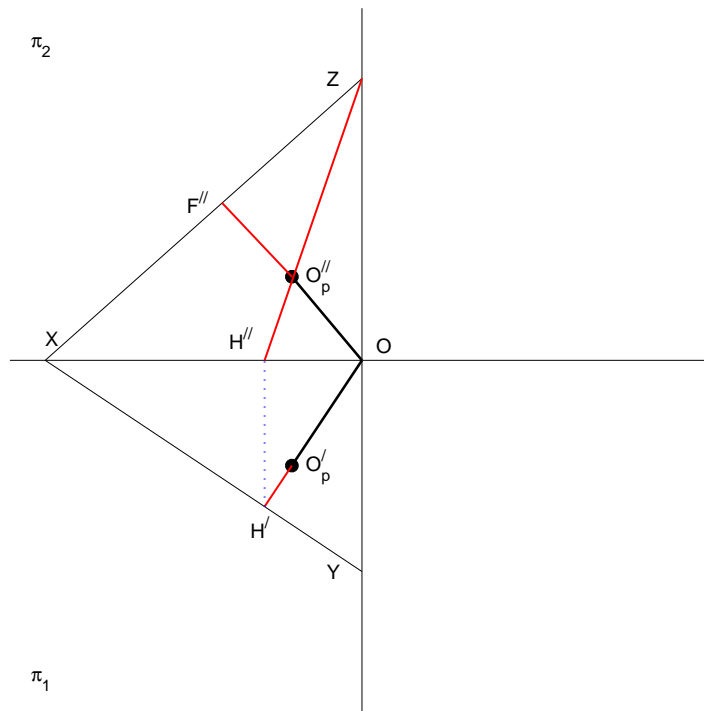


Figure 6.6: Sketch corresponding to an axonometric projection

Pavanelli, Miliani and Marchesini (1999).

6.10 Exercises

Exercise 1 - Vector normal to the projection plane

Referring to the Appendix and to Figure 6.3, show that the vector product $\overrightarrow{Y_0Z_0} \times \overrightarrow{Z_0X_0}$ yields the same vector normal to the projection plane as Equation 6.12.

Exercise 2 - The sketch of the perpendicular on the projection plane

In Figure 6.6 find the point O_p by using the plane that projects O_p on π_2 . Verify that you find the same point as in Example 1. Next, build the projection on π_3 and verify that OP''' is perpendicular on the trace YZ of Δ on π_3 .

6.11 Appendix - Origin of coordinates in axonometric projection

In Figure 6.3 the projection plane intercepts the x -axis at X , the y -axis at Y , and the z -axis at Z . It is easy to verify that the equation of the projection plane is

$$\frac{x}{X} + \frac{y}{Y} + \frac{z}{Z} = 1 \quad (6.9)$$

Let us find now the equation of a vector normal to the projection plane. To do this we need the expressions of two vectors that belong to the projection plane. Thus, from

$$\begin{aligned} \overrightarrow{OX_0} + \overrightarrow{X_0Y_0} &= \overrightarrow{OY_0} \\ \overrightarrow{OY_0} + \overrightarrow{Y_0Z_0} &= \overrightarrow{OZ_0} \end{aligned} \quad (6.10)$$

we deduce

$$\overrightarrow{X_0Y_0} = -X\mathbf{i} + Y\mathbf{j} \quad (6.11)$$

$$\overrightarrow{Y_0Z_0} = -Y\mathbf{j} + Z\mathbf{k}$$

The vector normal to the projection plane is obtained from the vector product

$$\mathbf{N} = \overrightarrow{X_0Y_0} \times \overrightarrow{Y_0Z_0} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -X & Y & 0 \\ 0 & Y & 0 \\ 0 & -Y & Z \end{vmatrix} = YZ\mathbf{i} + XZ\mathbf{j} + XY\mathbf{k} \quad (6.12)$$

We want now to find the equations of the line that passes through the origin, $O(0, 0, 0)$, and is parallel to the vector \mathbf{N} . This is the line that projects the origin, O , to O_p , the origin on the projection plane. Given a point P on the line we are seeking, there is a scalar t such that

$$\overrightarrow{OP} = t\mathbf{N}$$

or

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = YZt\mathbf{i} + XZt\mathbf{j} + XYt\mathbf{k} \quad (6.13)$$

Equating the coefficients of \mathbf{i} , \mathbf{j} , and \mathbf{k} extracting t and equating the three expressions that yield it, we obtain

$$\frac{x}{YZ} = \frac{y}{XZ} = \frac{z}{XY} \quad (6.14)$$

These are the *symmetric equations* of the line (see Fuller and Tarwater, 1992, pp. 327-8). In fact, we have here three equations that define three

planes perpendicular to the coordinate planes. Their intersection is the line that passes through the origin and is normal to the projection plane, Δ .

The intersection of this line with the projection plane defined by

Equation 6.9 is the projection, O_p , of the origin on the projection plane, Δ .

Solving the system defined by Equations 6.9 and 6.14 we obtain the

coordinates of the point O_p

$$\begin{aligned} x &= \frac{1}{\frac{1}{X} + \frac{X}{Y^2} + \frac{X}{Z^2}} \\ y &= \frac{1}{\frac{Y}{X^2} + \frac{1}{Y} + \frac{Y}{Z^2}} \\ z &= \frac{1}{\frac{Y}{X^2} + \frac{Z}{Y^2} + \frac{1}{Z}} \end{aligned} \tag{6.15}$$

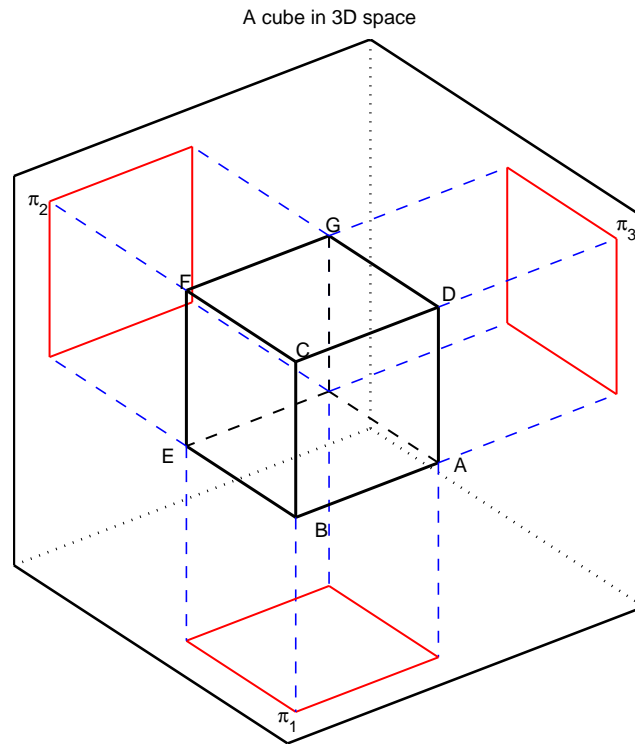


Figure 7.1: A cube in 3D space

Chapter 7

Cubes and prisms

7.1 The cube

7.2 Euler's formula

7.3 The projections of a cube

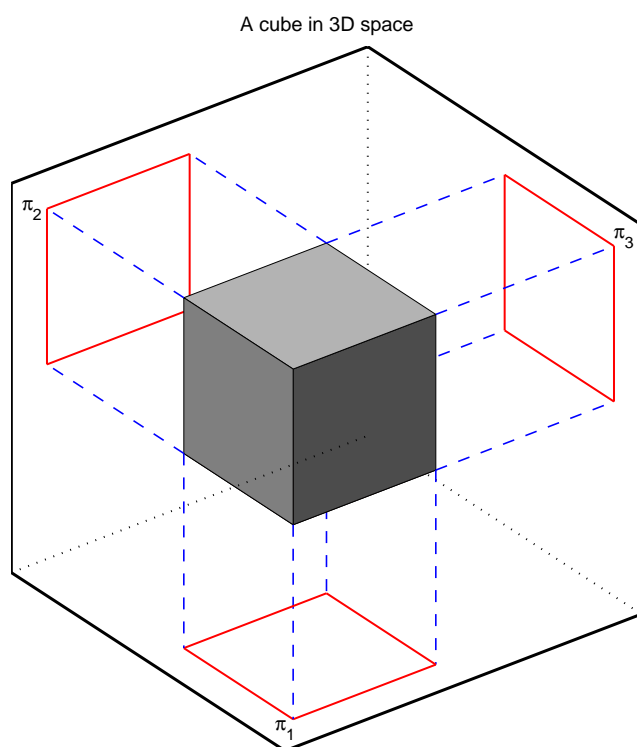


Figure 7.2: The same cube shaded

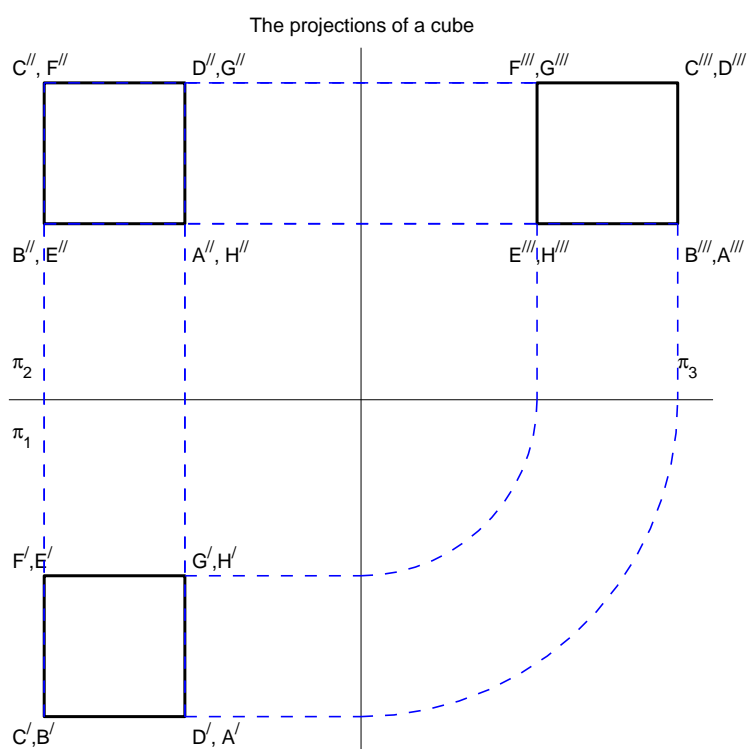


Figure 7.3: The projections of a cube

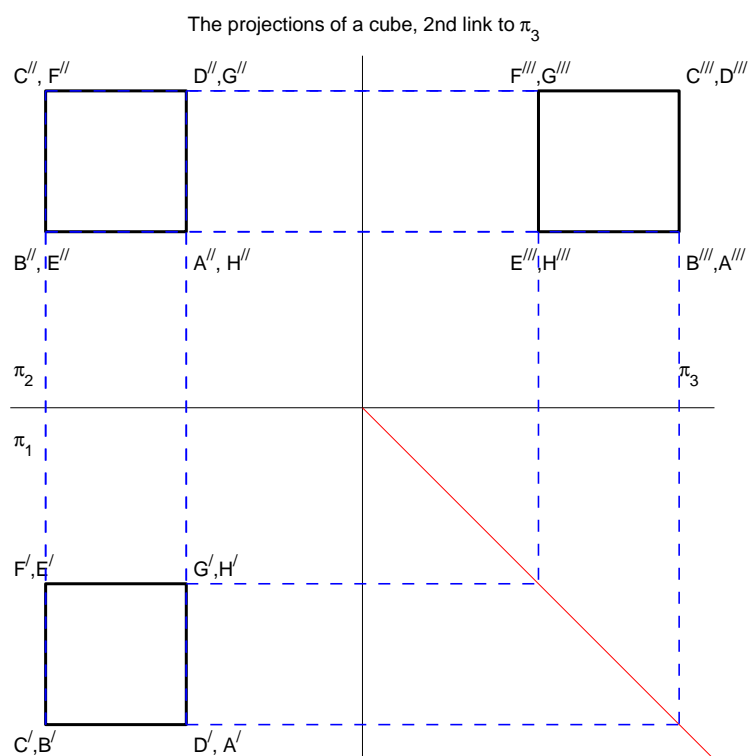


Figure 7.4: The projections of a cube, second link to π_3 .

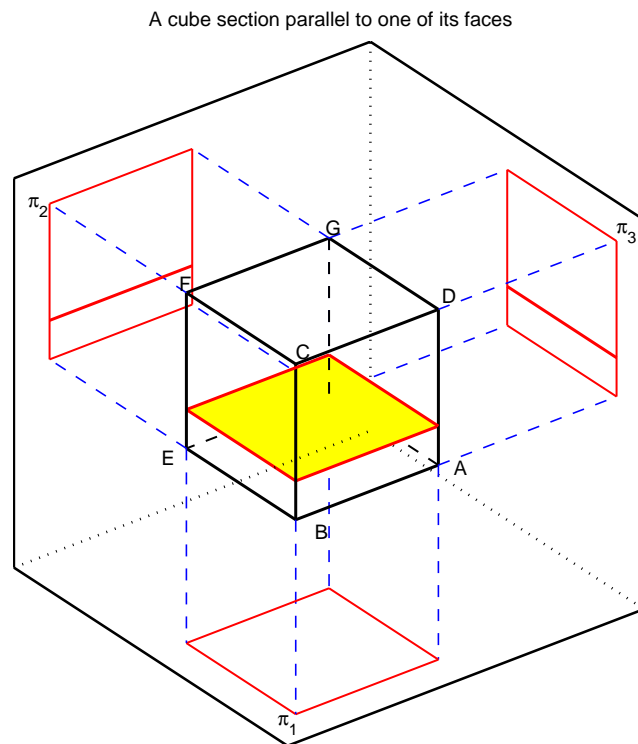


Figure 7.5: Cube section parallel to one of its faces

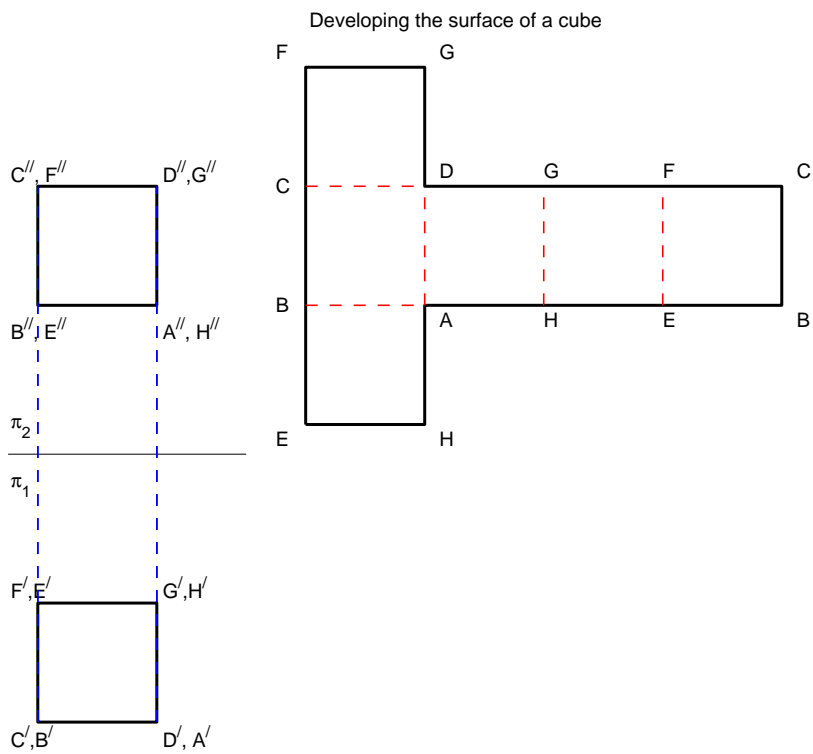


Figure 7.6: The development of a cube

Developing the surface of a cube, another solution

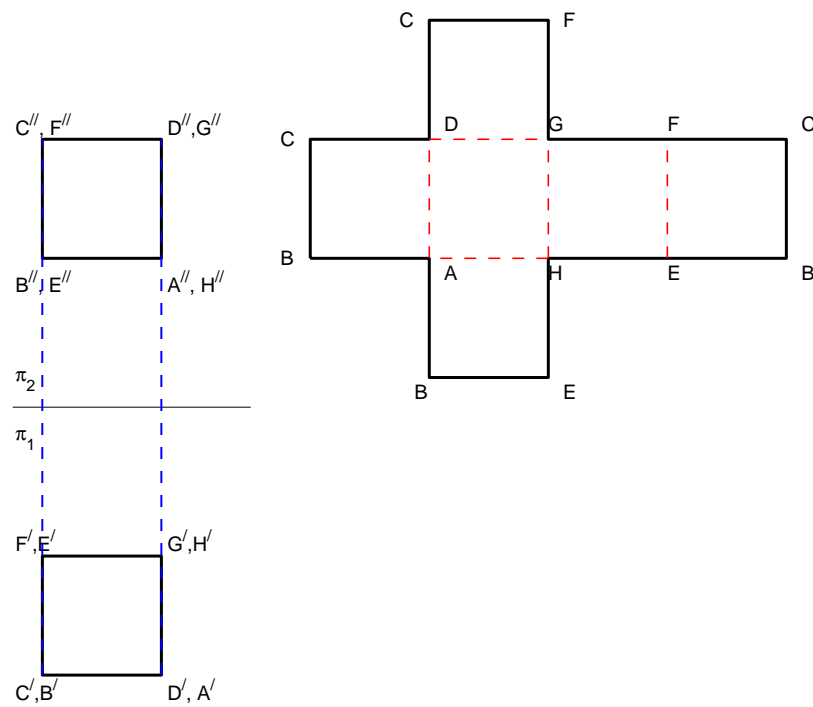


Figure 7.7: The development of a cube, another solution

Chapter 8

The pyramid

8.1 Definition

Defining a pyramid

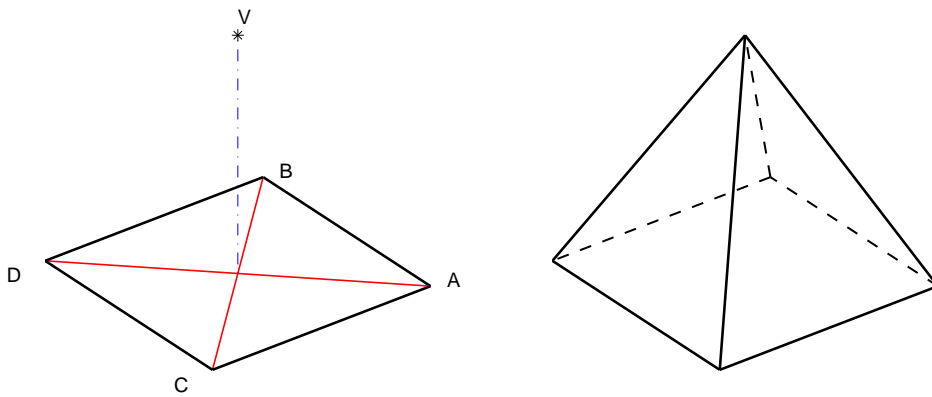
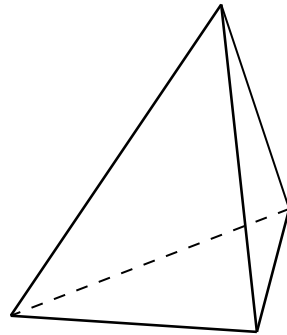


Figure 8.1: Defining a pyramid

Pyramid with triangular base



Pyramid with hexagonal base

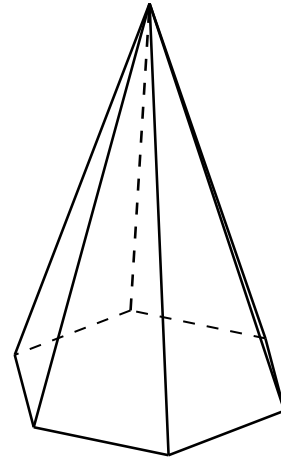


Figure 8.2: Defining a pyramid

8.2 Projections

8.3 Sections

8.3.1 Section parallel to base

8.3.2 Oblique sections

8.4 Developing a pyramid

8.4.1 Regular pyramid

8.4.2 Truncated pyramid

8.5 Exercises

Pyramid sections parallel to their bases

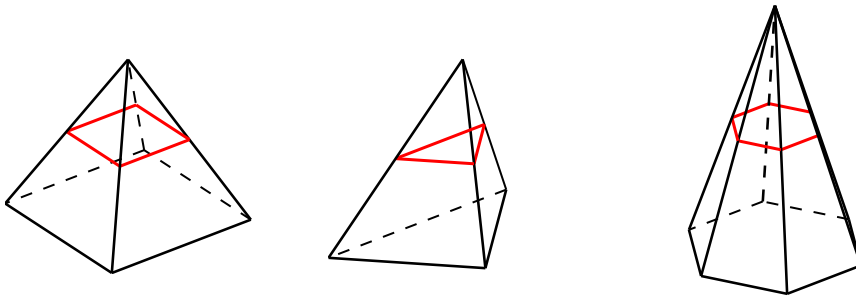


Figure 8.3: Pyramid, sections parallel to base

Pyramid truncated parallel to their bases

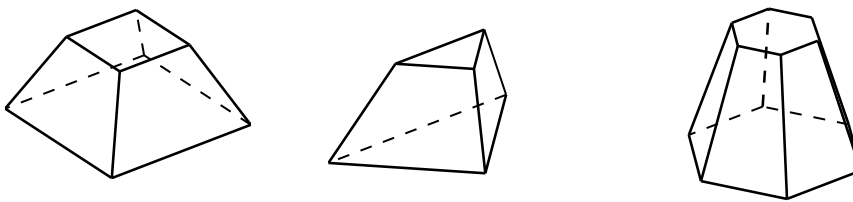


Figure 8.4: Pyramid, sections parallel to base

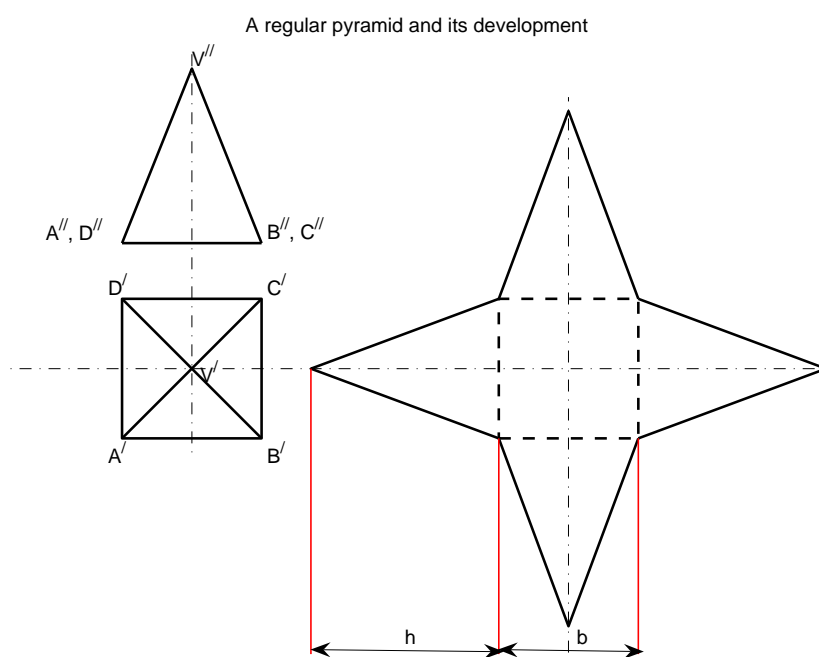


Figure 8.5: Developing a pyramid

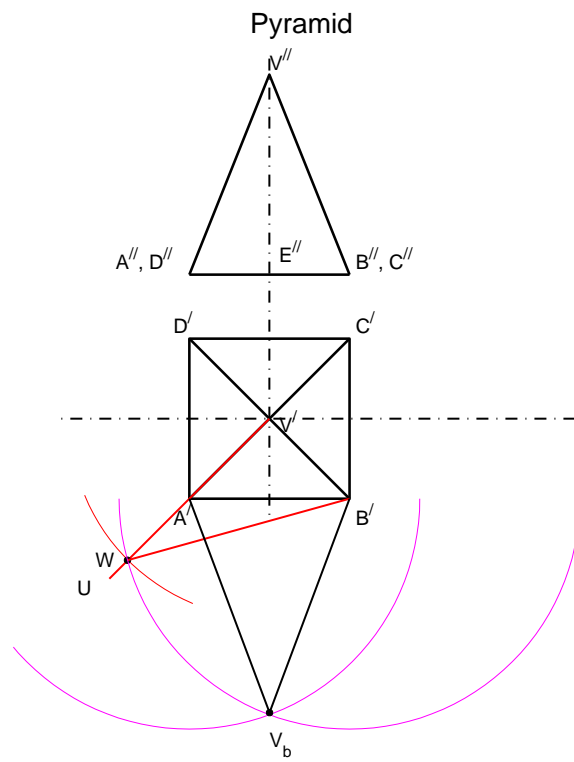


Figure 8.6: Finding the true size of a face

Pyramid development, a second solution

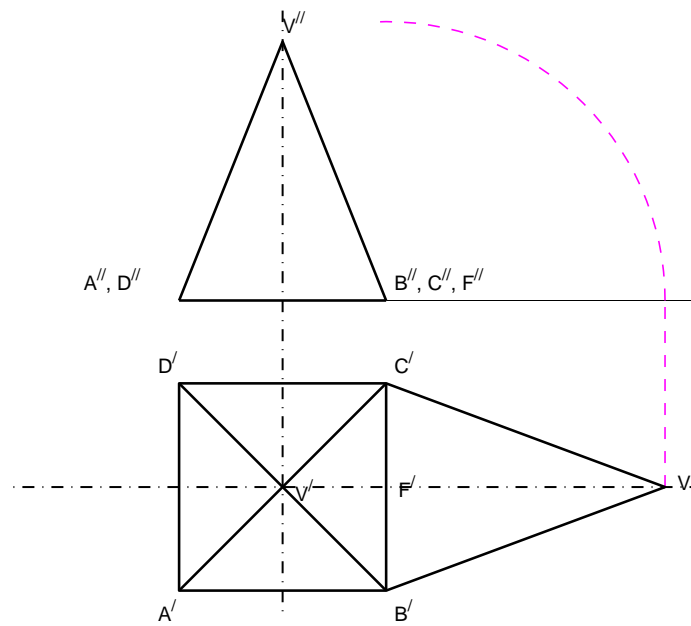


Figure 8.7: Finding the true size of a face, second solution

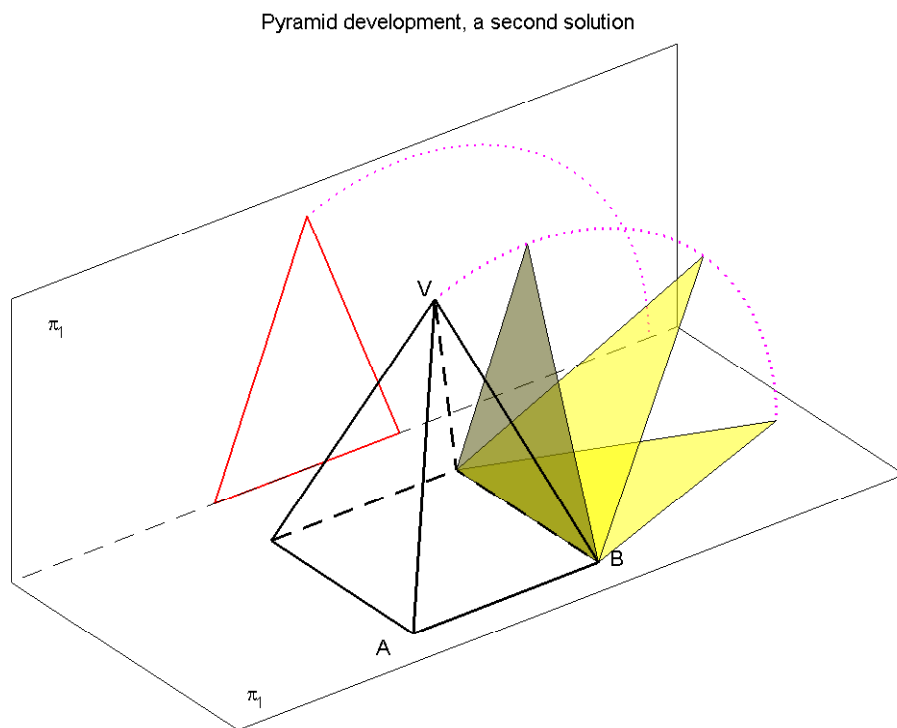


Figure 8.8: Finding the true size of a face, second solution

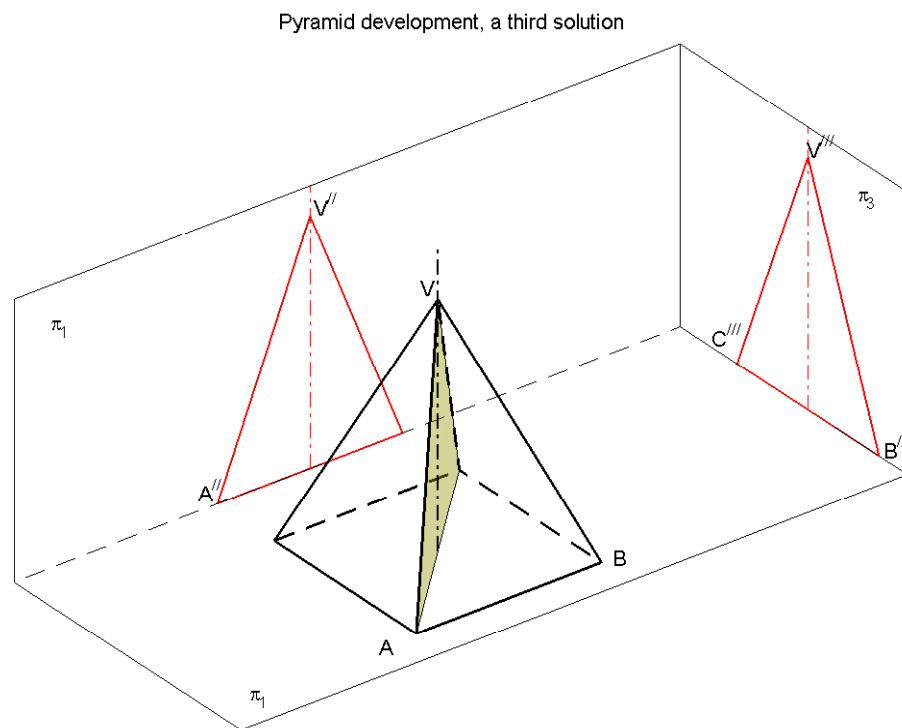


Figure 8.9: Finding the true size of a face, third solution

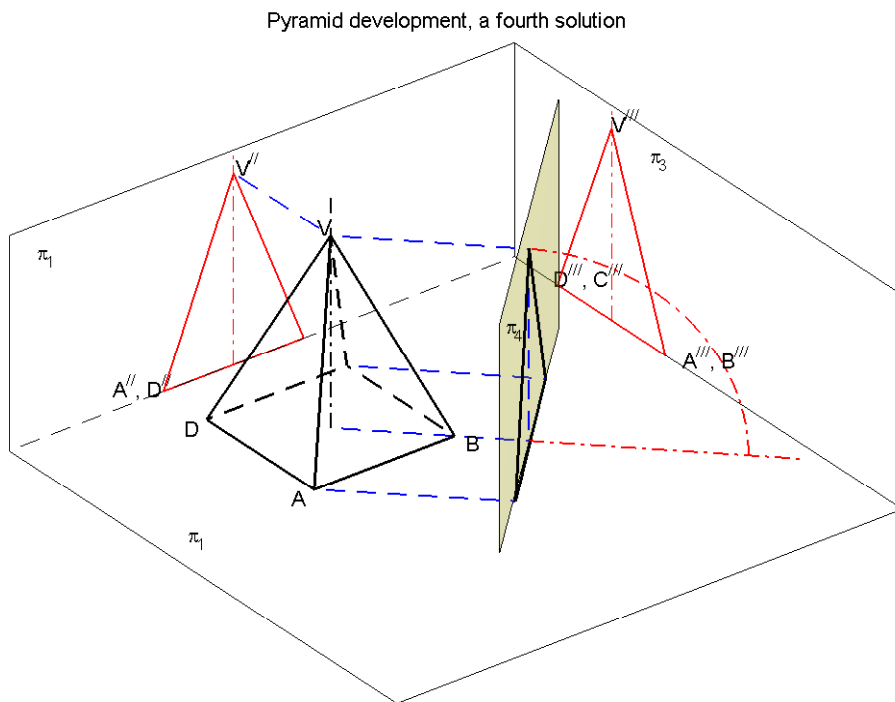


Figure 8.10: Finding the true size of a face, fourth solution

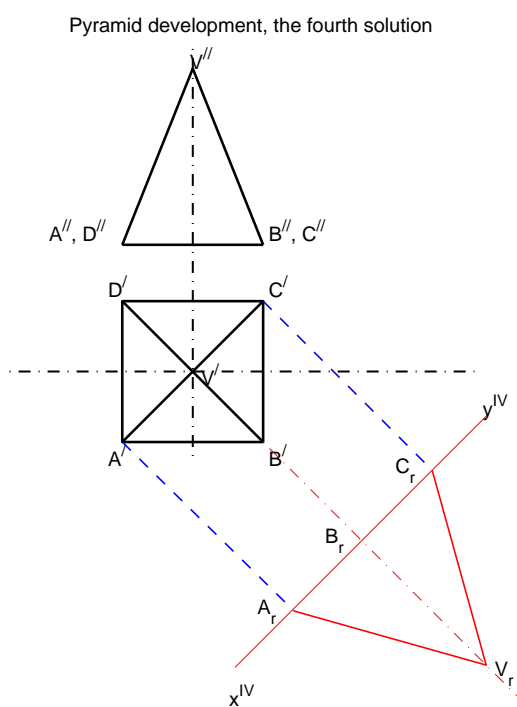


Figure 8.11: Finding the true size of a face, fourth solution

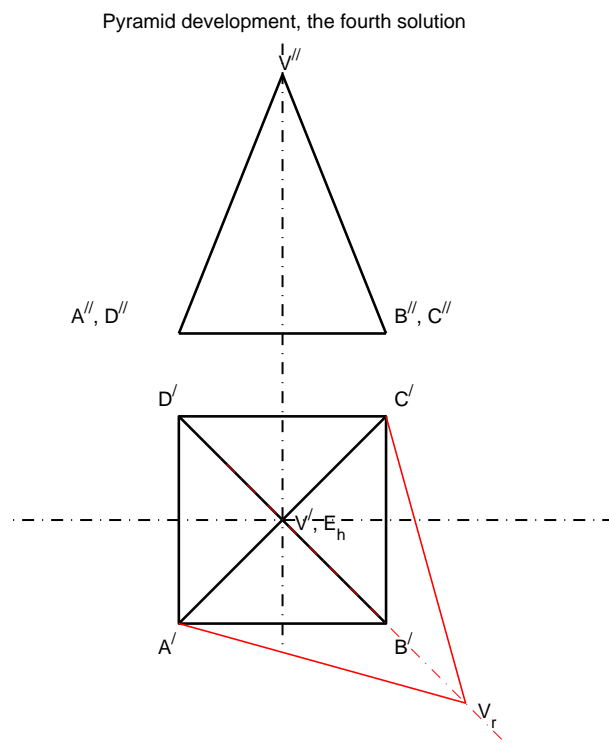


Figure 8.12: Finding the true size of a face, fourth solution

Circular cylinder

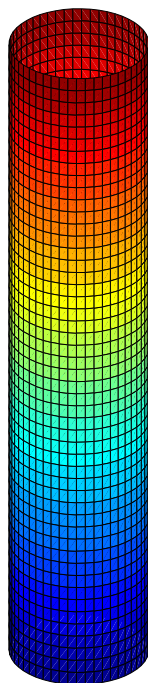


Figure 9.1: Defining a cylinder

Chapter 9

The cylinder

9.1 Definition

9.2 Point on the surface of a circular, right cylinder

9.2.1 The geometric aspect

Chapter 10

The helix

10.1 Introduction

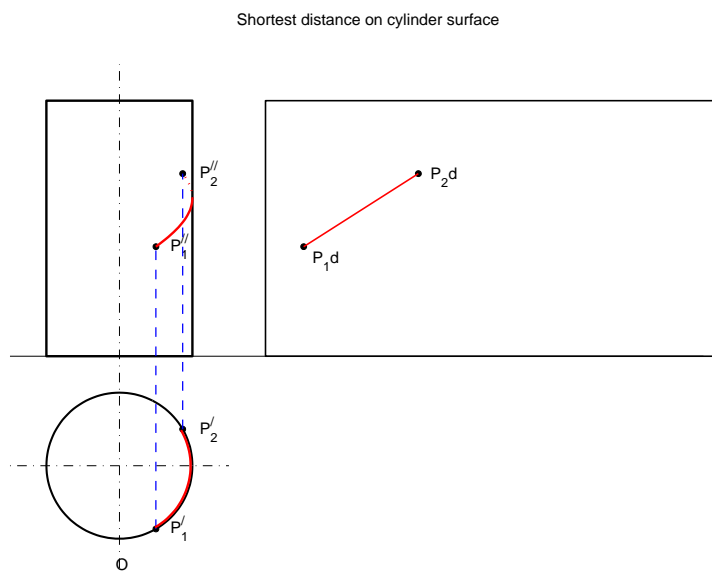


Figure 10.1: The shortest path on a cylinder surface

Figure 10.1 shows two projections of a cylinder and of two points, P_1 and P_2 , that lie on the cylinder surface. What is the curve along which we can travel the shortest distance between the two points? To solve the problem we can cut the cylinder, for example along the generator which passes through O , and develop the surface. The developed surface is shown in the right-hand part of Figure 10.1. On the developed surface the points P_1 and P_2 appear as P_1d and P_2d . The obvious solution is obtained by joining P_1d and P_2d by a straight line and bending back the developed surface into the given cylinder. The shortest path between P_1 and P_2 appears in the frontal projection as the red curve that joins P_1'' and P_2'' . The horizontal projection coincides with the arc of the base circle limited by the points P_1' and P_2' . The curve we are looking for is a **helix**.

The helix has important applications in technology and science. The surfaces of screw threads and of many springs are generated by points lying on helices. So are parts of the surfaces of screw propellers and fans. In biology, the DNA model is based on two helices.

10.2 The definition of the helix

For the definition of the helix we refer to Figure 10.2. Given the axis Oz , the *helix* is the curve described by a point, P , that rotates around the axis Oz at constant radius, r , and for each complete rotation advances in the direction Oz by a constant distance, p , called **pitch**. From the above

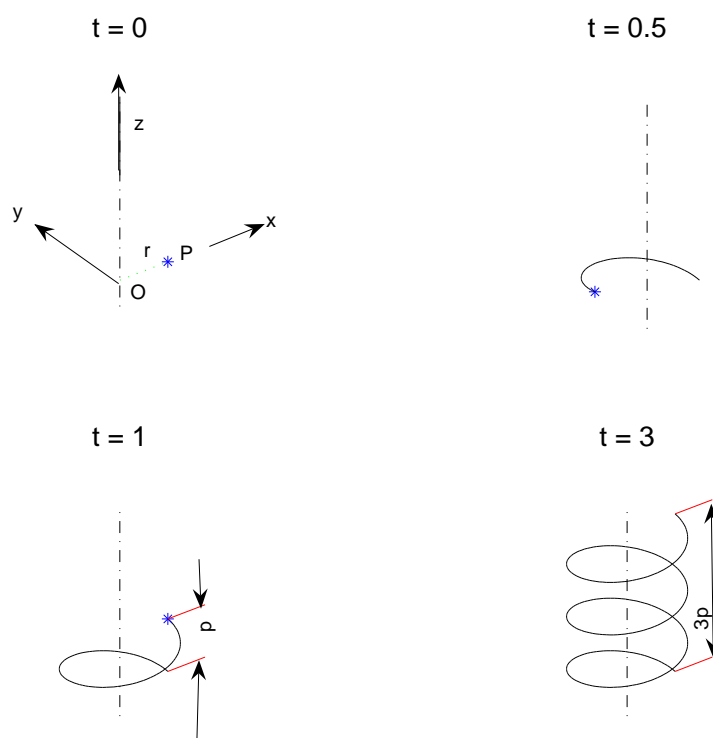


Figure 10.2: Defining a helix

definition we deduce the parametric equations of the helix

$$x = r \cos 2\pi t \quad (10.1)$$

$$y = r \sin 2\pi t$$

$$z = tp$$

where t is a parameter that for one complete rotation runs from 0 to 1. In

Figure 10.2 we see various stages of the generation of the helix.

If the generating point rotates clockwise and advances in the positive z

direction, we say that the helix is *right-handed*, otherwise we say that the helix is *left-handed*. Most screws used in technology are right-handed.

A few translations of the terms introduced in this section are

	helix	pitch	right-handed	left-handed
French	hélice	pas	à droite	à gauche
Italian	elica	passo	destrorso	sinistrorso
German	Schraubenlinie	Ganghöhe	rechtsgewundene	linksgewundene

10.3 The projections of the helix

To draw the projections of a helix we proceed as in Figure 10.3. We first sketch the projections of the cylindrical surface on which the helix is defined. Next, in the horizontal projection (that is, on π_1) we divide the base circle into a number of equal arcs, in this example twelve. In the frontal projection (that is, on π_2) and side projection (on π_3) we divide the pitch height into the same number of equal segments. For good order we number the divisions. In this example the divisions are numbered from 0 to 12, turning on the base circle in the direct trigonometric sense. In Figure 10.3 corresponding points of the three projections are connected by blue, pointed lines.

Our helix starts from the first division marked 0. Therefore, in the frontal projection the first point is found at the intersection of the vertical, blue line rising from 0 in the π_1 -plane and the horizontal, blue line starting

from the first division (also marked 0) of the pitch in the π_2 -plane. From here on, as the point rotates one twelfth of a circle, it also progresses in the z direction one twelfth of the pitch height. Therefore, the second point in the frontal projection lies at the intersection of the vertical, blue line rising from 1 in the π_1 plane and the horizontal, blue line corresponding to the second pitch division (also marked 1). We proceed so until the thirteenth point, corresponding to the division marked 12. The side projection (that is, on the π_3 plane) is obtained in the obvious way from the horizontal and the frontal projections.

Connecting the points found above we obtain curves that must seem familiar to us: the graphs of a cosine and of a sine function. Below we show that this is, indeed, the case.

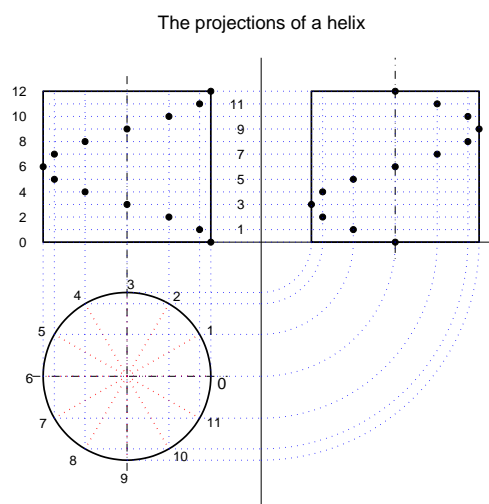


Figure 10.3: The projections of a helix

Figure 10.4 shows three projections of the helix defined by Equations 10.2

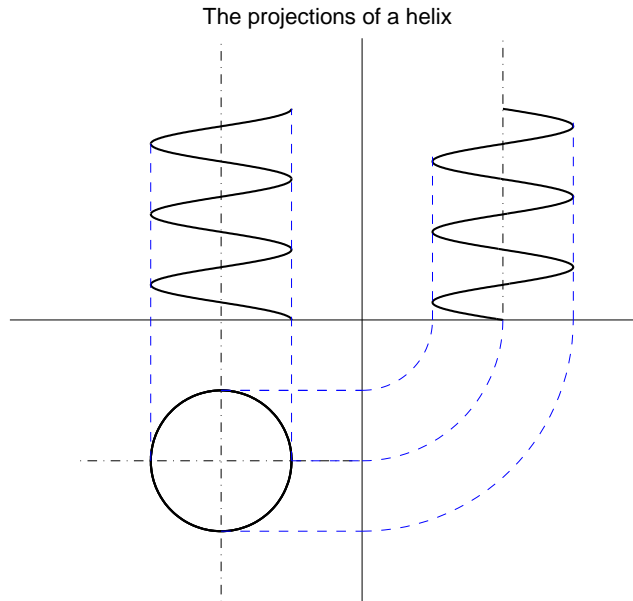


Figure 10.4: The projections of a helix

for $t = 0 \dots 3$. The horizontal projection is given by

$$x = r \cos 2\pi t \tag{10.2}$$

$$y = r \sin 2\pi t \tag{10.3}$$

Dividing both sides of the first equation by $\cos 2\pi t$, both sides of the second equation by $\sin 2\pi t$, raising the resulting equations to the second power and adding them we obtain

$$x^2 + y^2 = r^2 \tag{10.4}$$

This is the equation of a circle with the centre in the origin of the coordinates and the radius r .

The frontal projection, that is the projection on π_2 , is given by the equations

$$\begin{aligned}x &= r \cos 2\pi t & (10.5) \\z &= tp\end{aligned}$$

Extracting the value of t from the second equation and substituting into the first equation yields

$$x = r \cos \frac{2\pi}{p} z \quad (10.6)$$

As r is a constant, and $2\pi/p$ another one, we deduce that the frontal projection of the helix is a cosine line.

The side projection, that is the projection on π_3 , is given by the equations

$$\begin{aligned}y &= r \sin 2\pi t & (10.7) \\z &= tp\end{aligned}$$

From the above equations we obtain

$$y = r \sin \frac{2\pi}{p} z \quad (10.8)$$

We conclude that the side projection is a sine line.

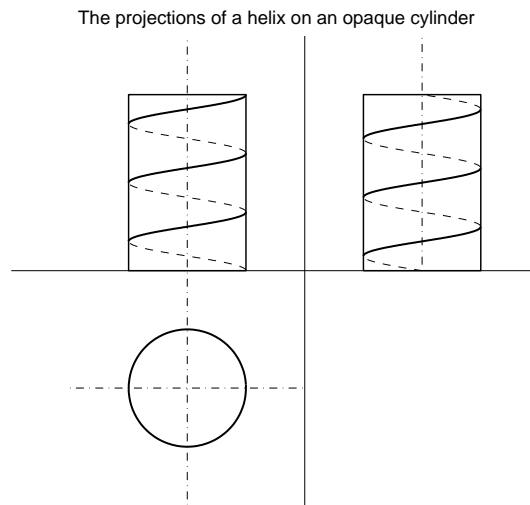


Figure 10.5: The projections of a helix

In Figure 10.4 we see only the helix. Figure 10.5 shows a cylinder with a helix on it. The cylinder is opaque so that part of the helix is visible and part hidden. It is easy to check that the helix is right-handed.

10.4 Developing the helix

To develop the helix we begin by developing the cylindrical surface on which the helix lies. Thus, in Figure 10.6, the developed cylindrical surface is shown at the right of the sketch of the cylinder with base radius r , and height equal to the pitch p . Next, we divide the perimeter of the base, in the horizontal projection, into a number of equal arcs, in this example 12. We divide the length of the development into the same number of segments. Further, we divide the height, both in the frontal projection and in the

Preparing to develop a helix

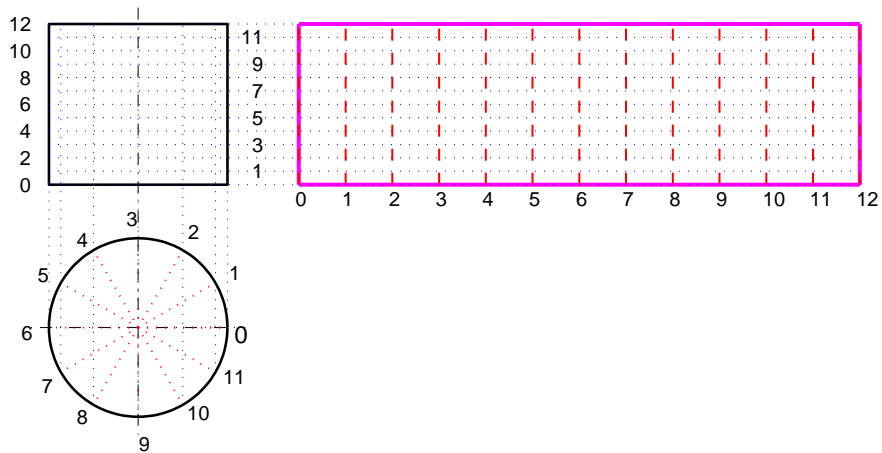


Figure 10.6: Preparing to develop the helix

developed surface, into the same number of segments. For good order we assign numbers to the various divisions. In this example we assign the numbers $0, 1, \dots, 12$ progressing in the direct trigonometric sense. If we consider the external surface of the surface, the numbers of the divisions of developed length are as shown in Figure 10.6.

The starting point corresponds to the division marked 0. Then, the first point of the developed helix is defined by the intersection of the horizontal, blue line marked 0 in the frontal projection, and the vertical, blue line marked 0 on the developed surface. As the point advances one division in the horizontal direction, it also rises one division in the vertical direction.

Developing a helix

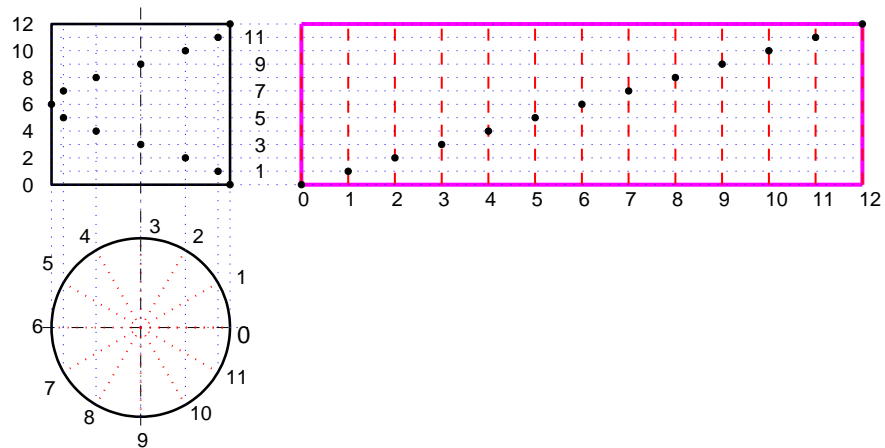


Figure 10.7: Developing the helix

Therefore, the second point of the developed helix is defined by the intersection of the horizontal, blue line marked 1 in the frontal projection, and the vertical, blue line marked 1 on the developed surface. Continuing in the same way we find the sequence of points shown in Figure 10.7. It is easy to check that all points lie on a straight line.

We can prove analytically that the developed helix is, indeed, a straight line. To do this we assume that the projections of the helix are described in the same system of coordinates as in Figure 10.2 and in Equations 10.2, while the developed helix is represented in a system in which the horizontal

coordinate is ξ , and the vertical coordinate is η . We immediately see that

$$\eta = z = tp \tag{10.9}$$

The horizontal coordinate of a point on the developed helix is equal to the developed length of the arc of the horizontal projection of the same point. For a rotation angle equal to $2\pi t$, the arc length is $2\pi r t$, and

$$\xi = 2\pi r t \tag{10.10}$$

Combining Equations 10.9 and 10.10 we obtain

$$\eta = \frac{p}{2\pi r} \xi \tag{10.11}$$

that is, the equation of a straight line with the constant slope

$$\frac{d\eta}{d\xi} = \frac{p}{2\pi r} \tag{10.12}$$

The knowledge of this slope is important in applications such as *power screws* and *worm gears* and according to its value the mechanism can be reversible or not. Moreover, the efficiency of the mechanism depends on the slope value.

10.5 The tangent to the helix

10.6 Technical applications

Appendix A

The MATLAB code that produced the fourth plot in Figure 10.2 is

```
r = 1; % cylinder radius
p = 1; % helix pitch
subplot(2, 2, 4)
    plot3([ 0 0 ], [ 0, 0 ], [ -0.5*p, 3.5*p ], 'k-.')
    hold on
    t = 0: 0.01: 3;
    x = r*cos(2*pi*t);
    y = r*sin(2*pi*t);
    z = t*p;
    plot3(x, y, z, 'k-')
    axis([ -3*r 3*r -3*r 3*r -p 4*p ])
    axis equal, axis off
    Ht = title('t = 3');
    set(Ht, 'FontSize', 14)
    plot3([ r 2.8*r ], [ 0 0 ], [ 0 0 ], 'r-')
    plot3([ r 2.8*r ], [ 0 0 ], [ 3*p 3*p ], 'r-')
```

```
plot3([ 3*r 3*r ], [ 0 0 ], [ 0 1.5*p ], 'k:')  
text(2*r, 0, p/2, '3p', 'FontSize', 10, 'Rotation', 90, 'VerticalAlignment', 'B')
```

The arrows used in dimensioning are added by using the `Double Arrow` facility of the figure toolbar.

10.7 Exercises

Exercise 1 - Developed helix

Develop the helix shown in Figure 10.4, cut the developed rectangle leaving an area for gluing and bend it into a cylinder. Turn the cylinder around to experiment the sensation of an advancing screw thread. Check that this thread is right-handed.

Chapter 11

The cone and the conic sections

11.1 Definition

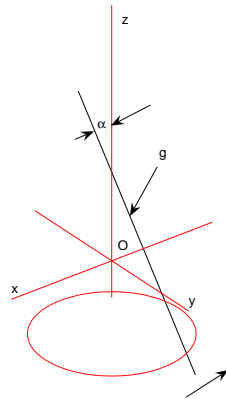


Figure 11.1: Generating a circular, right cone

An elliptic cone

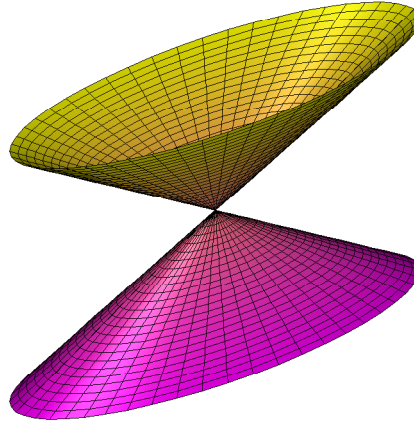


Figure 11.2: An elliptic, right cone

11.2 The projections of a cone

11.3 Point on conic section

11.3.1 Finding the horizontal projection using a section perpendicular to the cone axis

11.3.2 Finding the horizontal projection using a generator

11.4 Conic sections

11.4.1 The point

11.4.2 The circle

11.4.3 The ellipse

Sketch of circular, oblique cone

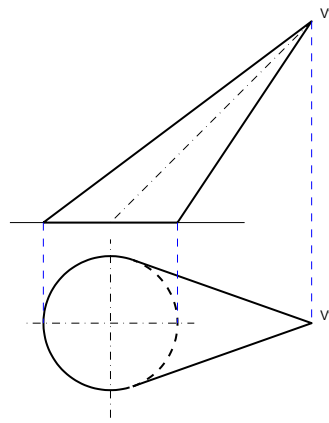


Figure 11.3: The sketch of an oblique cone

11.4.4 The parabola

11.4.5 The hyperbola

11.4.6 Straight-line sections

11.4.7 Dandelin's spheres

11.5 Developing a conic surface

11.5.1 The circular, right cone

11.5.2 The truncated circular, right cone

11.6 The cone as a two-dimensional manifold

11.7 The isometric view of the cone

11.8 Quadrics

11.9 Appendix A - The parabola

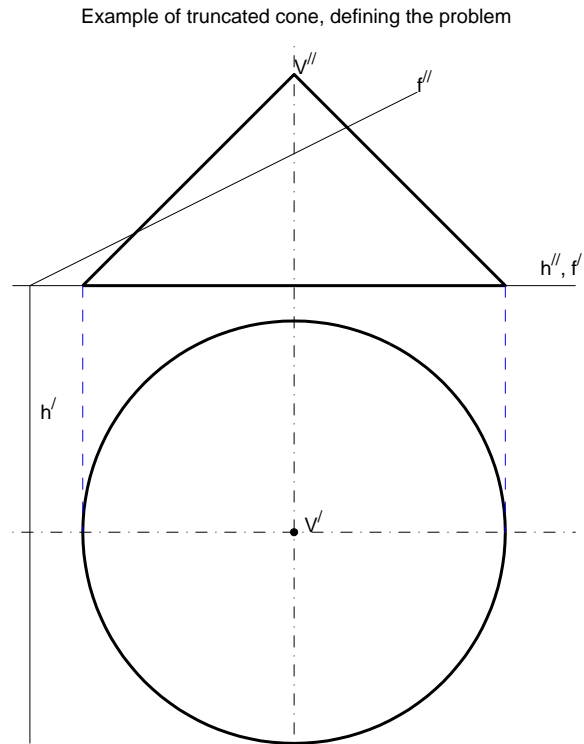


Figure 11.4: Truncating a circular, right cone

```
%VACUUM_BALLISTICS PLOTS TRAJECTORY IN VACUUM
```

```
% define parameters
```

```
V0    = 200;                % initial velocity, m/s
g      = 9.81;              % acceleration of gravity, m/s^2
alpha  = pi/4;              % initial angle
tmax   = 2*V0*sin(alpha)/g; % time until falling on horizont
adeg   = 180*alpha/pi;
t      = 0: 0.25: tmax;
```

Example of truncated cone, horizontal projection of intersection line

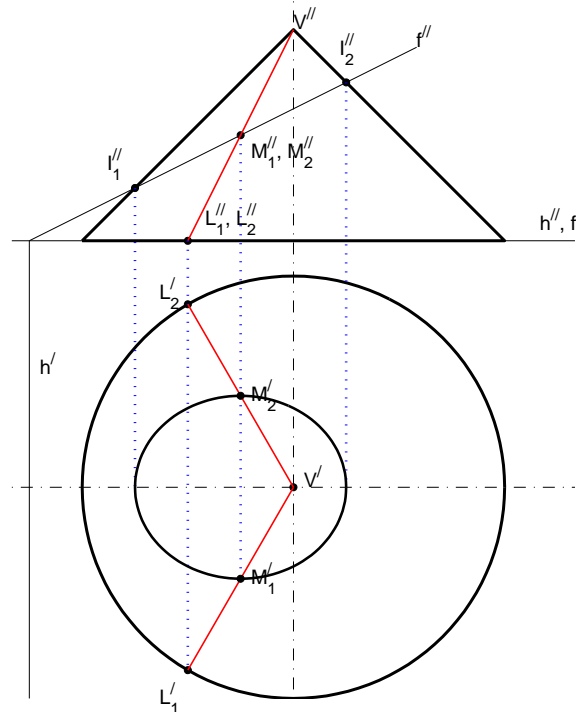


Figure 11.5: Horizontal projection of the intersection line

```

x      = V0*cos(alpha)*t;
y      = V0*sin(alpha)*t - g*t.^2/2;
plot(x, y, 'k-') axis equal, grid
title(['Vacuum ballistics, V_0 = ' num2str(V0) 'ms^{-1}, \alpha = ' num2str(ad)
xlabel('Horizontal distance, m')
ylabel('Height, m')

```

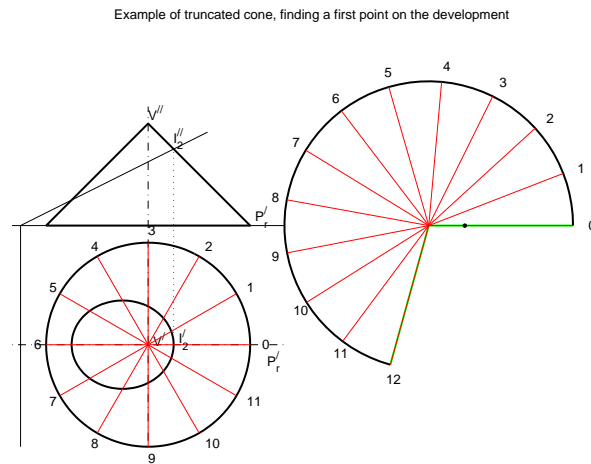


Figure 11.6: Developing a truncated cone

11.9.4 How to draw a parabola

11.10 Appendix B - The hyperbola

11.10.1 Equations

11.10.2 A few properties

11.10.3 Scientific and technical applications

11.10.4 How to draw a hyperbola

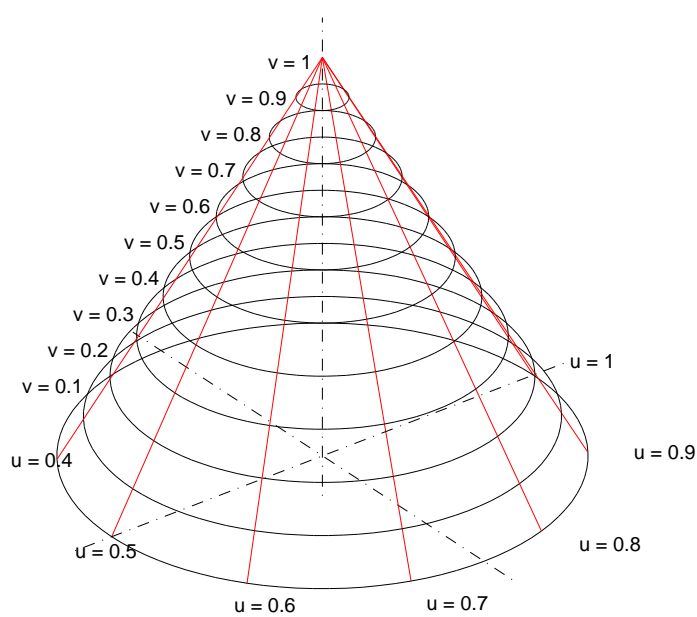


Figure 11.7: The cone as a two-dimensional surface

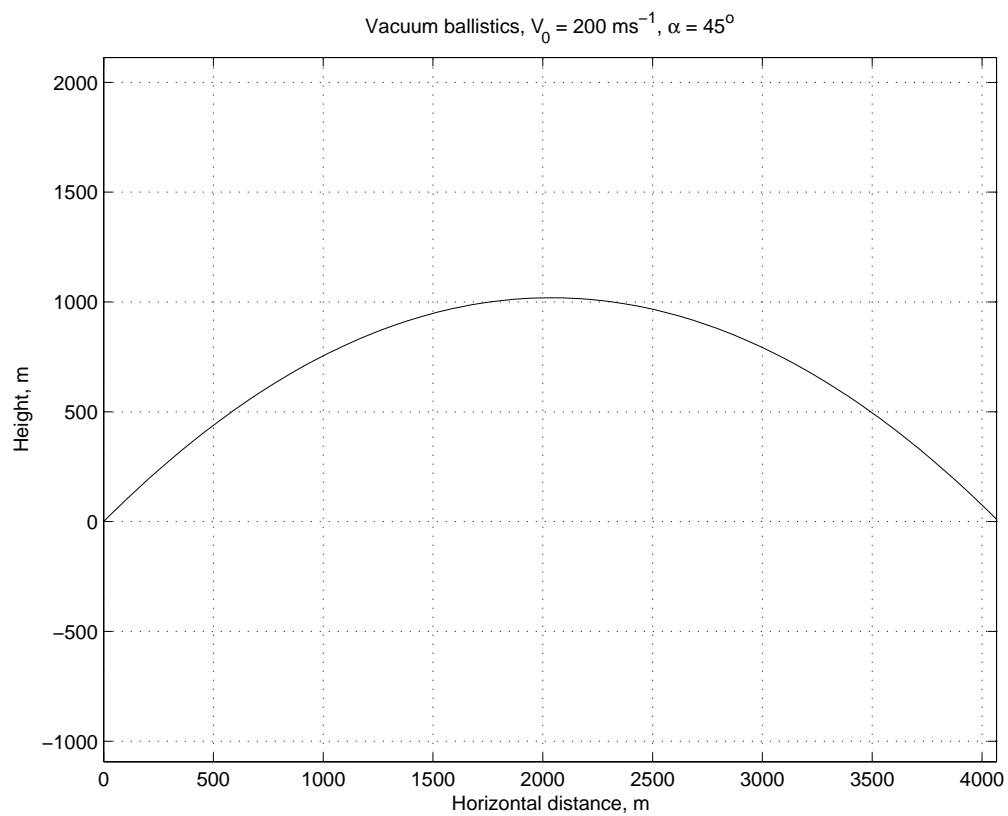


Figure 11.8: Ballistic trajectory in vacuum

Chapter 12

The sphere

12.1 Definition

12.2 Plane sections

12.2.1 Great circles

12.2.2 Small circles

12.3 Coordinates on spherical surface

12.4 The shortest path on a sphere

12.5 Straight line intersecting a sphere

12.6 The isometric view of the sphere

12.7 Geographic maps as projections

12.8 Exercises

Chapter 13

Axonometry II

Chapter 14

Complements

14.1 Curvature

14.1.1 Curvature of a plane curve

14.1.2 The curvatures of a surface

14.2 Ruled and developable surfaces

14.2.1 Ruled surfaces

14.2.2 Developable surfaces

14.3 More curves and surfaces

14.3.1 Parametric equations

14.3.2 Splines

14.3.3 Barycentric coordinates

14.3.4 Bézier curves

14.3.5 B-splines

Chapter 15

Advanced techniques

15.1 Edge view

15.1.1 Definition

15.1.2 Applications

15.2 Rotation about a horizontal axis (Rebating)

15.2.1 Rotating about an axis perpendicular on π_1

15.2.2 Rotating about an arbitrary horizontal axis

15.2.3 Applications

15.2.4 Theoretical significance

15.3 Exercises

Chapter 16

Stereo vision

16.1 The principles of stereo vision

16.2 Methods of viewing stereo pairs

16.3 The geometry of stereo vision

16.4 Algorithms for stereo vision

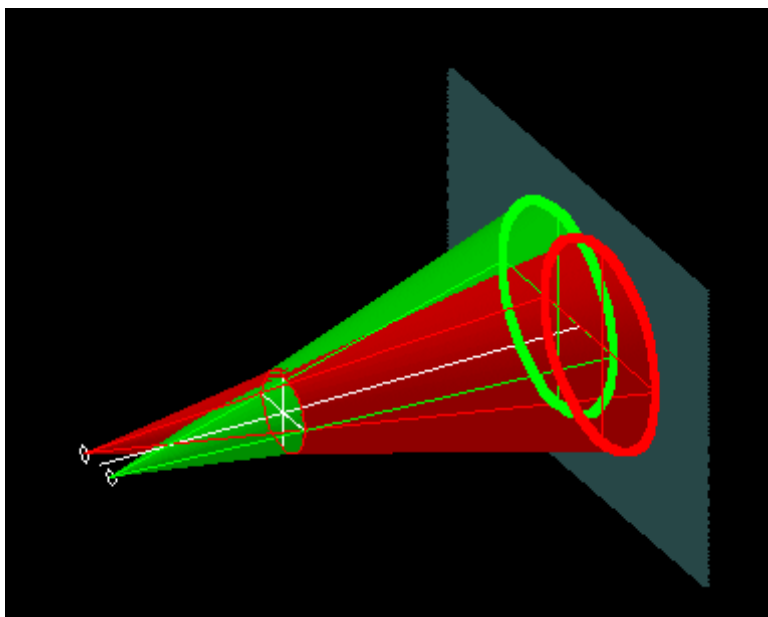


Figure 16.1: The principle of anaglyphs

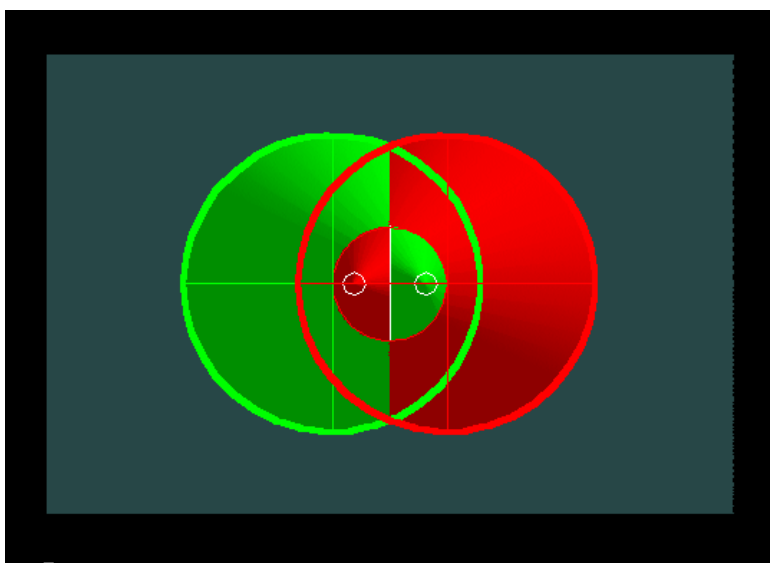


Figure 16.2: Stereoscopic projection of a ring in space

Bibliography

Abate, M. (1996), *Geometria*, Milano: McGraw Hill Libri Italia.

Anonymous (2002), *Technical drawings, Vol. 1, Technical drawings in general*, 4th edition, Genève: International Organization for Standardization.

Banchoff, T., and Wermer, J. (1983), *Linear algebra through geometry*, Berlin: Springer Verlag.

Bertoline, G.R., and Wiebe, E.N., (2003), *Technical graphics communications*, 3d edition, Boston: McGraw Hill.

Brannan, D.A., Esplen, M.F., and Gray, J.J. (1999), *Geometry*, Cambridge: Cambridge University Press.

Chirone, E. and Tornincasa, S. (2002), *Disegno tecnico industriale*, Vol. 1, Torino: Il Capitello.

Coolidge, J.L. (1963), *A history of geometrical methods*, New York: Dover Publications.

Delachet, A. (1964), *La géométrie projective*, Que sais-je 1103, Paris: Presses Universitaires de France.

Devlin, K. (2000), *The maths gene — Why everyone has it, but most people don't use it*, London: Phoenix.

Egerton, PA., and Hall, W.S. (1999), *Computer Graphics — Mathematical First Steps*, Harlow, England: Prentice Hall.

Flocon, A., and Taton, R. (1984), *La perspective*, Que sais-je 1050, Paris: Presses Universitaires de France.

Fuller, G., and Tarwater, D. (1992), *Analytic Geometry*, 7th edition, Reading, Massachusetts: Addison-Wesley Publishing Company.

ISO 5455-1979(E) (1979), *Technical drawings — Scales*, ISO

ISO 5456-1:1996(E) (1996), *Technical drawings — Projection methods — Part 1: Synopsis*, ISO

ISO 5456-2:1996(E) (1996), *Technical drawings — Projection methods — Part 2: Orthographic representations*, ISO

ISO 5456-3:1996(E) (1996), *Technical drawings — Projection methods — Part 3: Axonometric representations*, ISO

ISO 5456-4:1996(E) (1996), *Technical drawings — Projection methods — Part 4: Central projection*, ISO

ISO 10209-2:1993(E/F) (1993), *Technical product documentation — Vocabulary — Part 2: Terms relating to projection methods*, *Documentation technique du produit — Vocabulaire — Partie 2: Termes relatifs aux méthodes de projection*, ISO

Pavanelli, F., Miliani, M. and Marchesini, I. (1999), *Lezioni di disegno*, Vol. 3, *Assonometria*, Milano: Editore Ulrico Hoepli.

Reutter, F. 1972, *Darstellende Geometrie*, Karlsruhe: Verlag G. Braun.

Schmidt, R. (1993), *Geometria descriptiva con figuras estereoscopicas*, translated from German, Barcelona: Editorial Reverté.

Simmons and Maguire (2003), *Manual of Engineering Graphics*, Elsevier (Butterworth-Heinemann).

Whitehead, J. (1985), *Graphic communication*, London: Heinemann, Made Simple Books.

Glossary

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