THERMOELASTICITY

ME 036069

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Spring 2009

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1. Introduction

Many modern machines, fixtures and structures are designed to maintain extremely accurate tolerances. For example, machines that cut silicone wafers for the microelectronics industry have to be designed to maintain tolerances to within a few microns (10^{-6} m). Consequently, even small temperature changes can cause thermal distortions which are unacceptably large. Also, many microchips are layered structures manufactured using different materials for each layer. Thus, residual stresses, thermal stresses, and stress concentrations can be formed either during the manufacturing process when materials shrink at different rates or during cooling and heating cycles associated with normal usage. Such inhomogeneities in stress can lead to shortened fatigue life and premature failure.

The main objective of this course in applied thermoelasticity is to present the general theory of a thermoelastic material within the context of small strains and small temperature variations. Due to their general nature, the basic balance laws apply to a number of physical phenomena which include: purely mechanical response at constant temperature; purely thermal response at constant deformation; coupled thermomechanical response; and static and dynamic response.

The word applied in the title of this course is used to indicate that special emphasis will be placed on more practical aspects of the theoretical material. Even though the theory presented here is a linear theory, solutions of the partial differential equations for realistic practical problems are often too complicated to obtain analytically. Therefore, commercial computer numerical codes are usually used in industry to obtain numerical solutions. However, it is well known that the computer will only solve the problem that the user formulates. Consequently, special emphasis will be placed in this course on the proper formulation of thermoelastic boundary values problems. In particular, a number of simple analytical examples will be solved and analyzed to expose the main physical phenomena that can occur in thermoelastic materials.

2. Indicial notation and basic tensor operations

When an engineer aligns a complicated machine with very high precision, or when an experimentalist attempts to make a precise measurement of some physical phenomena, or when a theoretician attempts to formulate and solve a complicated problem, it is extremely important to use the proper tools. In mechanics, mathematical equations are developed to predict the response of materials to mechanical and thermal loads. It is well known from the study of statics and dynamics that there are a number of arbitrary choices made by the engineer to formulate a particular problem. For example, the choice of the origin of the coordinate system and the type of coordinates used are arbitrary choices. On the other hand, it is also well known that the physical response of a material cannot depend in any way on arbitrary mathematical choices. For this reason, it is essential to use mathematical tools that automatically incorporate this physical fact. In mechanics, these mathematical tools are called tensors. For convenience, this section reviews indicial notation and some basic tensor operations which will be used throughout the course.

In this text, attention will be confined to Euclidean three-dimensional space. In its printed form a vector will be denoted as a bold faced symbol like $\mathbf{a}$, whereas in its written form on the board, the same vector will be denoted by a symbol with a wavy line under the symbol like $\underset{\sim}{a}$. Similarly, a second or higher order tensor will be denoted as a bold faced symbol like $\mathbf{A}$, whereas in its written form on the board, the same tensor will be denoted by a symbol with a wavy line under the symbol like $\underset{\sim}{A}$.

BASE VECTORS OF A RECTANGULAR CARTESIAN COORDINATE SYSTEM

An arbitrary vector in three-dimensional space case be written as a linear combination of any three linearly independent vectors. As a special simple case, the base vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of a rectangular Cartesian coordinate system are taken to be constant orthonormal vectors which form a right-handed system. The notions of linear independence and right-handedness can be written in the mathematical form

$$\mathbf{e}_1 \times \mathbf{e}_2 \bullet \mathbf{e}_3 > 0 \quad , \tag{2.1}$$

where • denotes the usual scalar dot product operation and × denotes the usual cross product operation between two vectors. Also, the notion that these base vectors are orthonormal vectors indicates that they satisfy the restrictions

$$\begin{aligned} &\mathbf{e}_1 \bullet \mathbf{e}_1 = 1 \ , \ \mathbf{e}_1 \bullet \mathbf{e}_2 = 0 \ , \ \mathbf{e}_1 \bullet \mathbf{e}_3 = 0 \ , \\ &\mathbf{e}_2 \bullet \mathbf{e}_1 = 0 \ , \ \mathbf{e}_2 \bullet \mathbf{e}_2 = 1 \ , \ \mathbf{e}_2 \bullet \mathbf{e}_3 = 0 \ , \\ &\mathbf{e}_3 \bullet \mathbf{e}_1 = 0 \ , \ \mathbf{e}_3 \bullet \mathbf{e}_2 = 0 \ , \ \mathbf{e}_3 \bullet \mathbf{e}_3 = 1 \ . \end{aligned} \tag{2.2}$$

Moreover, it follows that these base vectors satisfy the additional equations

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3 \ , \ \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2 \ , \ \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1 \ . \tag{2.3}$$

COMPONENTS OF A VECTOR

The components $\{a_1, a_2, a_3\}$ of a general vector **a** can be used together with the base vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to express **a** in the form

$$\mathbf{a} = a_1 \, \mathbf{e}_1 + a_2 \, \mathbf{e}_2 + a_3 \, \mathbf{e}_3 \ . \tag{2.4}$$

Now, using the dot product it follows that the components $\{a_1, a_2, a_3\}$ of **a** are the projections of **a** in the directions of the base vectors

$$a_1 = \mathbf{a} \bullet \mathbf{e}_1 \ , \ a_2 = \mathbf{a} \bullet \mathbf{e}_2 \ , \ a_3 = \mathbf{a} \bullet \mathbf{e}_3 \ . \tag{2.5}$$

INDICIAL NOTATION

Often, it is convenient to use a short hand notation called indicial notation to write the components of vector quantities. Quantities written in indicial notation will have a finite number of indices attached to them. Since the number of indices can be zero a quantity with no index can also be considered to be written in index notation. The language of indicial notation is quite simple because only two types of indices can appear in any term. The index is either a <u>free index</u> or it is a <u>repeated index</u>. Also, a simple summation convention will be defined which applies only to repeated indices. These two types of indices and the summation convention are defined as follows.

Free Indices: Indices that appear only once in a given term are known as free indices. For our purposes, each of these free indices will take the values (1,2,3). For example, i is a free index in each of the following expressions

$$(x_1 , x_2 , x_3) = x_i \quad (i=1,2,3) \quad , \tag{2.6a}$$

$$(\mathbf{e}_1 , \mathbf{e}_2 , \mathbf{e}_3) = \mathbf{e}_i \quad (i=1,2,3) \quad . \tag{2.6b}$$

Notice that the free index i in (2.6) refers to the group of three quantities defined by i taking the values 1,2,3.

Repeated Indices: Indices that appear twice in a given term are known as repeated indices. For example i and j are repeated indices in the following expressions

$$x_i \, \mathbf{e}_i \quad , \quad a_j \, \mathbf{e}_j \quad , \quad a_i \, b_i \quad . \tag{2.7}$$

It is important to emphasize that in the language of indicial notation an index can never appear more than twice in any term. Also, the notion of a term is each group of symbols which are separated by a plus sign, a minus sign or an equals sign.

Einstein Summation Convention: When an index appears as a repeated index in a term, that index is understood to take on the values (1,2,3) and the resulting terms are summed. Thus, for examples, the vectors **x** and **a** can be expressed in the forms

$$\mathbf{x} = x_i \, \mathbf{e}_i = x_1 \, \mathbf{e}_1 + x_2 \, \mathbf{e}_2 + x_3 \, \mathbf{e}_3 \quad , \tag{2.8a}$$

$$\mathbf{a} = a_j \, \mathbf{e}_j = a_1 \, \mathbf{e}_1 + a_2 \, \mathbf{e}_2 + a_3 \, \mathbf{e}_3 \quad , \tag{2.8b}$$

Because of this summation convention, repeated indices are also known as dummy indices since their replacement by any other letter, not appearing as a free index and also not appearing as another repeated index, does not change the meaning of the term in which they occur. For examples,

$$x_i \, \mathbf{e}_i = x_j \, \mathbf{e}_j \quad , \quad a_i \, b_i = a_j \, b_j \quad . \tag{2.9}$$

It is important to emphasize that the same free indices must appear in each term in an equation so that, for example, the vector equation

$$\mathbf{c} = \mathbf{a} + \mathbf{b} \quad , \tag{2.10}$$

can be written in index form in terms of the components of **a**,**b**,**c** as

$$c_i = a_i + b_i \quad . \tag{2.11}$$

Kronecker Delta: The Kronecker delta symbol δ_{ij} is defined by

$$\delta_{ij} = \mathbf{e}_i \bullet \mathbf{e}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad . \tag{2.12}$$

Since the Kronecker delta δ_{ij} vanishes unless i=j it exhibits the following exchange property

$$\delta_{ij}\, x_j = (\, \delta_{1j}\, x_j \,,\, \delta_{2j}\, x_j \,,\, \delta_{3j}\, x_j \,) = (\, x_1 \,,\, x_2 \,,\, x_3 \,) = x_i \quad . \tag{2.13}$$

Notice that the Kronecker symbol can be removed by replacing the repeated index j in (2.13) by the free index i.

Recalling that an arbitrary vector **a** in Euclidean 3-Space can be expressed as a linear combination of the base vectors $\mathbf{e}_i$ such that

$$\mathbf{a} = a_i\, \mathbf{e}_i \quad , \tag{2.14}$$

it follows that the components a_i of **a** can be calculated using the Kronecker delta

$$a_i = \mathbf{e}_i \bullet \mathbf{a} = \mathbf{e}_i \bullet (a_m\, \mathbf{e}_m) = (\mathbf{e}_i \bullet \mathbf{e}_m)\, a_m = \delta_{im}\, a_m = a_i \quad . \tag{2.15}$$

Notice that when the expression (2.14) for **a** was substituted into (2.15) it was necessary to change the repeated index i in (2.15) to another letter (m) because the letter i already appeared in (2.15) as a free index. It also follows that the Kronecker delta can be used to calculate the dot product between two vectors **a** and **b** with components a_i and b_i, respectively, by

$$\mathbf{a} \bullet \mathbf{b} = (a_i\, \mathbf{e}_i) \bullet (b_j\, \mathbf{e}_j) = a_i\, (\mathbf{e}_i \bullet \mathbf{e}_j)\, b_j = a_i\, \delta_{ij}\, b_j = a_i\, b_i \quad . \tag{2.16}$$

<u>Permutation symbol</u>: The permutation symbol ε_{ijk} is defined by

$$\varepsilon_{ijk} = \mathbf{e}_i \times \mathbf{e}_j \bullet \mathbf{e}_k = \begin{cases} 1 & \text{if (i,j,k) are an even permutation of (1,2,3)} \\ -1 & \text{if (i,j,k) are an odd permutation of (1,2,3)} \\ 0 & \text{if at least two of (i,j,k) have the same value} \end{cases} \tag{2.17}$$

From the definition (2.17), it appears that the permutation symbol can be used in calculating the vector product between two vectors. In particular, it can be shown that

$$\mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk}\, \mathbf{e}_k \quad . \tag{2.18}$$

Now, using (2.18) it follows that the vector product between the vectors **a** and **b** can be represented in the form

$$\mathbf{a} \times \mathbf{b} = (a_i\, \mathbf{e}_i) \times (b_j\, \mathbf{e}_j) = (\mathbf{e}_i \times \mathbf{e}_j)\, a_i\, b_j = \varepsilon_{ijk}\, a_i\, b_j\, \mathbf{e}_k \quad . \tag{2.19}$$

Contraction: Contraction is the process of identifying two free indices in a given expression together with the implied summation convention. For example, it is possible to contract on the free indices i,j in δ_{ij} to obtain

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3 \quad . \tag{2.20}$$

Note that contraction on the set of $9=3^2$ quantities T_{ij} can be performed by multiplying T_{ij} by δ_{ij} to obtain

$$T_{ij}\, \delta_{ij} = T_{ii} \quad . \tag{2.21}$$

HIGHER ORDER TENSORS

A scalar is sometimes referred to as a zero order tensor and a vector is sometimes referred to as a first order tensor. Here, higher order tensors are defined deductively starting with the notion of a first order tensor or vector.

A second order tensor: The quantity **T** is called a second order tensor if it is a linear operator whose domain is the space of all vectors **v** and whose range **Tv** or **vT** is a vector. For example, if **T** is the stress tensor and **n** is the unit outward normal to a surface of a body, then the traction vector **t** is given by

$$\mathbf{t} = \mathbf{T}\,\mathbf{n} \quad . \tag{2.22}$$

A third order tensor: The quantity **T** is called a third order tensor if it is a linear operator whose domain is the space of all vectors **v** and whose range **Tv** or **vT** is a second order tensor.

A fourth order tensor: The quantity **T** is called a fourth order tensor if it is a linear operator whose domain is the space of all vectors **v** and whose range **Tv** or **vT** is a third order tensor.

Addition and Subtraction: The usual rules of addition and subtraction of two tensors **A** and **B** apply when the two tensors have the same order. It should be emphasized that tensors of different orders cannot be added or subtracted.

TENSOR PRODUCT

The tensor product operation is denoted by the symbol $\otimes$ and it is defined so that the tensor product $\mathbf{a}_1 \otimes \mathbf{a}_2$ is a special second order tensor having the following properties

$$(\mathbf{a}_1 \otimes \mathbf{a}_2)\, \mathbf{b}_1 = \mathbf{a}_1\, (\mathbf{a}_2 \bullet \mathbf{b}_1) \quad , \quad \mathbf{b}_1\, (\mathbf{a}_1 \otimes \mathbf{a}_2) = (\mathbf{b}_1 \bullet \mathbf{a}_1)\, \mathbf{a}_2 \quad , \tag{2.23}$$

where $\mathbf{a}_i$ and $\mathbf{b}_i$ are vectors. The tensor product operation can be used to form a string of more than two vectors that also becomes a tensor. For example, tensor product $\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3$ is a special third order tensor having the following properties

$$(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3)\, \mathbf{b}_1 = (\mathbf{a}_1 \otimes \mathbf{a}_2)\, (\mathbf{a}_3 \bullet \mathbf{b}_1) \ ,$$

$$\mathbf{b}_1\, (\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3) = (\mathbf{b}_1 \bullet \mathbf{a}_1)\, (\mathbf{a}_2 \otimes \mathbf{a}_3) \ . \tag{2.24}$$

<u>Dot Product (Special Case)</u>: The dot product operation between two vectors can be generalized to an operation between any two tensors (including higher order tensors). For example the dot product of two second order tensors becomes a scalar

$$(\mathbf{a}_1 \otimes \mathbf{a}_2) \bullet (\mathbf{b}_1 \otimes \mathbf{b}_2) = (\mathbf{a}_1 \bullet \mathbf{b}_1)\, (\mathbf{a}_2 \bullet \mathbf{b}_2) \ ,$$

$$(\mathbf{b}_1 \otimes \mathbf{b}_2) \bullet (\mathbf{a}_1 \otimes \mathbf{a}_2) = (\mathbf{b}_1 \bullet \mathbf{a}_1)\, (\mathbf{b}_2 \bullet \mathbf{a}_2) \ , \tag{2.25}$$

the dot product of a third order tensor with a second order tensor becomes a vector

$$(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3) \bullet (\mathbf{b}_1 \otimes \mathbf{b}_2) = \mathbf{a}_1\, (\mathbf{a}_2 \bullet \mathbf{b}_1)\, (\mathbf{a}_3 \bullet \mathbf{b}_2) \ ,$$

$$(\mathbf{b}_1 \otimes \mathbf{b}_2) \bullet (\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3) = (\mathbf{a}_1 \bullet \mathbf{b}_1)\, (\mathbf{a}_2 \bullet \mathbf{b}_2)\, \mathbf{a}_3 \ , \tag{2.26}$$

the dot product of a third order tensor with a third order tensor becomes a scalar

$$(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3) \bullet (\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3) = (\mathbf{a}_1 \bullet \mathbf{b}_1)\, (\mathbf{a}_2 \bullet \mathbf{b}_2)\, (\mathbf{a}_3 \bullet \mathbf{b}_3) \ ,$$

$$(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3) \bullet (\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3) = (\mathbf{a}_1 \bullet \mathbf{b}_1)\, (\mathbf{a}_2 \bullet \mathbf{b}_2)\, (\mathbf{a}_3 \bullet \mathbf{b}_3) \ , \tag{2.27}$$

the dot product of a fourth order tensor with a second order tensor becomes a second order tensor

$$(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3 \otimes \mathbf{a}_4) \bullet (\mathbf{b}_1 \otimes \mathbf{b}_2) = (\mathbf{a}_1 \otimes \mathbf{a}_2)\, (\mathbf{a}_3 \bullet \mathbf{b}_1)\, (\mathbf{a}_4 \bullet \mathbf{b}_2) \ ,$$

$$(\mathbf{b}_1 \otimes \mathbf{b}_2) \bullet (\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3 \otimes \mathbf{a}_4) = (\mathbf{a}_1 \bullet \mathbf{b}_1)\, (\mathbf{a}_2 \bullet \mathbf{b}_2)\, (\mathbf{a}_3 \otimes \mathbf{a}_4) \ , \tag{2.28}$$

and the dot product of a fourth order tensor with a fourth order tensor becomes a scalar

$$(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3 \otimes \mathbf{a}_4) \bullet (\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3 \otimes \mathbf{b}_4) = (\mathbf{a}_1 \bullet \mathbf{b}_1)\, (\mathbf{a}_2 \bullet \mathbf{b}_2)\, (\mathbf{a}_3 \bullet \mathbf{b}_3)\, (\mathbf{a}_4 \bullet \mathbf{b}_4) \ ,$$

$$(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3 \otimes \mathbf{b}_4) \bullet (\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3 \otimes \mathbf{a}_4) = (\mathbf{a}_1 \bullet \mathbf{b}_1)\, (\mathbf{a}_2 \bullet \mathbf{b}_2)\, (\mathbf{a}_3 \bullet \mathbf{b}_3)\, (\mathbf{a}_4 \bullet \mathbf{b}_4) \ . \tag{2.29}$$

In particular, notice from (2.25), (2.27) and (2.29), that the dot product of a tensor with another tensor of the same order is commutative, whereas from (2.26) and (2.28) it can be seen that the dot product of a tensor with another tensor of different order is not commutative.

Cross Product (Special Case): The cross product of a second order tensor with a vector becomes a second order tensor

$$(\mathbf{a}_1 \otimes \mathbf{a}_2) \times \mathbf{b}_1 = \mathbf{a}_1 \otimes (\mathbf{a}_2 \times \mathbf{b}_1) \ ,$$

$$\mathbf{b}_1 \times (\mathbf{a}_1 \otimes \mathbf{a}_2) = (\mathbf{b}_1 \times \mathbf{a}_1) \otimes \mathbf{a}_2 \ , \tag{2.30}$$

the cross product of a second order tensor with another second order tensor becomes a second order tensor

$$(\mathbf{a}_1 \otimes \mathbf{a}_2) \times (\mathbf{b}_1 \otimes \mathbf{b}_2) = (\mathbf{a}_1 \times \mathbf{b}_1) \otimes (\mathbf{a}_2 \times \mathbf{b}_2) \ ,$$

$$(\mathbf{b}_1 \otimes \mathbf{b}_2) \times (\mathbf{a}_1 \otimes \mathbf{a}_2) = (\mathbf{b}_1 \times \mathbf{a}_1) \otimes (\mathbf{b}_2 \times \mathbf{a}_2) \ , \tag{2.31}$$

the cross product of a third order tensor with a second order tensor becomes a third order tensor

$$(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3) \times (\mathbf{b}_1 \otimes \mathbf{b}_2) = \mathbf{a}_1 \otimes (\mathbf{a}_2 \times \mathbf{b}_1) \otimes (\mathbf{a}_3 \times \mathbf{b}_2) \ ,$$

$$(\mathbf{b}_1 \otimes \mathbf{b}_2) \times (\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3) = (\mathbf{b}_1 \times \mathbf{a}_1) \otimes (\mathbf{b}_2 \times \mathbf{a}_2) \otimes \mathbf{a}_3 \ , \tag{2.32}$$

and the cross product of a third order tensor with another third order tensor becomes a third order tensor

$$(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3) \times (\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3) = (\mathbf{a}_1 \times \mathbf{b}_1) \otimes (\mathbf{a}_2 \times \mathbf{b}_2) \otimes (\mathbf{a}_3 \times \mathbf{b}_3) \ ,$$

$$(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3) \times (\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3) = (\mathbf{b}_1 \times \mathbf{a}_1) \otimes (\mathbf{b}_2 \times \mathbf{a}_2) \otimes (\mathbf{b}_3 \times \mathbf{a}_3) \ . \tag{2.33}$$

Juxtaposition (Special Case): The operation of juxtaposition of a second order tensor with another second order tensor is a second order tensor (2=2+2–2)

$$(\mathbf{a}_1 \otimes \mathbf{a}_2)\ (\mathbf{b}_1 \otimes \mathbf{b}_2) = (\mathbf{a}_2 \bullet \mathbf{b}_1)\ (\mathbf{a}_1 \otimes \mathbf{b}_2) \ ,$$

$$(\mathbf{b}_1 \otimes \mathbf{b}_2)\ (\mathbf{a}_1 \otimes \mathbf{a}_2) = (\mathbf{b}_2 \bullet \mathbf{a}_1)\ (\mathbf{b}_1 \otimes \mathbf{a}_2) \ , \tag{2.34}$$

and the juxtaposition of a third order tensor with a second order tensor is a third order tensor (3=3+2–2)

$$(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3)\ (\mathbf{b}_1 \otimes \mathbf{b}_2) = (\mathbf{a}_3 \bullet \mathbf{b}_1)\ (\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{b}_2) \ ,$$

$$(\mathbf{b}_1 \otimes \mathbf{b}_2)\ (\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3) = (\mathbf{b}_2 \bullet \mathbf{a}_1)\ (\mathbf{b}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3) \ . \tag{2.35}$$

Transpose (Special Case): The left transpose operation is denoted by a superscript LT on the left-hand side of the tensor, such that the left transpose of a second order tensor is defined by

$$^{LT}(\mathbf{a}_1\otimes\mathbf{a}_2) = (\mathbf{a}_2\otimes\mathbf{a}_1) \quad , \tag{2.36}$$

the left transpose of a third order tensor is defined by

$$^{LT}(\mathbf{a}_1\otimes\mathbf{a}_2\otimes\mathbf{a}_3) = (\mathbf{a}_2\otimes\mathbf{a}_1)\otimes\mathbf{a}_3 \quad , \tag{2.37}$$

and the left transpose of a fourth order tensor is defined by

$$^{LT}(\mathbf{a}_1\otimes\mathbf{a}_2\otimes\mathbf{a}_3\otimes\mathbf{a}_4) = (\mathbf{a}_2\otimes\mathbf{a}_1)\otimes(\mathbf{a}_3\otimes\mathbf{a}_4) \quad , \tag{2.38}$$

Similarly, the right transpose operation is denoted by a superscript T on the right-hand side of the tensor, such that the right transpose of a second order tensor is defined by

$$(\mathbf{a}_1\otimes\mathbf{a}_2)^T = (\mathbf{a}_2\otimes\mathbf{a}_1) \quad , \tag{2.39}$$

the right transpose of a third order tensor is defined by

$$(\mathbf{a}_1\otimes\mathbf{a}_2\otimes\mathbf{a}_3)^T = \mathbf{a}_1\otimes(\mathbf{a}_3\otimes\mathbf{a}_2) \quad , \tag{2.40}$$

and the right transpose of a fourth order tensor is defined by

$$(\mathbf{a}_1\otimes\mathbf{a}_2\otimes\mathbf{a}_3\otimes\mathbf{a}_4)^T = (\mathbf{a}_1\otimes\mathbf{a}_2)\otimes(\mathbf{a}_4\otimes\mathbf{a}_3) \quad . \tag{2.41}$$

In particular, notice that the transpose operations change the order of the two vectors closest to the side of operation of the operator. In discussing the strain energy of an elastic material it is necessary to consider higher order symmetry of the elastic moduli. Specifically, it is convenient to introduce higher order transpose operators like LT(2) and T(2) which interchange groups of two vectors, such that the second order left transpose LT(2) of a fourth order tensor is defined by

$$^{LT(2)}(\mathbf{a}_1\otimes\mathbf{a}_2\otimes\mathbf{a}_3\otimes\mathbf{a}_4) = (\mathbf{a}_3\otimes\mathbf{a}_4)\otimes(\mathbf{a}_1\otimes\mathbf{a}_2) \quad , \tag{2.42}$$

and the second order left transpose LT(2) of a fifth order tensor is defined by

$$^{LT(2)}(\mathbf{a}_1\otimes\mathbf{a}_2\otimes\mathbf{a}_3\otimes\mathbf{a}_4\otimes\mathbf{a}_5) = (\mathbf{a}_3\otimes\mathbf{a}_4)\otimes(\mathbf{a}_1\otimes\mathbf{a}_2)\otimes\mathbf{a}_5 \quad . \tag{2.43}$$

Similarly, the second order right transpose T(2) of a fourth order tensor is defined by

$$(\mathbf{a}_1\otimes\mathbf{a}_2\otimes\mathbf{a}_3\otimes\mathbf{a}_4)^{T(2)} = (\mathbf{a}_3\otimes\mathbf{a}_4)\otimes(\mathbf{a}_1\otimes\mathbf{a}_2) \quad , \tag{2.44}$$

and the second order right transpose T(2) of a fifth order tensor is defined by

$$(\mathbf{a}_1\otimes\mathbf{a}_2\otimes\mathbf{a}_3\otimes\mathbf{a}_4\otimes\mathbf{a}_5)^{T(2)} = \mathbf{a}_1\otimes(\mathbf{a}_4\otimes\mathbf{a}_5)\otimes(\mathbf{a}_2\otimes\mathbf{a}_3) \quad . \tag{2.45}$$

From these examples, it can be seen that the second order left transpose operator LT(2) considers the first four vectors in the tensor string on the left-hand side of the tensor as two groups of two vectors. The order of the two vectors in each of these groups remains

unchanged but the order of the groups is reversed. Similarly, the second order right transpose operator T(2) considers the first four vectors in the tensor string on the right-hand side of the tensor as two groups of two vectors. Again, the order of the two vectors in each of these groups remains unchanged but the order of the groups is reversed. Since these operators are applied to tensor products of at least four vectors, they can be applied only to tensors which are fourth order or higher.

BASE TENSORS AND COMPONENTS OF HIGHER ORDER TENSORS

The space of second order tensors is spanned by the 9 ($=3^2$) base tensors ($\mathbf{e}_i \otimes \mathbf{e}_j$), such that an arbitrary second order tensor **T** can be expressed in the form

$$\mathbf{T} = T_{ij}\,(\mathbf{e}_i \otimes \mathbf{e}_j) \quad , \tag{2.46}$$

where T_{ij} are the 9 ($=3^2$) components of **T** with respect to the rectangular Cartesian base vectors $\mathbf{e}_i$. This equation is a natural generalization of the representation (2.14) for a vector. Similarly, the equation (2.15) for calculating the components of a vector can be generalized to a second order tensor, such that

$$T_{ij} = \mathbf{T} \bullet (\mathbf{e}_i \otimes \mathbf{e}_j) \quad . \tag{2.47}$$

It then follows by deduction, that ($\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_m$) are the 21 ($=3^3$) base tensors of an arbitrary third order tensor **T** which has 21 ($=3^3$) components T_{ijm}, such that

$$\mathbf{T} = T_{ijm}\,(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_m) \quad , \quad T_{ijm} = \mathbf{T} \bullet (\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_m) \quad . \tag{2.48}$$

Also, ($\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_m \otimes \mathbf{e}_n$) are the 81 ($=3^4$) base tensors of an arbitrary fourth order tensor **T** which has 81 ($=3^4$) components T_{ijmn}, such that

$$\mathbf{T} = T_{ijmn}\,(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_m \otimes \mathbf{e}_n) \quad , \quad T_{ijmn} = \mathbf{T} \bullet (\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_m \otimes \mathbf{e}_n) \quad . \tag{2.49}$$

From the representation (2.46), it can be seen that a general second order tensor has 9 independent components. Consequently, the second order tensor $\mathbf{a} \otimes \mathbf{b}$, which is determined by the tensor product of two vectors **a** and **b**, is only a special case of a second order tensor. Specifically, since each of the vectors **a** and **b** has only three independent components, the tensor $\mathbf{a} \otimes \mathbf{b}$ has only 6 independent components

$$\mathbf{T} = \mathbf{a} \otimes \mathbf{b} \quad , \quad T_{ij} = \mathbf{T} \bullet (\mathbf{e}_i \otimes \mathbf{e}_j) = a_i\, b_j \quad , \tag{2.50}$$

instead of nine components of a general tensor

Given the definitions (2.46)-(2.49), it should be emphasized that when the tensor is written in direct notion **T** it can describe a physical quantity, which by definition, should be independent of the arbitrary choice of coordinates. However, the components T_{ij}, T_{ijm} and T_{ijmn} are explicitly dependent on the orientation of the chosen base vectors $\mathbf{e}_i$. For this reason, tensors are the proper mathematical entities to formulate mathematical equations for physical laws.

Moreover, in the expressions (2.46)-(2.49), the components of a tensor are treated as scalars and the base tensors are strings of vectors. Therefore, all of the tensor operations defined above for the special case of a string of tensor products of vectors, apply to the base tensors and thus also apply to the general tensors. For example, the transpose of the second order tensor **T** takes the form

$$\mathbf{T} = T_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j \quad , \quad \mathbf{T}^T = T_{ij}\, (\mathbf{e}_i \otimes \mathbf{e}_j)^T = T_{ij}\, \mathbf{e}_j \otimes \mathbf{e}_i \quad . \tag{2.51a,b}$$

Also, it can be shown that for a general second order tensor **T** and a general vector **v**, that

$$\mathbf{T}\, \mathbf{v} = \mathbf{v}\, \mathbf{T}^T \quad , \quad T_{ij}\, v_j = v_j\, T_{ij} \quad . \tag{2.52a,b}$$

Furthermore, given the vectors **a** and **b** and the second order tensors **A** and **B**, it can be shown that

$$\mathbf{Aa} \bullet \mathbf{Bb} = \mathbf{a} \bullet \mathbf{A}^T\mathbf{Bb} = \mathbf{A}^T\mathbf{B} \bullet (\mathbf{a} \otimes \mathbf{b}) = A_{im}\, a_m\, B_{in}\, b_n \quad . \tag{2.53}$$

ADDITIONAL DEFINITIONS AND RESULTS

In order to better understand the definition of juxtaposition and in order to connect this definition with the usual rules for matrix multiplication, let **A**, **B**, **C** be second order tensors with components A_{ij}, B_{ij}, C_{ij}, respectively, and define **C** by

$$\mathbf{C} = \mathbf{AB} \quad . \tag{2.54}$$

Using the representation (2.46) for each of these tensors, it follows that

$$\mathbf{C} = A_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j\, B_{mn}\, \mathbf{e}_m \otimes \mathbf{e}_n = A_{ij}\, B_{mn}\, (\mathbf{e}_j \bullet \mathbf{e}_m)\, \mathbf{e}_i \otimes \mathbf{e}_n = A_{im}\, B_{mn}\, \mathbf{e}_i \otimes \mathbf{e}_n \quad , \tag{2.55a}$$

$$C_{ij} = \mathbf{C} \bullet \mathbf{e}_i \otimes \mathbf{e}_j = A_{rm}\, B_{mn}\, (\mathbf{e}_r \otimes \mathbf{e}_n) \bullet (\mathbf{e}_i \otimes \mathbf{e}_j) = A_{im}\, B_{mj} \quad . \tag{2.55b}$$

Examination of the result (2.55b) indicates that the second index of **A** is summed with the first index of **B**, which is consistent with the usual operation of row times column inherent in the definition of matrix multiplication.

Symmetric Tensor: The second order tensor $\mathbf{T}$, with the $9=3^2$ components T_{ij} referred to the base vectors $\mathbf{e}_i$, is said to be symmetric if

$$\mathbf{T} = \mathbf{T}^T \ , \ T_{ij} = T_{ji} \ . \tag{2.56a,b}$$

Since this equations imposes three restrictions on $\mathbf{T}$, it follows that there are only six independent components of a symmetric tensor. Moreover, using (2.52) it can be shown that if $\mathbf{T}$ is symmetric and $\mathbf{v}$ is an arbitrary vector with components v_i, then

$$\mathbf{T}\,\mathbf{v} = \mathbf{v}\,\mathbf{T} \ , \ T_{ij}\, v_j = v_j\, T_{ji} \ . \tag{2.57a,b}$$

Skew-Symmetric Tensor: The second order tensor $\mathbf{T}$, with the $9=3^2$ components T_{ij} referred to the base vectors $\mathbf{e}_i$, is said to be skew-symmetric if

$$\mathbf{T} = -\,\mathbf{T}^T \ , \ T_{ij} = -\,T_{ji} \ . \tag{2.58a,b}$$

Since this equations imposes six restrictions on $\mathbf{T}$, it follows that there are only three independent components of a skew-symmetric tensor. In particular, the diagonal components of $\mathbf{T}$ vanish. Moreover, using (2.52) it can be shown that if $\mathbf{T}$ is skew-symmetric and $\mathbf{v}$ is an arbitrary vector with components v_i, then

$$\mathbf{T}\,\mathbf{v} = -\,\mathbf{v}\,\mathbf{T} \ , \ T_{ij}\, v_j = -\,v_j\, T_{ji} \ . \tag{2.59a,b}$$

Using these definitions, it can be observed that an arbitrary second order tensor $\mathbf{T}$, with components T_{ij}, can be separated uniquely into its symmetric part denoted by $\mathbf{T}_{sym}$, with components $T_{(ij)}$, and its skew-symmetric part denoted by $\mathbf{T}_{skew}$, with components $T_{[ij]}$, such that

$$\mathbf{T} = \mathbf{T}_{sym} + \mathbf{T}_{skew} \ , \ T_{ij} = T_{(ij)} + T_{[ij]} \ , \tag{2.60a,b}$$

$$\mathbf{T}_{sym} = \frac{1}{2}(\mathbf{T} + \mathbf{T}^T) = \mathbf{T}_{sym}^T \ , \ T_{(ij)} = \frac{1}{2}(T_{ij} + T_{ji}) = T_{(ji)} \ , \tag{2.60c,d}$$

$$\mathbf{T}_{skew} = \frac{1}{2}(\mathbf{T} - \mathbf{T}^T) = -\,\mathbf{T}_{skew}^T \ , \ T_{[ij]} = \frac{1}{2}(T_{ij} - T_{ji}) = -\,T_{[ji]} \ . \tag{2.60e,f}$$

Trace: The trace operation is defined as the dot product of an arbitrary second order tensor $\mathbf{T}$ with the second order identity tensor $\mathbf{I}$. Letting T_{ij} be the components of $\mathbf{T}$ it follows that

$$\mathbf{T} \bullet \mathbf{I} = T_{ij}\,(\mathbf{e}_i \otimes \mathbf{e}_j) \bullet (\mathbf{e}_m \otimes \mathbf{e}_m) = T_{ij}\,(\mathbf{e}_i \bullet \mathbf{e}_m)(\mathbf{e}_j \bullet \mathbf{e}_m) = T_{ij}\,\delta_{im}\,\delta_{jm} \ , \tag{2.61a}$$

$$\mathbf{T} \bullet \mathbf{I} = T_{ij}\, \delta_{ij} = T_{jj} \quad . \tag{2.61b}$$

<u>Deviatoric Tensor</u>: The second order tensor **T**, with the $9=3^2$ components T_{ij} referred to the base vectors $\mathbf{e}_i$, is said to be deviatoric if

$$\mathbf{T} \bullet \mathbf{I} = 0 \quad , \quad T_{mm} = 0 \quad . \tag{2.62a,b}$$

<u>Spherical and Deviatoric Parts</u>: Using these definitions it can be observed that an arbitrary second order tensor **T**, with components T_{ij}, can be separated uniquely into its spherical part denoted by T **I**, with components T δ_{ij}, and its deviatoric part denoted by **T'**, with components T'_{ij}, such that

$$\mathbf{T} = T\, \mathbf{I} + \mathbf{T}' \quad , \quad T_{ij} = T\, \delta_{ij} + T_{ij}' \quad , \tag{2.63a,b}$$

$$\mathbf{T}' \bullet \mathbf{I} = 0 \quad , \quad T'_{mm} = 0 \quad . \tag{2.63c,d}$$

Taking the dot product of (2.63a) with the second order identity **I**, it can be shown that T is the mean value of the diagonal terms of **T**

$$T = \frac{1}{3}\, \mathbf{T} \bullet \mathbf{I} = \frac{1}{3}\, T_{mm} \quad . \tag{2.64}$$

When **T** is the stress tensor, this spherical part is related to the pressure p in the body, such that

$$p = -\frac{1}{3}\, \mathbf{T} \bullet \mathbf{I} \quad . \tag{2.65}$$

Also, the von Mises stress σ_e, which is a measure of elastic distortion of the material, is defined in terms of the deviatoric stress **T'**, such that

$$\sigma_e^2 = \frac{3}{2}\, \mathbf{T}' \bullet \mathbf{T}' \quad . \tag{2.66}$$

For the simplest model of plasticity of metals, plastic deformation is possible only when σ_e attains the value Y of yield strength (in uniaxial stress). Consequently, the material remains elastic whenever

$$\sigma_e < Y \quad . \tag{2.67}$$

For later convenience, it is useful to consider properties of the dot product between strings of second order tensors and vectors. To this end, Let **a**, **b** be vectors with

components a_i abd b_i, and let **A**, **B**, **C**, **D** be second order tensors, with components A_{ij}, B_{ij}, C_{ij}, D_{ij}, respectively. Then, it can be shown that

$$\mathbf{A}\,\mathbf{a} \bullet \mathbf{B}\,\mathbf{b} = \mathbf{a} \bullet \mathbf{A}^T\mathbf{B}\,\mathbf{b} = \mathbf{A}^T\mathbf{B} \bullet \mathbf{a}\otimes\mathbf{b} = A_{im}a_m B_{in} b_n \ , \tag{2.68a}$$

$$\mathbf{A} \bullet (\mathbf{BCD}) = A_{ij}B_{im}C_{mn}D_{nj} \ , \quad \mathbf{A} \bullet (\mathbf{BCD}) = (\mathbf{B}^T\mathbf{A}) \bullet (\mathbf{CD}) \ , \tag{2.68b,c}$$

$$\mathbf{A} \bullet (\mathbf{BCD}) = (\mathbf{AD}^T) \bullet (\mathbf{BC}) \ , \quad \mathbf{A} \bullet (\mathbf{BCD}) = (\mathbf{B}^T\mathbf{AD}^T) \bullet \mathbf{C} \ . \tag{2.68d,e}$$

<u>Gradient</u>: Let x_i be the components of the position vector **x** associated with the rectangular Cartesian base vectors $\mathbf{e}_i$. The gradient of a scalar function f with respect to the position **x** is a vector denoted by grad f and represented by

$$\text{grad } f = \nabla f = \partial f/\partial\mathbf{x} = \partial f/\partial x_m\, \mathbf{e}_m = f_{,m}\, \mathbf{e}_m \ , \tag{2.69}$$

where for convenience a comma is used to denote partial differentiation. Also, the gradient of a tensor function **T** is denoted by grad **T** and is represented by

$$\text{grad } \mathbf{T} = \partial\mathbf{T}/\partial\mathbf{x} = \partial\mathbf{T}/\partial x_m \otimes \mathbf{e}_m = \mathbf{T}_{,m} \otimes \mathbf{e}_m \ . \tag{2.70}$$

Note that the derivative $\partial\mathbf{T}/\partial\mathbf{x}$ is written on the same line to indicate the order of the quantities. To see the importance of this, let **T** be a second order tensor with components T_{ij} so that

$$\text{grad } \mathbf{T} = \partial\mathbf{T}/\partial\mathbf{x} = \partial[T_{ij}\, \mathbf{e}_i\otimes\mathbf{e}_j]/\partial x_m\otimes\mathbf{e}_m = T_{ij,m}\, \mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_m \ . \tag{2.71}$$

<u>Divergence:</u> The divergence of a tensor **T** is a vector denoted by div **T** which is represented by

$$\text{div } \mathbf{T} = \frac{\partial\mathbf{T}}{\partial x_k} \bullet \mathbf{e}_k = \mathbf{T}_{,k} \bullet \mathbf{e}_k \ . \tag{2.72}$$

For example, if **T** is a second order tensor, then from (2.46) and (2.72) it follows that

$$\text{div } \mathbf{T} = T_{ij,j}\, \mathbf{e}_i \ . \tag{2.73}$$

<u>Curl</u>: The curl of a vector **v** with components v_i is a vector denoted by curl **v** which is represented by

$$\text{curl } \mathbf{v} = -\frac{\partial\mathbf{v}}{\partial x_j} \times \mathbf{e}_j = -\, v_{i,j}\, \varepsilon_{ijk}\, \mathbf{e}_k = v_{i,j}\, \varepsilon_{jik}\, \mathbf{e}_k \ . \tag{2.74}$$

Also, the curl of a tensor **T** is a tensor denoted by curl **T** which is represented by

$$\text{curl } \mathbf{T} = -\frac{\partial\mathbf{T}}{\partial x_k} \times \mathbf{e}_k \ . \tag{2.75}$$

For example, if **T** is a second order tensor with components T_{ij}, then

$$\text{curl } \mathbf{T} = - T_{ij,k}\, \varepsilon_{jkm}\, \mathbf{e}_i \otimes \mathbf{e}_m \ . \tag{2.76}$$

<u>Laplacian</u>: The Laplacian of a tensor **T** is a tensor denoted by $\nabla^2\mathbf{T}$ which is represented by

$$\nabla^2\mathbf{T} = \text{div} (\text{grad } \mathbf{T}) = [\ \mathbf{T}_{,i} \otimes \mathbf{e}_i\ ]_{,j} \bullet \mathbf{e}_j = \mathbf{T}_{,mm} \quad . \tag{2.77}$$

<u>Biharmonic Operator</u>: The biharmonic operator of a tensor **T** is a tensor denoted by $\nabla^2\nabla^2\mathbf{T}$ which is represented by

$$\nabla^2\nabla^2\mathbf{T} = \mathbf{T}_{,mmnn} \quad . \tag{2.78}$$

<u>Divergence Theorem</u>: Let **n** be the unit outward normal to a surface ∂P of a region P, da be the element of area of ∂P, dv be the element of volume of P, and **T** be an arbitrary tensor of any order. Then, the divergence theorem states that

$$\int_{\partial P} \mathbf{T}\, \mathbf{n}\, da = \int_P \text{div } \mathbf{T}\, dv \ . \tag{2.79}$$

HIERARCHY OF TENSOR OPERATIONS

To simplify the notation and reduce the need for using parentheses to clarify mathematical equations, it is convenient to define the hierarchy of the tensor operations according to Table 2.1, with level 1 operations being performed before level 2 operations and so forth. Also, as is usual, the order in which operations in the same level are performed is determined by which operation appears in the most left-hand position in the equation.

Level	Tensor Operation
1	Left Transpose (LT) and Right Transpose (T)
2	Cross product ($\times$)
3	Juxtaposition and Tensor product ($\otimes$)
4	Dot product ($\bullet$)
5	Addition and Subtraction

Table 2.1 Hierarchy of tensor operations

3. Kinematics: position vector, displacement vector, strain tensor, strain-displacement relations, rotation tensor, homogeneous deformations, rigid body motion, compatibility conditions.

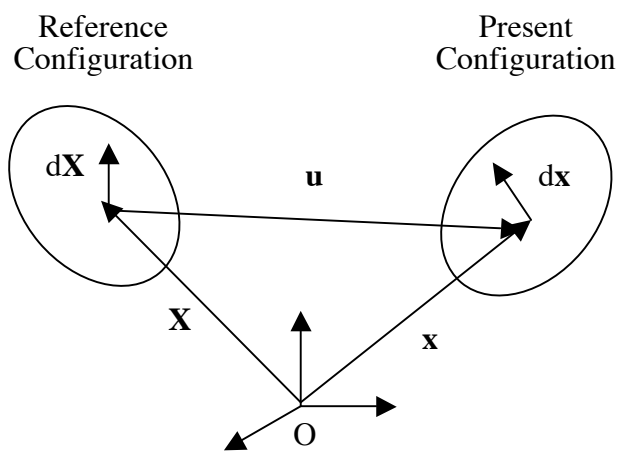

Fig. 3.1 Reference and present configurations, showing the position and displacement vectors.

POSITION VECTOR

In order to describe the motion of a body it is convenient to first identify the location of a material point in the body in a fixed reference configuration by the position vector **X**, relative to a fixed origin O. In the present (deformed) configuration at time t, the same material point is located by the position vector **x**. Consequently, a motion of the body is characterized by the vector function

$$\mathbf{x} = \mathbf{x}(\mathbf{X},t) \quad . \tag{3.1}$$

This vector function is presumed to be one-to-one and invertible at any point in the body and at any time.

DISPLACEMENT VECTOR

The displacement vector $\mathbf{u}(\mathbf{X},t)$ is a vector field that represents the location of a material point in the present configuration relative to its location in the reference configuration. Consequently, $\mathbf{u}$ is defined by (see Fig. 3.1)

$$\mathbf{x} = \mathbf{X} + \mathbf{u}(\mathbf{X},t) \quad . \tag{3.2}$$

STRAIN TENSOR

Figure 3.1 shows a material line element $d\mathbf{X}$ in the reference configuration which is deformed into the material line element $d\mathbf{x}$ in the present configuration. The simple notion of the strain of this line element can be used to motivate the definition of the strain tensor. Specifically, let $d\mathbf{X}$ have length dS and direction $\mathbf{S}$, and let $d\mathbf{x}$ have length ds and direction $\mathbf{s}$, such that

$$d\mathbf{X} = \mathbf{S}\, dS \quad , \quad \mathbf{S} \bullet \mathbf{S} = 1 \quad ,$$
$$d\mathbf{x} = \mathbf{s}\, ds \quad , \quad \mathbf{s} \bullet \mathbf{s} = 1 \quad , \tag{3.3}$$

where $\mathbf{S}$ and $\mathbf{s}$ are unit vectors. Now, the stretch λ of the line element $d\mathbf{X}$ is defined as the ratio of the lengths ds and dS, such that

$$\lambda = \frac{ds}{dS} \quad . \tag{3.4}$$

Consequently, the strain E of this line element is defined by

$$E = \frac{ds - dS}{dS} = \lambda - 1 \quad . \tag{3.5}$$

Next, using the chain rule of differentiation, it follows that the deformation gradient $\mathbf{F}$ characterizes the deformation of the line element $d\mathbf{X}$ into $d\mathbf{x}$, such that

$$d\mathbf{x} = \mathbf{F}\, d\mathbf{X} \quad , \quad \mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X} \quad , \quad \lambda\, \mathbf{s} = \mathbf{F}\, \mathbf{S} \quad . \tag{3.6}$$

The deformation gradient $\mathbf{F}$ is a local quantity that is defined at the material point $\mathbf{X}$ at time t. Moreover, $\mathbf{F}$ characterizes both the extension and the rotation of the material line element. In order to determine the stretch λ it is most convenient to first calculate the length squared ds^2 of the line element $d\mathbf{x}$,

$$ds^2 = d\mathbf{x} \bullet d\mathbf{x} = \mathbf{F}\, d\mathbf{X} \bullet \mathbf{F}\, d\mathbf{X} = d\mathbf{X} \bullet \mathbf{F}^T\mathbf{F}\, d\mathbf{X} = d\mathbf{X} \bullet \mathbf{C}\, d\mathbf{X} \quad ,$$
$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{C}^T \tag{3.7}$$

where the symmetric tensor **C** is called the right Cauchy-Green deformation tensor. Now, with the help of the definitions (3.3), it follows that

$$\lambda^2 = \mathbf{S} \bullet \mathbf{C}\,\mathbf{S} = \mathbf{C} \bullet (\mathbf{S}\otimes\mathbf{S}) \quad . \tag{3.8}$$

Also, the Lagrangian strain **E** is defined in terms of **C** by

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) \quad , \quad \mathbf{C} = \mathbf{I} + 2\mathbf{E} \quad . \tag{3.9}$$

To see that **E** is a strain measure, this definition is substituted into (3.8) to obtain

$$\lambda^2 = 1 + 2\,\mathbf{E} \bullet (\mathbf{S}\otimes\mathbf{S}) \quad . \tag{3.10}$$

Moreover, the strain (3.5) becomes

$$\mathrm{E} = \sqrt{1 + 2\,\mathbf{E} \bullet (\mathbf{S}\otimes\mathbf{S})} - 1 \quad . \tag{3.11}$$

In particular, for small values of the strain tensor **E**, this expression can be expanded in a Taylor series to obtain

$$\mathrm{E} \approx \mathbf{E} \bullet (\mathbf{S}\otimes\mathbf{S}) \quad . \tag{3.12}$$

STRAIN-DISPLACEMENT RELATIONS

The strain-displacement relations can be obtained by substituting (3.3) into the definitions (3.6) for the deformation gradient **F** and (3.9) for the strain **E** to obtain

$$\mathbf{F} = \mathbf{I} + \partial\mathbf{u}/\partial\mathbf{X} \quad ,$$

$$\mathbf{C} = \left[\mathbf{I} + \partial\mathbf{u}/\partial\mathbf{X}\right]^{\mathrm{T}} \left[\mathbf{I} + \partial\mathbf{u}/\partial\mathbf{X}\right] = \mathbf{I} + \partial\mathbf{u}/\partial\mathbf{X} + (\partial\mathbf{u}/\partial\mathbf{X})^{\mathrm{T}} + (\partial\mathbf{u}/\partial\mathbf{X})^{\mathrm{T}}(\partial\mathbf{u}/\partial\mathbf{X})$$

$$\mathbf{E} = \mathbf{e} + \frac{1}{2}(\partial\mathbf{u}/\partial\mathbf{X})^{\mathrm{T}}(\partial\mathbf{u}/\partial\mathbf{X}) \quad , \tag{3.13}$$

where the symmetric tensor **e** is the strain tensor associated with small displacements. In particular, if the quadratic terms in the displacements are neglected then

$$\mathbf{E} \approx \mathbf{e} \quad , \quad \mathbf{e} = \frac{1}{2}\left[\partial\mathbf{u}/\partial\mathbf{X} + (\partial\mathbf{u}/\partial\mathbf{X})^{\mathrm{T}}\right] = \mathbf{e}^{\mathrm{T}} \; . \tag{3.14}$$

Moreover, for small displacements there is no distinction between differentiation of the displacement **u** with respect to **X** or **x**, so that

$$\partial\mathbf{u}/\partial\mathbf{X} \approx \partial\mathbf{u}/\partial\mathbf{x} \quad ,$$

$$\mathbf{e} = \frac{1}{2}\left[\partial\mathbf{u}/\partial\mathbf{x} + (\partial\mathbf{u}/\partial\mathbf{x})^{\mathrm{T}}\right] = \mathbf{e}^{\mathrm{T}} \; , \quad e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = e_{ji} \quad . \tag{3.15}$$

Also, using this approximation the expression (3.12) reduces to

$$E = \mathbf{e} \bullet (\mathbf{S} \otimes \mathbf{S}) = e_{ij}\, S_i\, S_j \quad . \tag{3.16}$$

In particular, notice that the six components e_{ij} of the strain tensor **e** are defined at a material point. The formula (3.16) indicates that at the same material point, different line elements (specified by different direction **S**) have different strains.

<u>Physical interpretation of the diagonal components of strain</u>: Using (3.16) and considering line elements that were directed in the $\mathbf{e}_i$ directions in the reference configuration, it follows that

$$\begin{aligned} E &= e_{11} = u_{1,1} \quad \text{for } \mathbf{S} = \mathbf{e}_1 \quad , \\ E &= e_{22} = u_{2,2} \quad \text{for } \mathbf{S} = \mathbf{e}_2 \quad , \\ E &= e_{33} = u_{3,3} \quad \text{for } \mathbf{S} = \mathbf{e}_3 \quad . \end{aligned} \tag{3.17}$$

This means that the diagonal components of the strain tensor characterize the strains of the line elements that were directed in the $\mathbf{e}_i$ directions in the reference configuration. Also, it can be seen from (3.17) that factor (1/2) in the definition (3.9) causes the linearized strain **e** to be consistent with the simple definition of the strain of a line element ($e_{11} = u_{1,1}$) which is the differential form of the definition (3.9). However, for a material line element in a general direction **S**, both the diagonal and off-diagonal components of **e** contribute to the strain E.

<u>Physical interpretation of the off-diagonal components of strain</u>: In order to discuss the physical interpretation of the off-diagonal components of strain it is convenient to consider two different line elements which are characterized by the directions $\mathbf{S}^{(1)}$, $\mathbf{S}^{(2)}$ in the reference configuration; the directions $\mathbf{s}^{(1)}$, $\mathbf{s}^{(2)}$ in the present configuration, and the strains $E^{(1)}$, $E^{(2)}$, respectively. Specifically, using (3.3)-(3.9) it follows that

$$\left[1 + E^{(1)}\right] \mathbf{s}^{(1)} = \mathbf{F}\, \mathbf{S}^{(1)} \quad , \quad \left[1 + E^{(2)}\right] \mathbf{s}^{(2)} = \mathbf{F}\, \mathbf{S}^{(2)} \quad . \tag{3.18}$$

Thus,

$$\mathbf{s}^{(1)} \bullet \mathbf{s}^{(2)} = \frac{(\mathbf{I} + 2\mathbf{E}) \bullet \{\mathbf{S}^{(1)} \otimes \mathbf{S}^{(2)}\}}{\{1 + E^{(1)}\}\{1 + E^{(2)}\}} \quad . \tag{3.19}$$

Next, for small displacements, quadratic terms in the strains can be neglected and (3.19) can be approximated by

$$\mathbf{s}^{(1)} \bullet \mathbf{s}^{(2)} = \left[1 - E^{(1)} - E^{(2)}\right] \mathbf{S}^{(1)} \bullet \mathbf{S}^{(2)} + 2\mathbf{e} \bullet \{\mathbf{S}^{(1)} \otimes \mathbf{S}^{(2)}\} \quad ,$$

$$\cos\theta = \left[1 - E^{(1)} - E^{(2)}\right] \cos\Theta + 2\mathbf{e} \bullet \{\mathbf{S}^{(1)} \otimes \mathbf{S}^{(2)}\} \quad , \tag{3.20}$$

where Θ is the angle between $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$, and θ is the angle between $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$. As a special case, if $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$ are orthogonal ($\Theta=\pi/2$), $\theta=\pi/2-\gamma$, and γ is small, then

$$\cos\theta = \cos(\pi/2-\gamma) = \sin\gamma \approx \gamma = 2\mathbf{e} \bullet \{\mathbf{S}^{(1)} \otimes \mathbf{S}^{(2)}\} \quad ,$$

$$\gamma = 2\mathbf{e} \bullet \{\mathbf{S}^{(1)} \otimes \mathbf{S}^{(2)}\} \quad \text{for } \mathbf{S}^{(1)} \bullet \mathbf{S}^{(2)} = 0 \quad . \tag{3.21}$$

More specifically, if $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$ are in the directions $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively, then

$$\gamma = 2\, e_{12} \quad \text{for } \mathbf{S}^{(1)} = \mathbf{e}_1 \text{ and } \mathbf{S}^{(2)} = \mathbf{e}_2 \quad . \tag{3.22}$$

This means that the reduction γ in the angle between the two line elements which were in the directions $\mathbf{e}_1$ and $\mathbf{e}_2$ in the reference configuration is related directly to the off-diagonal component e_{12} of the strain tensor. Similar results can be derived for the other off-diagonal components of strain. Moreover, γ is often called the engineering shear strain and e_{12} is the tensorial shear strain. As a specific example, consider the case of simple shear (Fig. 3.2) which is characterized by

$$u_1 = \gamma\, x_2 \quad , \quad u_2 = 0 \quad , \quad u_3 = 0 \quad . \tag{3.23}$$

Notice that the line element that was in the $\mathbf{e}_1$ direction remains in the $\mathbf{e}_1$ direction. Whereas, the line element that was in the $\mathbf{e}_2$ direction is rotated clockwise about the $\mathbf{e}_3$ axis through the angle γ.

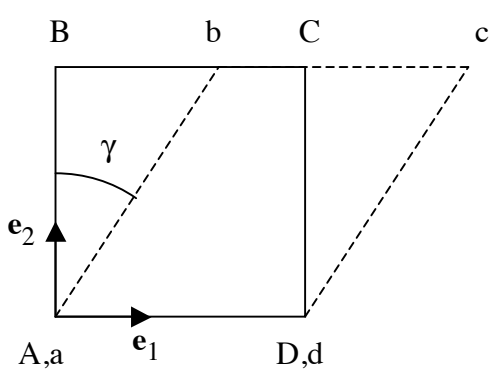

Fig. 3.2 Simple shear. The square ABCD represents the reference configuration and the parallelogram abcd represents the deformed present configuration.

A pure measure of dilatation (volume change): In order to derive a pure measure of dilatation, consider a set of right-handed line elements $\{d\mathbf{X}^{(1)}, d\mathbf{X}^{(2)}, d\mathbf{X}^{(3)}\}$ in the reference configuration, which form a parallelepiped that is deformed to the parallelepiped associated with the set of line elements $\{d\mathbf{x}^{(1)}, d\mathbf{x}^{(2)}, d\mathbf{x}^{(3)}\}$ in the present configuration. The volumes dV and dv of these parallelepipeds are given, respectively, by

$$dV = d\mathbf{X}^{(1)} \times d\mathbf{X}^{(2)} \bullet d\mathbf{X}^{(3)} \quad , \quad dv = d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)} \bullet d\mathbf{x}^{(3)} \quad . \tag{3.24}$$

Next, using (3.6) it follows that

$$dv = \{\mathbf{F}\, d\mathbf{X}^{(1)}\} \times \{\mathbf{F}\, d\mathbf{X}^{(2)}\} \bullet \{\mathbf{F}\, d\mathbf{X}^{(3)}\} \quad . \tag{3.25}$$

Moreover, it can be shown that for an arbitrary nonsingular tensor $\mathbf{F}$ ($\det\mathbf{F} \neq 0$), and arbitrary vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, that

$$\mathbf{F}\,\mathbf{a} \times \mathbf{F}\,\mathbf{b} = J\,\mathbf{F}^{-T}\,(\mathbf{a} \times \mathbf{b}) \quad , \quad J = \det \mathbf{F} \quad , \tag{3.26}$$

where $\mathbf{F}^{-T}$ is the inverse of the transpose of $\mathbf{F}$, and J is the determinant of $\mathbf{F}$. Thus, using this result, (3.25) can be reduced to

$$dv = J\,\mathbf{F}^{-T}\,\{d\mathbf{X}^{(1)} \times d\mathbf{X}^{(2)}\} \bullet \{\mathbf{F}\, d\mathbf{X}^{(3)}\} = J\,\{d\mathbf{X}^{(1)} \times d\mathbf{X}^{(2)}\} \bullet \mathbf{F}^{-1}\,\{\mathbf{F}\, d\mathbf{X}^{(3)}\} \quad ,$$

$$dv = J\, d\mathbf{X}^{(1)} \times d\mathbf{X}^{(2)} \bullet d\mathbf{X}^{(3)} = J\, dV \quad . \tag{3.27}$$

This means that J is a pure measure of dilatation since it is a pure measure of volume change. Next, using the definition of the determinant, it follows that

$$J = \mathbf{F}\,\mathbf{e}_1 \times \mathbf{F}\,\mathbf{e}_2 \bullet \mathbf{F}\,\mathbf{e}_3 \quad . \tag{3.28}$$

Therefore, for small deformations, (3.13) and (3.15) can be used and quadratic terms in the displacement can be neglected to obtain

$$J = \{\mathbf{e}_1 + \partial\mathbf{u}/\partial\mathbf{x}\,\mathbf{e}_1\} \times \{\mathbf{e}_2 + \partial\mathbf{u}/\partial\mathbf{x}\,\mathbf{e}_2\} \bullet \{\mathbf{e}_3 + \partial\mathbf{u}/\partial\mathbf{x}\,\mathbf{e}_3\} \quad ,$$

$$J = 1 + \{\partial\mathbf{u}/\partial\mathbf{x}\,\mathbf{e}_1\} \bullet (\mathbf{e}_2 \times \mathbf{e}_3) + \{\partial\mathbf{u}/\partial\mathbf{x}\,\mathbf{e}_2\} \bullet (\mathbf{e}_3 \times \mathbf{e}_1) + \{\partial\mathbf{u}/\partial\mathbf{x}\,\mathbf{e}_3\} \bullet (\mathbf{e}_1 \times \mathbf{e}_2),$$

$$J = 1 + \{\partial\mathbf{u}/\partial\mathbf{x}\,\mathbf{e}_1\} \bullet \mathbf{e}_1 + \{\partial\mathbf{u}/\partial\mathbf{x}\,\mathbf{e}_2\} \bullet \mathbf{e}_2 + \{\partial\mathbf{u}/\partial\mathbf{x}\,\mathbf{e}_3\} \bullet \mathbf{e}_3 \quad ,$$

$$J = 1 + \partial\mathbf{u}/\partial\mathbf{x} \bullet (\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3) = 1 + \partial\mathbf{u}/\partial\mathbf{x} \bullet \mathbf{I} = 1 + \mathbf{e} \bullet \mathbf{I} \quad . \tag{3.29}$$

Thus, the trace of the strain tensor is a pure measure of dilatation

$$\mathbf{e} \bullet \mathbf{I} = e_{ii} = J - 1 = \frac{dv - dV}{dV} \quad . \tag{3.30}$$

A pure measure of distortion: The strain tensor **e** describes the dilatation and distortion of the material at each material point. It has been shown (3.30) that the trace of **e** is a pure measure of dilatation. It therefore, follows that the deviatoric part **e**' of **e** is a pure measure of distortion since its trace vanishes

$$\mathbf{e}' = \mathbf{e} - \frac{1}{3}(\mathbf{e} \bullet \mathbf{I})\,\mathbf{I} \quad , \quad \mathbf{e}' \bullet \mathbf{I} = 0 \quad ,$$

$$e'_{ij} = e_{ij} - \frac{1}{3} e_{mm}\,\delta_{ij} \quad , \quad e'_{mm} = 0 \quad . \tag{3.31}$$

ROTATION TENSOR

The displacement gradient $\partial\mathbf{u}/\partial\mathbf{x}$ is a general tensor that can be separated into its symmetric part **e** and its skew-symmetric part $\boldsymbol{\omega}$, such that

$$\partial\mathbf{u}/\partial\mathbf{x} = \mathbf{e} + \boldsymbol{\omega} \quad ,$$

$$\mathbf{e} = \frac{1}{2}\left[\partial\mathbf{u}/\partial\mathbf{X} + (\partial\mathbf{u}/\partial\mathbf{X})^T\right] = \mathbf{e}^T \quad , \quad e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = e_{ji} \quad ,$$

$$\boldsymbol{\omega} = \frac{1}{2}\left[\partial\mathbf{u}/\partial\mathbf{X} - (\partial\mathbf{u}/\partial\mathbf{X})^T\right] = -\,\boldsymbol{\omega}^T \quad , \quad \omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) = -\,\omega_{ji} \quad . \tag{3.32}$$

The symmetric part **e** has already been identified as the strain tensor and its physical meaning has been discussed. Here, it will be shown that the skew-symmetric part $\boldsymbol{\omega}$ has the physical meaning of a rotation tensor. To this end, it is recalled from (3.13) and (3.15) that for small deformations, the deformation gradient becomes

$$\mathbf{F} = \mathbf{I} + \mathbf{e} + \boldsymbol{\omega} \quad . \tag{3.33}$$

Moreover, with the help of (3.3), (3.4), (3.5) and (3.12), it follows that for small deformations

$$(1 + E)\,\mathbf{s} = (\mathbf{I} + \mathbf{e} + \boldsymbol{\omega})\,\mathbf{S} \quad , \quad E = (\mathbf{e} \bullet \mathbf{S}\otimes\mathbf{S}) \quad ,$$

$$\mathbf{s} - \mathbf{S} = \boldsymbol{\omega}\,\mathbf{S} + \left[\mathbf{e} - (\mathbf{e} \bullet \mathbf{S}\otimes\mathbf{S})\,\mathbf{I}\right]\mathbf{S} \quad ,$$

$$s_i = S_i + \omega_{ij}\,S_j + \left[e_{ij} - e_{mn}\,S_m\,S_n\,\delta_{ij}\right]S_j \quad . \tag{3.34}$$

Since both **S** and **s** are unit vectors, the vector **s** can only rotate relative to the vector **S**. Mathematically, it can be shown that since $\boldsymbol{\omega}$ is a skew-symmetric tensor, that

$$\boldsymbol{\omega} \bullet (\mathbf{S} \otimes \mathbf{S}) = 0 \ , \tag{3.35}$$

so that neglecting second order quantities in the displacements yields

$$(\mathbf{s} - \mathbf{S}) \bullet \mathbf{S} = 0 \ . \tag{3.36}$$

Physically, this means that the change in **S** is perpendicular to **S**, which is consistent with a small rotation.

Next, it is observed from (3.34) that, for a general line element **S**, the rotation of the line element depends on both the rotation tensor $\boldsymbol{\omega}$ and the strain tensor **e**. However, for the special case, when **S** is chosen in the direction of an eigenvector of the strain **e**, the rotation is totally controlled by the rotation tensor $\boldsymbol{\omega}$

$$\mathbf{s} - \mathbf{S} = \boldsymbol{\omega}\, \mathbf{S} \quad \text{for } \mathbf{e}\, \mathbf{S} = \mathrm{E}\, \mathbf{S} \ , \ \mathrm{E} = \mathbf{e} \bullet (\mathbf{S} \otimes \mathbf{S}) \ . \tag{3.37}$$

Appendix A provides details of the determination of the eigenvalues and eigenvectors of a real second order symmetric tensor.

HOMOGENEOUS DEFORMATIONS

A deformation is said to be homogeneous if the deformation gradient **F** is independent of the position **X**. Within the context of the small deformation theory this means that the displacement gradient $\partial \mathbf{u}/\partial \mathbf{x}$ is independent of the position **x**. Specifically, take

$$\partial \mathbf{u}/\partial \mathbf{x} = \mathbf{H}(t) \ , \quad u_{i,j} = H_{ij}(t) \ , \tag{3.38}$$

where **H** is a general tensor function of time t only. Integrating (3.38) with respect to space, it follows that the displacement field **u** for homogeneous deformation becomes

$$\mathbf{u} = \mathbf{c}(t) + \mathbf{H}(t)\, \mathbf{x} \ , \tag{3.39}$$

where **c**(t) is a vector function of time only. It also follows that the strain and rotation tensors associated with this homogeneous become

$$\mathbf{e} = \frac{1}{2}(\mathbf{H} + \mathbf{H}^{\mathrm{T}}) \ , \ \boldsymbol{\omega} = \frac{1}{2}(\mathbf{H} - \mathbf{H}^{\mathrm{T}}) \ . \tag{3.40}$$

Notice that there are twelve degrees of freedom associated with a homogeneous deformation: three associated with translation vector **c**; three associated with the rotation tensor $\boldsymbol{\omega}$; and six associated with the strain tensor **e**.

RIGID BODY MOTION

A rigid body is a body for which the length between any two material points remains constant. It then follows that rigid body motion is a special case of homogeneous deformation for which the strain of any material line element vanishes. Consequently, since the strain E in (3.16) must vanish for any material point **x** and any material direction **N**, it can be shown that for rigid body motion the strain tensor **e** must vanish at all points

$$\mathbf{e} = 0 \quad \text{for rigid body motion.} \tag{3.41}$$

Thus, the tensor **H** in (3.39) must be a skew-symmetric tensor

$$\mathbf{H}^T = -\mathbf{H} \ , \quad \boldsymbol{\omega} = \mathbf{H} \ , \tag{3.42}$$

so that (3.39) reduces to

$$\mathbf{u} = \mathbf{c}(t) + \boldsymbol{\omega}(t)\,\mathbf{x} \quad \text{for rigid body motion.} \tag{3.43}$$

Physically, this means that a rigid body has six degrees of freedom: three associated with translation **c**; and three associated with rotation $\boldsymbol{\omega}$ (with $\boldsymbol{\omega}$ being skew-symmetric).

COMPATIBILITY CONDITIONS

In order to understand the notion of compatibility from a physical point of view, it is convenient to consider a body that has been divided into a finite number of tetrahedrons which just fit together in its stress-free undeformed reference configuration. Now, mark each tetrahedron so that when these parts are separated they can be put back together with the same topology (i.e. the same neighbors). Next, separate the parts and deform each tetrahedron in an arbitrary manner. Obviously, it is not reasonable to expect that these deformed parts will fit together without gaps between the parts. This is an example when the strain field is not compatible. However, if the strains in each of these tetrahedrons is suitably restricted, then the parts will fit together to form a deformed intact body.

Mathematically, this means that the strain field e_{ij} must satisfy certain restrictions for a displacement field to exist. More specifically, it is noted that the strain-displacement equations (3.15) indicate that the six independent components e_{ij} of the strain are derived

from only three components u_i of the displacement vector. To derive these restrictions, consider integration of the displacement gradient over an arbitrary curve C in space (Sokolnikoff, 1956)

$$u_i = \int_C u_{i,m}\, dx_m = \int_C (e_{im} + \omega_{im})\, dx_m \quad , \tag{3.44}$$

where (3.32) had been used to express $u_{i,m}$ in terms of the strain and rotation tensors. However, the rotation tensor can be rewritten in the form

$$\omega_{im} = \left[(x_n - x_n^0)\, \omega_{in}\right]_{,m} - (x_n - x_n^0)\, \omega_{in,m} \quad , \tag{3.45}$$

and the gradient $\omega_{in,m}$ of ω_{im} can be expressed in terms of derivatives of the strain tensor using the expressions

$$\omega_{in,m} = \frac{1}{2}(u_{i,nm} - u_{n,im}) + \frac{1}{2}(u_{m,in} - u_{m,in}) \quad ,$$

$$\omega_{in,m} = \frac{1}{2}(u_{i,nm} + u_{m,in}) - \frac{1}{2}(u_{n,im} + u_{m,in}) = e_{im,n} - e_{mn,i} \quad . \tag{3.46}$$

Thus, with the help of (3.45) and (3.46), it follows that (3.44) can be integrated to obtain

$$u_i = u_i^0 + (x_n - x_n^0)\, \omega_{in}(x_m^0) + \int_C U_{im}\, dx_m$$

$$U_{im} = e_{im} - (x_n - x_n^0)\, (e_{im,n} - e_{mn,i}) \quad , \tag{3.47}$$

where u_i^0 are the components of the displacement and $\omega_{in}(x_m^0)$ are the components of the rotation tensor ω_{in} at the location x_n^0. Also, U_{im} has been introduced for convenience.

The displacement field u_i will be single valued if the integral in (3.47) is zero for all closed curves C. The necessary and sufficient condition for u_i to be a single valued function of x_i, is that U_{im} is the derivative of a potential function and that its curl vanishes,

$$U_{im,j} = U_{ij,m} \quad . \tag{3.48}$$

Thus, substitution of (3.47) into (3.48) yields the condition that

$$e_{im,j} - (e_{im,j} - e_{mj,i}) - (x_n - x_n^0)\, (e_{im,nj} - e_{mn,ij}) =$$

$$e_{ij,m} - (e_{ij,m} - e_{jm,i}) - (x_n - x_n^0)\, (e_{ij,nm} - e_{jn,im}) \quad , \tag{3.49}$$

which simplifies to

$$(x_n - x_n^0)\,(e_{ij,nm} + e_{mn,ij} - e_{im,nj} - e_{jn,im}) = 0 \quad . \tag{3.50}$$

However, since this equation must be valid for all values of x_n^0, it follows that the linearized Riemann curvature tensor R_{ijmn} must vanish

$$R_{ijmn} = e_{ij,mn} + e_{mn,ij} - e_{im,nj} - e_{jn,im} = 0 \quad , \tag{3.51}$$

at each point in the body. It is obvious, from this definition that R_{ijmn} is a double symmetric tensor having the symmetries

$$R_{ijmn} = R_{jimn} = R_{ijnm} = R_{mnij} \quad . \tag{3.52}$$

This means that R_{ijmn} has only 21 independent components instead of 81 ($=3^4$). However, only 6 of these 21 components are truly independent. This is because three of the components have the forms

$$R_{mmmm} = e_{mm,mm} + e_{mm,mm} - e_{mm,mm} - e_{mm,mm} = 0 \ \text{(no sum on m)} \tag{3.53}$$

which are automatically satisfied; six of the components have the forms

$$R_{immm} = e_{im,mm} + e_{mm,im} - e_{im,mm} - e_{mm,im} = 0 \ , \ \text{(no sum on m)} \tag{3.54}$$

which are automatically satisfied; the components automatically satisfy the three equations

$$R_{iijj} = e_{ii,jj} + e_{jj,ii} - e_{ij,ij} - e_{ij,ij} = -\,R_{ijij} \ , \ \text{(no sum on i or j)} \tag{3.55}$$

and three additional equations

$$R_{imjm} = e_{im,jm} + e_{jm,im} - e_{ij,mm} - e_{mm,ij} = -\,R_{imjm} \ \text{(no sum on m and } i \neq j \neq m) \ . \tag{3.56}$$

More specifically, these 6 conditions can be written in the forms

$$R_{1122} = \frac{\partial^2 e_{11}}{\partial x_2^2} + \frac{\partial^2 e_{22}}{\partial x_1^2} - 2\,\frac{\partial^2 e_{12}}{\partial x_1 \partial x_2} = 0 \ ,$$

$$R_{2233} = \frac{\partial^2 e_{22}}{\partial x_3^2} + \frac{\partial^2 e_{33}}{\partial x_2^2} - 2\,\frac{\partial^2 e_{23}}{\partial x_2 \partial x_3} = 0 \ ,$$

$$R_{1133} = \frac{\partial^2 e_{33}}{\partial x_1^2} + \frac{\partial^2 e_{11}}{\partial x_3^2} - 2\,\frac{\partial^2 e_{13}}{\partial x_1 \partial x_3} = 0 \ ,$$

$$R_{1123} = \frac{\partial^2 e_{11}}{\partial x_2 \partial x_3} + \frac{\partial^2 e_{23}}{\partial x_1^2} - \frac{\partial^2 e_{12}}{\partial x_1 \partial x_3} - \frac{\partial^2 e_{13}}{\partial x_1 \partial x_2} = 0 \ ,$$

$$R_{2213} = \frac{\partial^2 e_{22}}{\partial x_1 \partial x_3} + \frac{\partial^2 e_{13}}{\partial x_2^2} - \frac{\partial^2 e_{12}}{\partial x_2 \partial x_3} - \frac{\partial^2 e_{23}}{\partial x_1 \partial x_2} = 0 \ ,$$

$$R_{3312} = \frac{\partial^2 e_{33}}{\partial x_1 \partial x_2} + \frac{\partial^2 e_{12}}{\partial x_3^2} - \frac{\partial^2 e_{13}}{\partial x_2 \partial x_3} - \frac{\partial^2 e_{23}}{\partial x_1 \partial x_3} = 0 \ . \tag{3.57}$$

Alternatively, by contracting on m and n in the expression (3.51) for R_{ijmn}, it can be shown that

$$R_{ij} = R_{ijmm} = e_{ij,mm} + e_{mm,ij} - e_{im,mj} - e_{jm,im} = 0 \ . \tag{3.58}$$

To see that these 6 equations are equivalent to the 6 equations (3.57), use is made of the symmetry conditions (3.52) and the restrictions (3.53)-(3.56) to deduce that

$$R_{11} = R_{1111} + R_{1122} + R_{1133} = R_{1122} + R_{1133} = 0 \ ,$$

$$R_{22} = R_{2211} + R_{2222} + R_{2233} = R_{1122} + R_{2233} = 0 \ ,$$

$$R_{33} = R_{3311} + R_{3322} + R_{3333} = R_{1133} + R_{2233} = 0 \ ,$$

$$R_{12} = R_{1211} + R_{1222} + R_{1233} = R_{3312} = 0 \ ,$$

$$R_{13} = R_{1311} + R_{1322} + R_{1333} = R_{2213} = 0 \ ,$$

$$R_{23} = R_{2311} + R_{2322} + R_{2333} = R_{1123} = 0 \ . \tag{3.59}$$

Thus, when the compatibility conditions (3.57) or (3.58) are satisfied, the existence of the displacement field is guaranteed.

4. Basic balance laws: Conservation of mass; the balances of linear momentum, entropy, angular momentum and energy (first law of thermodynamics); and the reduced energy equation.

This section presents the basic balance laws controlling the thermomechanical response of simple continua. It is important to emphasize that these balance laws are valid for all simple continuum so they are valid for a wide class of materials which include: inviscid fluids, viscous fluids, Non-Newtonian fluids, thermoelastic solids, elastic-plastic solids, elastic-viscoplastic solids, etc. The equations that characterize the response of a particular material are called constitutive equations. In this course, attention will be focused on thermoelastic solids and the constitutive equations for these materials will be discussed in a later section.

Following the work of Green and Naghdi (1977,1978), the balance laws will be separated into two groups. One group includes: the conservation of mass, the balance of linear momentum and the balance of entropy, which are used to determine the mass density ρ (mass per unit volume), the position $\mathbf{x}$ (or displacement $\mathbf{u}$) of a material point, and the absolute temperature θ. The second group includes: the balance of angular momentum and the balance of energy, which are assumed to be satisfied identically and are used to impose restrictions on constitutive assumptions.

In the following, P denotes a material region which can be any part of the body under consideration. Also, ∂P denotes the smooth closed boundary of P, and $\mathbf{n}$ denotes the unit outward normal vector to ∂P.

CONSERVATION OF MASS

The conservation of mass requires the total mass of the material region P to remain constant

$$\int_P \rho \, dv = \int_{P_0} \rho_0 \, dV \quad , \tag{4.1}$$

where P_0 is the region in the reference configuration associated with P, and ρ_0 is the mass density in the reference configuration. Next, using the result (3.27), it follows that the integral over P_0 can be converted to an integral over P to obtain

$$\int_P \left[\rho - \rho_0 \, J^{-1}\right] dv = 0 \quad . \tag{4.2}$$

Assuming that this expression is valid for arbitrary parts P and that the integrand is continuous, the local form of the conservation of mass becomes

$$\rho = \rho_0 \, J^{-1} \quad , \tag{4.3}$$

which must be satisfied at each point of P. Moreover, using the result (3.29) associated with the small deformation theory, it follows that (4.3) can be rewritten in the form

$$\rho \; = \rho_0 \, (1 + \mathbf{e} \bullet \mathbf{I})^{-1} = \rho_0 \, (1 - \mathbf{e} \bullet \mathbf{I}) \quad . \tag{4.4}$$

This means that the density ρ decreases when the volume increases ($\mathbf{e} \bullet \mathbf{I} > 0$), which is consistent with simple physical experience.

BALANCE OF LINEAR MOMENTUM

The balance of linear momentum is a direct generalization of Newton's second law for a particle. In words, it states that the rate of change of linear momentum is equal to the total force applied to the body. This physical concept is translated into the mathematical expression

$$\frac{d}{dt} \int_P \rho \, \mathbf{v} \, dv = \int_P \rho \, \mathbf{b} \, dv + \int_{\partial P} \mathbf{t} \, da \quad , \tag{4.5}$$

where $\mathbf{v}$ is the absolute velocity of a material point,

$$\mathbf{v} = \dot{\mathbf{x}} = \dot{\mathbf{u}} \quad , \tag{4.6}$$

a superposed dot (•) denote material time differentiation holding $\mathbf{X}$ fixed, $\mathbf{b}$ is the external specific (per unit mass) body force (e.g. gravity), and $\mathbf{t}$ is the stress vector (force per unit area da) applied to the boundary ∂P of the body. In order to develop the local form of this equation, it is first recalled that the element of mass dm can be expressed as

$$dm = \rho \, dv \quad . \tag{4.7}$$

Therefore, the first integral in (4.5) can be written in the form

$$\int_P \rho \, \mathbf{v} \, dv = \int \mathbf{v} \, dm \quad . \tag{4.8}$$

However, since the mass is constant, the time differentiation can be interchanged with the integration over mass to deduce that

$$\frac{d}{dt} \int_P \rho \, \mathbf{v} \, dv = \frac{d}{dt} \int \mathbf{v} \, dm = \int \dot{\mathbf{v}} \, dm = \int_P \rho \, \dot{\mathbf{v}} \, dv = \int_P \rho \, \ddot{\mathbf{u}} \, dv \quad . \tag{4.9}$$

Next, it is recalled that the stress vector **t** is related to the stress tensor **T** by the expression

$$\mathbf{t}(\mathbf{x},t;\mathbf{n}) = \mathbf{T}(\mathbf{x},t)\,\mathbf{n} \quad , \tag{4.10}$$

where it is important to emphasize that the stress vector **t** depends linearly on the unit outward normal **n**, but the stress tensor **T** only depends on position and time. This means that the stress tensor **T** characterizes the state of stress at a point in the body, whereas the stress vector characterizes the state of stress applied to a specific surface through a point in the body. Now, using the divergence theorem (2.79) it follows that

$$\int_{\partial P} \mathbf{T}\,\mathbf{n}\,da = \int_P \operatorname{div} \mathbf{T}\,dv \quad . \tag{4.11}$$

Thus, with the help of (4.9) and (4.11), the balance of linear momentum (4.5) can be written in the form

$$\int_P \left[\rho\,\dot{\mathbf{v}} - \rho\,\mathbf{b} - \operatorname{div} \mathbf{T}\right] dv = 0 \quad . \tag{4.12}$$

Again, assuming that the integrand is continuous and that this equation is valid for arbitrary parts P, it can be shown that the local form of the balance of linear momentum becomes

$$\rho\,\dot{\mathbf{v}} = \rho\,\mathbf{b} + \operatorname{div} \mathbf{T} \quad , \tag{4.13}$$

which must be satisfied at each point of P. Moreover, for the small deformation theory it is assumed that **u**, **b** and **T** are small, so that the density ρ can be replaced by its reference value ρ_0 and (4.13) reduces to

$$\rho_0\,\ddot{\mathbf{u}} = \rho_0\,\mathbf{b} + \operatorname{div} \mathbf{T} \quad , \quad \rho_0\,\ddot{u}_i = \rho_0\,b_i + T_{ij,j} \quad . \tag{4.14}$$

Also, for the small deformation theory, material differentiation reduces to partial differentiation with respect to time

$$\dot{\mathbf{u}} = \frac{\partial \mathbf{u}}{\partial t} \quad . \tag{4.15}$$

BALANCE OF ENTROPY

It is well known that many thermomechanical processes proceed in a specific direction. For example, it is known that heat flows from hot regions to cold regions and not the reverse. This means that if a hot body is put into thermal contact with a cold body

and the two bodies are insulated from their surroundings, then the hot body will get colder and the cold body will get hotter until they reach an equilibrium state where both bodies are at the same temperature. One of the main reasons for introducing the notion of entropy is to develop mathematical equations which can be used to quantify this type of irreversible process.

From the point of view of continuum mechanics, it is assumed that the specific entropy η is defined at each material point. In words, the balance of entropy states that the rate of change of entropy is equal to the external rate of supply of entropy plus the internal rate of production of entropy. This physical concept is translated into the mathematical expression

$$\frac{d}{dt}\int_P \rho\, \eta\, dv = \left[\int_P \rho\, s\, dv - \int_{\partial P} \mathbf{p} \bullet \mathbf{n}\, da \right] + \int_P \rho\, \xi\, dv \quad , \tag{4.16}$$

where s denotes the specific external rate of supply of entropy at a point in the body, $\mathbf{p}$ denotes the external rate of entropy flux vector through the boundary ∂P, and ξ denotes the internal rate of entropy production (Green and Naghdi, 1977,1978). The minus sign is used here because $\mathbf{p} \bullet \mathbf{n}$ denotes the entropy flux in the direction of $\mathbf{n}$, which indicates that the entropy is expelled from the body instead of supplied to the body.

Following similar arguments to those used to develop the local form of the balance of linear momentum, it can be shown that

$$\frac{d}{dt}\int_P \rho\, \eta\, dv = \int_P \rho\, \dot{\eta}\, dv \quad , \quad \int_{\partial P} \mathbf{p} \bullet \mathbf{n}\, da = \int_P \operatorname{div} \mathbf{p}\, dv \quad , \tag{4.17}$$

so that (4.16) reduces to

$$\int_P \left[\rho\, \dot{\eta} - \rho\, s + \operatorname{div} \mathbf{p} - \rho\, \xi \right] dv = 0 \quad . \tag{4.18}$$

Again, assuming that the integrand is continuous and that this equation is valid for arbitrary parts P, it can be shown that the local form of the balance of entropy becomes

$$\rho\, \dot{\eta} = \rho\, s - \operatorname{div} \mathbf{p} + \rho\, \xi \quad , \tag{4.19}$$

which must be satisfied at each point of P. Furthermore, if η, s, ξ are small quantities, then ρ can be replaced by ρ_0, and (4.19) reduces to

$$\rho_0\, \dot{\eta} = \rho_0\, s - \operatorname{div} \mathbf{p} + \rho_0\, \xi \quad , \quad \rho_0\, \dot{\eta} = \rho_0\, s - p_{j,j} + \rho_0\, \xi \quad , \tag{4.20}$$

where p_j are the components of the entropy flux $\mathbf{p}$ relative to the base vectors $\mathbf{e}_j$.

BALANCE OF ANGULAR MOMENTUM

The balance of angular momentum is also a direct generalization of the balance of angular momentum for a rigid body. In words, it states that the rate of change of angular momentum about a fixed point is equal to the total moment applied to the body about the same fixed point. Taking the fixed point as the fixed origin O, this physical concept is translated into the mathematical expression

$$\frac{d}{dt}\int_P \mathbf{x}\times\rho\,\mathbf{v}\,dv = \int_P \mathbf{x}\times\rho\,\mathbf{b}\,dv + \int_{\partial P}\mathbf{x}\times\mathbf{t}\,da \quad . \tag{4.21}$$

Next, using the fact that

$$\text{div}\,(\mathbf{x}\times\mathbf{T}) = (\mathbf{x}\times\mathbf{T})_{,j}\bullet\mathbf{e}_j = \mathbf{x}\times\mathbf{T}_{,j}\bullet\mathbf{e}_j + \mathbf{x}_{,j}\times\mathbf{T}\,\mathbf{e}_j = \mathbf{x}\times\text{div}\,\mathbf{T} + \mathbf{e}_j\times\mathbf{T}\,\mathbf{e}_j \quad , \tag{4.22}$$

and following similar arguments to those used to develop the local form of the balance of linear momentum, it can be shown that

$$\frac{d}{dt}\int_P \mathbf{x}\times\rho\,\mathbf{v}\,dv = \int_P \rho\,\overline{\mathbf{x}\times\mathbf{v}}^{\bullet}\,dv = \int_P \rho\,[\,\mathbf{v}\times\mathbf{v} + \mathbf{x}\times\dot{\mathbf{v}}]\,dv = \int_P \mathbf{x}\times\rho\,\dot{\mathbf{v}}\,dv \quad ,$$

$$\int_{\partial P}\mathbf{x}\times\mathbf{t}\,da = \int_P [\mathbf{x}\times\text{div}\,\mathbf{T} + \mathbf{e}_j\times\mathbf{T}\,\mathbf{e}_j\,]\,dv \quad . \tag{4.23}$$

Then, (4.21) can be rewritten in the form

$$\int_P [\,\mathbf{x}\times\{\rho\,\dot{\mathbf{v}} - \rho\,\mathbf{b} - \text{div}\,\mathbf{T}\} - \mathbf{e}_j\times\mathbf{T}\,\mathbf{e}_j\,]\,dv = 0 \quad . \tag{4.24}$$

Again, assuming that the integrand is continuous, that this equation is valid for arbitrary parts P, and using the local form (4.13) of the balance of linear momentum, it follows that the local form of the balance of angular momentum becomes

$$\mathbf{e}_j\times\mathbf{T}\,\mathbf{e}_j = 0 \quad . \tag{4.25}$$

Moreover, it can be shown that this restriction requires the stress tensor to be symmetric

$$\mathbf{T}^T = \mathbf{T} \quad , \quad T_{ij} = T_{ji} \quad . \tag{4.26}$$

BALANCE OF ENERGY (FIRST LAW OF THERMODYNAMICS)

The balance of energy is usually called the first law of thermodynamics. In words, it states that the rate of change of internal energy and kinetic energy equals the total rate of

external work supplied to the body plus the total rate of external heat supplied to the body. It is important to emphasize that the first law of thermodynamics expresses the equivalence of the rates of work and heat.

In order to express this physical law in mathematical terms it is necessary to introduce a few more variables that characterize the state of the material. To this end, let ε be the specific internal energy and let $\mathcal{E}$ be the total internal energy in the part P

$$\mathcal{E} = \int_P \rho\, \varepsilon \, dv \ ; \tag{4.27}$$

let $\mathcal{K}$ be the total kinetic energy in the part P

$$\mathcal{K} = \int_P \frac{1}{2} \rho\, \mathbf{v} \bullet \mathbf{v} \, dv \ ; \tag{4.28}$$

let $\mathcal{W}$ be the total external rate of work done on the part P of the body due to body forces and surface tractions

$$\mathcal{W} = \int_P \rho\, \mathbf{b} \bullet \mathbf{v} \, dv + \int_{\partial P} \mathbf{t} \bullet \mathbf{v} \, da \ ; \tag{4.29}$$

and let $\mathcal{H}$ be the total external rate of heat supplied to the part P of the body due to the specific external rate of heat supply r (e.g. radiation) and due to the heat flux **q** vector per unit area of ∂P

$$\mathcal{H} = \int_P \rho\, r \, dv - \int_{\partial P} \mathbf{q} \bullet \mathbf{n} \, da \ . \tag{4.30}$$

The minus sign is used here because $\mathbf{q} \bullet \mathbf{n}$ denotes the heat flux in the direction of **n**, which indicates that the heat is expelled from the body instead of supplied to the body.

Using these definitions, the balance of energy becomes

$$\dot{\mathcal{E}} + \dot{\mathcal{K}} = \mathcal{W} + \mathcal{H} \ . \tag{4.31}$$

Also, using the divergence theorem it can be shown that

$$\int_{\partial P} \mathbf{t} \bullet \mathbf{v} \, da = \int_{\partial P} \mathbf{v} \bullet \mathbf{T}\, \mathbf{n} \, da = \int_P \operatorname{div} (\mathbf{v} \bullet \mathbf{T}) \, dv \ . \tag{4.32}$$

However,

$$\operatorname{div} (\mathbf{v} \bullet \mathbf{T}) = (\mathbf{v} \bullet \mathbf{T})_{,j} \bullet \mathbf{e}_j = \mathbf{v} \bullet \mathbf{T}_{,j} \bullet \mathbf{e}_j + \mathbf{v}_{,j} \bullet \mathbf{T}\, \mathbf{e}_j \ ,$$

$$\operatorname{div} (\mathbf{v} \bullet \mathbf{T}) = \mathbf{v} \bullet \operatorname{div} \mathbf{T} + \mathbf{T} \bullet \mathbf{L} \ , \tag{4.33}$$

where **L** is the velocity gradient

$$\mathbf{L} = \partial \mathbf{v}/\partial \mathbf{x} \quad . \tag{4.34}$$

Moreover, since **T** is symmetric

$$\mathbf{T} \bullet \mathbf{L} = \mathbf{T} \bullet \mathbf{D} \quad , \tag{4.35}$$

where **D** is the symmetric part of the velocity gradient

$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) = \mathbf{D}^T \quad . \tag{4.36}$$

Thus, following similar arguments to those used to develop the local form of the balance of linear momentum and using the results (4.33) and (4.36), it can be shown that

$$\dot{\mathcal{E}} = \int_P \rho \, \dot{\varepsilon} \, dv \quad , \quad \dot{\mathcal{K}} = \int_P \mathbf{v} \bullet \rho \, \dot{\mathbf{v}} \, dv \quad ,$$

$$\int_{\partial P} \mathbf{t} \bullet \mathbf{v} \, da = \int_P \left[\mathbf{v} \bullet \operatorname{div} \mathbf{T} + \mathbf{T} \bullet \mathbf{D} \right] dv \quad , \quad \int_{\partial P} \mathbf{q} \bullet \mathbf{n} \, da = \int_P \operatorname{div} \mathbf{q} \, dv \quad , \tag{4.37}$$

so that the balance of energy (4.31) can be written in the form

$$\int_P \left[\, \mathbf{v} \bullet \{ \rho \, \dot{\mathbf{v}} - \rho \, \mathbf{b} - \operatorname{div} \mathbf{T} \} + \{ \rho \, \dot{\varepsilon} - \rho \, r + \operatorname{div} \mathbf{q} - \mathbf{T} \bullet \mathbf{D} \} \, \right] dv = 0 \quad . \tag{4.38}$$

Again, assuming that the integrand is continuous, that this equation is valid for arbitrary parts P, and using the local form (4.13) of the balance of linear momentum, it can be shown that the local form of the balance of energy becomes

$$\rho \, \dot{\varepsilon} = \rho \, r - \operatorname{div} \mathbf{q} + \mathbf{T} \bullet \mathbf{D} \quad , \tag{4.39}$$

which must be satisfied at each point of P. Furthermore, if ε, r, **T** are small quantities, then ρ can be replaced by ρ_0, **D** can be replaced by the strain rate $\dot{\mathbf{e}}$, and (4.39) reduces to

$$\rho_0 \, \dot{\varepsilon} = \rho_0 \, r - \operatorname{div} \mathbf{q} + \mathbf{T} \bullet \dot{\mathbf{e}} \quad , \quad \rho_0 \, \dot{\varepsilon} = \rho_0 \, r - q_{j,j} + T_{ij} \, \dot{e}_{ij} \quad , \tag{4.40}$$

where q_j are the components of **q** relative to the base vectors $\mathbf{e}_j$.

THE REDUCED ENERGY EQUATION

Next, the absolute temperature θ is introduced and the external rate of entropy supply s and entropy flux **p** are related to the external rate of heat supply r and heat flux **q** by the relations

$$s = \frac{r}{\theta} \quad , \quad \mathbf{p} = \frac{\mathbf{q}}{\theta} \quad . \tag{4.41}$$

It then follows that

$$\text{div}\,\mathbf{p} = \mathbf{p}_{,j} \bullet \mathbf{e}_j = \frac{\mathbf{q}_{,j} \bullet \mathbf{e}_j}{\theta} - \frac{\mathbf{q}}{\theta^2} \bullet \theta_{,j}\,\mathbf{e}_j = \frac{\text{div}\,\mathbf{q}}{\theta} - \frac{\mathbf{p} \bullet \mathbf{g}}{\theta} \quad , \tag{4.42}$$

where $\mathbf{g}$ is the temperature gradient

$$\mathbf{g} = \partial\theta/\partial\mathbf{x} = \theta_{,j}\,\mathbf{e}_j \ . \tag{4.43}$$

Now, using the balance of entropy (4.19) and the expressions (4.41) and (4.43), it can be shown that

$$\rho\, r - \text{div}\,\mathbf{q} = \rho\theta\dot{\eta} - \mathbf{p} \bullet \mathbf{g} - \rho\theta\xi \quad , \tag{4.44}$$

Moreover, it is convenient to separate the internal rate of production of entropy ξ into two parts (Rubin, 1992)

$$\rho\theta\xi = -\,\mathbf{p} \bullet \mathbf{g} + \rho\theta\xi' \quad , \tag{4.45}$$

where the first term ($-\,\mathbf{p} \bullet \mathbf{g}$) is a thermal part related to entropy (or heat) flux and the second term is related to material dissipation. Then, (4.44) simplifies to

$$\rho\, r - \text{div}\,\mathbf{q} = \rho\theta\dot{\eta} - \rho\theta\xi' \quad . \tag{4.46}$$

Also, with the help of this expression, the energy equation (4.39) can be rewritten in the form

$$\rho\theta\xi' = \rho\theta\dot{\eta} - \rho\,\dot{\varepsilon} + \mathbf{T} \bullet \mathbf{D} \quad , \tag{4.47}$$

Since the derivative of the entropy appears in this equation, it is most appropriate to use this equation when entropy η is considered to be an independent variable. Alternatively, it is possible to introduce the definition of the Helmholtz free energy ψ

$$\psi = \varepsilon - \theta\eta \quad , \tag{4.48}$$

to rewrite (4.47) in the form

$$\rho\theta\xi' = -\,\rho\eta\dot{\theta} - \rho\dot{\psi} + \mathbf{T} \bullet \mathbf{D} \quad , \tag{4.49}$$

which is called the reduced energy equation. Specifically, the definition (4.48) transforms the equation (4.47) into one in which temperature θ is considered to be an independent variable. Furthermore, if ξ', η, ψ, and $\mathbf{T}$ are considered to be small

quantities, then ρ can be replaced by ρ_0 and $\mathbf{D}$ in (4.36) can be replaced by the strain rate $\dot{\mathbf{e}}$ to obtain the simplified form

$$\rho_0\theta\xi' = -\rho_0\eta\dot{\theta} - \rho_0\dot{\psi} + \mathbf{T}\bullet\dot{\mathbf{e}} \ , \quad \rho_0\theta\xi' = -\rho_0\eta\dot{\theta} - \rho_0\dot{\psi} + T_{ij}\,\dot{e}_{ij} \ . \tag{4.50}$$

In the remainder of this course attention will be focused on the forms of the equations which consider the strain $\mathbf{e}$ and the absolute temperature θ to be the independent variables. Moreover, it is noted that although the equation (4.50) has been developed using simplifications associated with small deformations, no assumptions have been used yet about the magnitude of the temperature changes. It will be seen later, that for the complete linearized theory, additional simplifications will be introduced which assume that the temperature θ remains close to its reference value θ_0.

For later convenience, the small deformation forms of the conservation of mass (4.4), the balance of linear momentum (4.14), and the balance of entropy are summarized here

$$\rho = \rho_0\,(1 - \mathbf{e}\bullet\mathbf{I}) \ , \quad \rho = \rho_0\,(1 - e_{mm}) \ , \tag{4.51a,b}$$

$$\rho_0\,\ddot{\mathbf{u}} = \rho_0\,\mathbf{b} + \mathrm{div}\,\mathbf{T} \ , \quad \rho_0\,\ddot{u}_i = \rho_0\,b_i + T_{ij,j} \ . \tag{4.51c,d}$$

$$\rho_0\,\dot{\eta} = \rho_0\,s - \mathrm{div}\,\mathbf{p} + \rho_0\,\xi \ , \quad \rho_0\,\dot{\eta} = \rho_0\,s - p_{j,j} + \rho_0\,\xi \ . \tag{4.51e,f}$$

Also, the reduced forms of the balance of angular momentum (4.26) and the balance of energy (4.50) are summarized here

$$\mathbf{T}^T = \mathbf{T} \ , \quad T_{ij} = T_{ji} \ . \tag{4.52a,b}$$

$$\rho_0\theta\xi' = -\rho_0\eta\dot{\theta} - \rho_0\dot{\psi} + \mathbf{T}\bullet\dot{\mathbf{e}} \ , \quad \rho_0\theta\xi' = -\rho_0\eta\dot{\theta} - \rho_0\dot{\psi} + T_{ij}\,\dot{e}_{ij} \ . \tag{4.52c,d}$$

Moreover, the quantities in these equations are related by the expressions (4.41), (4.43), (4.45) and (4.48), which are collected here

$$s = \frac{r}{\theta} \ , \quad \mathbf{p} = \frac{\mathbf{q}}{\theta} \ , \quad \mathbf{g} = \partial\theta/\partial\mathbf{x} = \theta_{,j}\,\mathbf{e}_j \ , \tag{4.53a,b,c}$$

$$\rho\theta\xi = -\mathbf{p}\bullet\mathbf{g} + \rho\theta\xi' \ , \quad \psi = \varepsilon - \theta\eta \ . \tag{4.53d,e}$$

In the thermodynamic procedures proposed by Green and Naghdi (1977,1978), the balance laws (4.51) are used to determine the density ρ, the displacement vector $\mathbf{u}$ and the temperature θ, and the balance laws (4.52) are used to place restrictions on

constitutive equations which will be described later. Alternatively, the energy equation (4.40)

$$\rho_0 \dot{\varepsilon} = \rho_0 \, r - \mathrm{div}\, \mathbf{q} + \mathbf{T} \bullet \dot{\mathbf{e}} \;, \quad \rho_0 \dot{\varepsilon} = \rho_0 \, r - q_{j,j} + T_{ij} \dot{e}_{ij} \;, \qquad (4.54a,b)$$

can be used instead of the balance of entropy to determine the temperature field.

5. Constitutive equations for an isotropic thermoelastic material within the context of the small deformation theory.

A thermoelastic material is considered to be an ideal material because it exhibits no material dissipation. Moreover, a material is said to be anisotropic if different samples, which are taken from different orientations relative to the material microstructure, exhibit different material responses. Single crystals of metal, silicone and composite materials are examples of such anisotropic materials. However, if all such different samples exhibit the same material response, then the material is said to be isotropic. For simplicity, attention will be confined in this course to the simplest case of isotropic thermoelastic materials within the context of the small deformation theory.

The constitutive equations for such materials can be developed by making the following assumptions:

(A1) The response functions

$$\{ \psi , \eta , \varepsilon , \xi' , \mathbf{T} \} \ , \tag{5.1}$$

depend only on the variables

$$\{ \mathbf{e} , \theta \} \ . \tag{5.2}$$

(A2) The response function

$$\mathbf{p} \ , \tag{5.3}$$

depends only on the variables (5.2) and on the temperature gradient

$$\mathbf{g} = \frac{\partial \theta}{\partial \mathbf{x}} \ . \tag{5.4}$$

Using the assumption (A1), the reduced form of the energy equation (4.52c) becomes

$$\rho_0 \theta \xi' = - \rho_0 [\ \eta + \frac{\partial \psi}{\partial \theta} \ ] \dot{\theta} + [\ \mathbf{T} - \rho_0 \frac{\partial \psi}{\partial \mathbf{e}} \ ] \bullet \dot{\mathbf{e}} \ . \tag{5.5}$$

Now, the assumption (A2) requires ξ' and the coefficients in square brackets to be explicitly independent of the rates

$$\{ \dot{\mathbf{e}} , \dot{\theta} \} \ . \tag{5.6}$$

Thus, since the reduced energy equation (5.5) must be valid for all thermomechanical processes, it follows that the constitutive equations for a thermoelastic material must satisfy the restrictions that

$$\eta = -\frac{\partial\psi}{\partial\theta} \ , \ \mathbf{T} = \rho_0 \frac{\partial\psi}{\partial\mathbf{e}} \ , \ \xi' = 0 \ . \tag{5.7a,b,c}$$

Thus, once a functional form for the Helmholtz free energy ψ is specified, the constitutive equations for the entropy η and the stress $\mathbf{T}$ are determined by mere differentiation. Also, since $\mathbf{e}$ is a symmetric tensor, the stress $\mathbf{T}$ given by (5.7b) is symmetric and thus automatically satisfies the restriction (4.52a) associated with the reduced form of angular momentum. Furthermore, the result (5.7c) proves that a thermoelastic material is nondissipative.

For an isotropic material, ψ must be an isotropic function of the strain $\mathbf{e}$, and $\mathbf{p}$ must be an isotropic function of the strain $\mathbf{e}$ and the temperature gradient $\mathbf{g}$. In particular, ψ can depend on $\mathbf{e}$ only through its invariants (see Appendix A), which can be taken to be

$$\mathbf{e} \bullet \mathbf{I} \ , \ \mathbf{e}' \bullet \mathbf{e}' \ , \ \det \mathbf{e} \ . \tag{5.8a,b,c}$$

For the simplest case, ψ is taken to be a quadratic function strain so the invariant (5.8c) is omitted. Also, $\mathbf{p}$ is taken to be independent of strain $\mathbf{e}$. Specifically, ψ and $\mathbf{p}$ are proposed in the forms

$$\rho_0\psi = \rho_0 C_v \left[(\theta-\theta_0) - \theta \ln(\theta/\theta_0) \right] + \frac{1}{2} K (\mathbf{e} \bullet \mathbf{I})^2 + \mu \, \mathbf{e}' \bullet \mathbf{e}' - 3 K\alpha (\theta-\theta_0) (\mathbf{e} \bullet \mathbf{I}) \ ,$$

$$\mathbf{p} = -\frac{\kappa}{\theta} \mathbf{g} \ , \tag{5.9a,b}$$

where C_v is the constant specific heat at constant deformation, θ_0 is the reference temperature, K is the constant bulk modulus, μ is the constant shear modulus, α is the constant coefficient of linear thermal expansion, and κ is the constant heat conduction coefficient. Notice that the term associated with C_v is purely thermal, the term associated with K is the strain energy of dilatational deformation, the term associated with μ is the strain energy of distortional deformation, and the term associated with α characterizes the coupled thermomechanical response to temperature and dilatation. Also, using (4.53b), the assumption (5.9b) leads to the usual form for Fourier heat conduction with

$$\mathbf{q} = -\kappa \, \mathbf{g} \ , \tag{5.10}$$

which indicates that heat flows in the direction parallel to the temperature gradient.

Now, using the definition (3.31) of the deviatoric strain tensor $\mathbf{e}'$, it can be shown that

$$\rho_0\dot{\psi} = -\left[\rho_0 C_v \ln(\theta/\theta_0) + 3K\alpha\,(\mathbf{e}\bullet\mathbf{I})\right]\dot{\theta}$$

$$+\left[\,K\{\mathbf{e}\bullet\mathbf{I} - 3\alpha(\theta-\theta_0)\}\mathbf{I} + 2\mu\,\mathbf{e}'\,\right]\bullet\dot{\mathbf{e}}\ . \qquad (5.11)$$

Thus, the entropy and stress associated with the constitutive assumptions (5.9) become

$$\rho_0\eta = \rho_0 C_v \ln(\theta/\theta_0) + 3K\alpha\,(\mathbf{e}\bullet\mathbf{I})\ , \qquad (5.12a)$$

$$\mathbf{T} = -\,p\,\mathbf{I} + \mathbf{T}'\ ,\quad p = -\,K\{\mathbf{e}\bullet\mathbf{I} - 3\alpha(\theta-\theta_0)\}\ ,\ \mathbf{T}' = 2\mu\,\mathbf{e}'\ , \qquad (5.12b,c,d)$$

where p is the pressure and $\mathbf{T}'$ is the deviatoric stress. Moreover, using the definition (4.53e) it can be shown that the internal energy ε associated with the constitutive assumption (5.9a) becomes

$$\rho_0\varepsilon = \rho_0 C_v\,(\theta-\theta_0) + 3K\alpha\theta_0\,(\mathbf{e}\bullet\mathbf{I}) + \frac{1}{2}K\,(\mathbf{e}\bullet\mathbf{I})^2 + \mu\,\mathbf{e}'\bullet\mathbf{e}'\ . \qquad (5.13)$$

It is clear from this functional form that C_v is the specific heat at constant deformation since when the strain remains constant $\dot{\varepsilon} = C_v\,\dot{\theta}$.

Often, an engineering approach is taken which generalizes the purely mechanical theory by defining the thermal strain $\mathbf{e}_\theta$ by

$$\mathbf{e}_\theta = \alpha(\theta-\theta_0)\,\mathbf{I}\ , \qquad (5.14)$$

and replacing the total strain $\mathbf{e}$ in the constitutive equation for stress by the quantity

$$\mathbf{e} - \mathbf{e}_\theta\ . \qquad (5.15)$$

However, if this is done in the (5.9a) for the Helmholtz free energy, then ψ is proposed in the form

$$\rho_0\psi = \rho_0\bar{\psi} = \rho_0 C_v\left[\,(\theta-\theta_0) - \theta\ln(\theta/\theta_0)\right]$$

$$+\frac{1}{2}K\{\mathbf{e}\bullet\mathbf{I} - 3\alpha(\theta-\theta_0)\}^2 + \mu\,\mathbf{e}'\bullet\mathbf{e}'\ , \qquad (5.16)$$

instead of the form (5.9a). Next, using this expression it can be shown that

$$\rho_0\dot{\bar{\psi}} = -\left[\rho_0 C_v \ln(\theta/\theta_0) + 3K\alpha\{(\mathbf{e}\bullet\mathbf{I}) - 3\alpha(\theta-\theta_0)\}\,\right]\dot{\theta}$$

$$+ \left[K\{\mathbf{e} \bullet \mathbf{I} - 3\alpha(\theta - \theta_0)\}\mathbf{I} + 2\mu\, \mathbf{e}' \right] \bullet \dot{\mathbf{e}} \ , \tag{5.17}$$

so that the constitutive equations for the entropy becomes

$$\rho_0 \eta = \rho_0 \bar{\eta} = \left[\rho_0 C_v \ln(\theta/\theta_0) + 3K\alpha\{(\mathbf{e} \bullet \mathbf{I}) - 3\alpha(\theta - \theta_0)\}\right] \ , \tag{5.18}$$

the stress is again given by (5.12b,c,d), and the internal energy becomes

$$\rho_0 \varepsilon = \rho_0 \bar{\varepsilon} = \rho_0 C_v(\theta - \theta_0) + \frac{1}{2} K\{\mathbf{e} \bullet \mathbf{I} - 3\alpha(\theta - \theta_0)\}\{\mathbf{e} \bullet \mathbf{I} + 3\alpha(\theta + \theta_0)\}$$

$$+ \mu\, \mathbf{e}' \bullet \mathbf{e}' \ . \tag{5.19}$$

However, since the expression (5.19) is more complicated than (5.13) and since the interpretation of C_v in (5.19) is not clear, the constitutive assumption (5.9a) is preferred over (5.16).

In addition, it is noted that for the purely mechanical theory of an anisotropic elastic material, the strain energy function (or Helmholtz free energy) is given by

$$\rho_0 \psi = \frac{1}{2} \mathbf{K} \bullet (\mathbf{e} \otimes \mathbf{e}) \ , \tag{5.20}$$

where **K** is a fourth order tensor having the following symmetries

$$\mathbf{K} = \mathbf{K}^T = {}^{LT}\mathbf{K} = \mathbf{K}^{T(2)} \ , \ K_{ijkl} = K_{ijlk} = K_{jikl} = K_{klij} \ . \tag{5.21}$$

Also, it can be shown that (5.7b) holds so that the stress is given by

$$\mathbf{T} = \rho_0 \frac{\partial \psi}{\partial \mathbf{e}} = \mathbf{K} \bullet \mathbf{e} \ , \ T_{ij} = K_{ijkl}\, e_{kl} \ . \tag{5.22}$$

Thus, the material properties of the this anisotropic material are determined by the stiffness tensor **K**. In general, since **K** is a fourth order tensor it has 3^4=81 independent components. However, the symmetry conditions (5.21) impose restrictions that reduce the number of independent components to 21

$$\begin{pmatrix} K_{1111} & K_{1112} & K_{1113} & K_{1122} & K_{1123} & K_{1133} & K_{1212} \\ K_{1213} & K_{1222} & K_{1223} & K_{1233} & K_{1313} & K_{1322} & K_{1323} \\ K_{1333} & K_{2222} & K_{2223} & K_{2233} & K_{2323} & K_{2333} & K_{3333} \end{pmatrix} \tag{5.23}$$

6. Summary of the basic equations

For convenience, the basic equations associated with the small deformation theory of a thermoelastic material are summarized as follows:

KINEMATICS

Strain-displacement relations (3.15)

$$\mathbf{e} = \frac{1}{2}\left[\partial\mathbf{u}/\partial\mathbf{x} + (\partial\mathbf{u}/\partial\mathbf{x})^T\right] = \mathbf{e}^T \ , \quad e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = e_{ji} \ . \tag{6.1a,b}$$

Deviatoric strain (3.31)

$$\mathbf{e}' = \mathbf{e} - \frac{1}{3}(\mathbf{e} \bullet \mathbf{I})\,\mathbf{I} \ , \quad e'_{ij} = e_{ij} - \frac{1}{3}\, e_{mm}\, \delta_{ij} \ . \tag{6.2c,d}$$

Compatibility (3.58)

$$e_{ij,mm} + e_{mm,ij} - e_{im,mj} - e_{jm,im} = 0 \ . \tag{6.3}$$

BALANCE LAWS

Conservation of mass (4.51a,b)

$$\rho = \rho_0\,(1 - \mathbf{e} \bullet \mathbf{I}) \ , \quad \rho = \rho_0\,(1 - e_{mm}) \ , \tag{6.4a,b}$$

Balance of linear momentum (4.51c,d)

$$\rho_0\,\ddot{\mathbf{u}} = \rho_0\,\mathbf{b} + \mathrm{div}\,\mathbf{T} \ , \quad \rho_0\,\ddot{u}_i = \rho_0\,b_i + T_{ij,j} \ . \tag{6.5a,b}$$

Balance of entropy (4.51e,f)

$$\rho_0\,\dot{\eta} = \rho_0\,s - \mathrm{div}\,\mathbf{p} + \rho_0\,\xi \ , \quad \rho_0\,\dot{\eta} = \rho_0\,s - p_{j,j} + \rho_0\,\xi \ . \tag{6.6a,b}$$

Balance of angular momentum (4.52a,b)

$$\mathbf{T}^T = \mathbf{T} \ , \quad T_{ij} = T_{ji} \ . \tag{6.7a,b}$$

Balance of energy (4.40)

$$\rho_0\,\dot{\varepsilon} = \rho_0\,r - \mathrm{div}\,\mathbf{q} + \mathbf{T} \bullet \dot{\mathbf{e}} \ , \quad \rho_0\,\dot{\varepsilon} = \rho_0\,r - q_{j,j} + T_{ij}\,\dot{e}_{ij} \ , \tag{6.8a,b}$$

CONSTITUTIVE EQUATIONS

Helmholtz free energy (5.9a)

$$\rho_0\psi = \rho_0 C_v \left[(\theta-\theta_0) - \theta \ln(\theta/\theta_0) \right] + \frac{1}{2} K (\mathbf{e} \bullet \mathbf{I})^2 + \mu\, \mathbf{e}' \bullet \mathbf{e}'$$

$$- 3\, K\alpha\, (\theta-\theta_0)\, (\mathbf{e} \bullet \mathbf{I}) \ , \tag{6.9a}$$

$$\rho_0\psi = \rho_0 C_v \left[(\theta-\theta_0) - \theta \ln(\theta/\theta_0) \right] + \frac{1}{2} K\, e_{mm}\, e_{nn} + \mu\, e'_{ij}\, e'_{ij}$$

$$- 3\, K\alpha\, (\theta-\theta_0)\, e_{mm} \ , \tag{6.9b}$$

Entropy flux vector (5.9b)

$$\mathbf{p} = -\frac{\kappa}{\theta}\, \mathbf{g} \ , \quad \mathbf{g} = \partial\theta/\partial\mathbf{x} \ , \quad p_i = -\frac{\kappa}{\theta}\, g_i \ , \quad g_i = \theta_{,i} \ . \tag{6.10a,b,c,d}$$

Heat flux vector (5.10)

$$\mathbf{q} = -\kappa\, \mathbf{g} \ , \quad \mathbf{g} = \partial\theta/\partial\mathbf{x} \ , \quad q_i = -\kappa\, g_i \ , \quad g_i = \theta_{,i} \ . \tag{6.11a,b,c,d}$$

Entropy (5.12a)

$$\rho_0\eta = \rho_0 C_v \ln(\theta/\theta_0) + 3K\alpha\, (\mathbf{e} \bullet \mathbf{I}) \ , \tag{6.12a}$$

$$\rho_0\eta = \rho_0 C_v \ln(\theta/\theta_0) + 3K\alpha\, e_{mm} \ , \tag{6.12b}$$

Internal rate of production of entropy (4.45), (5.7c)

$$\rho_0\theta\xi = -\, \mathbf{p} \bullet \mathbf{g} \ , \tag{6.13}$$

Stress (5.12b,c,d)

$$\mathbf{T} = -\, p\, \mathbf{I} + \mathbf{T}' \ , \quad p = -\, K\{\mathbf{e} \bullet \mathbf{I} - 3\alpha(\theta-\theta_0)\} \ , \quad \mathbf{T}' = 2\mu\, \mathbf{e}' \ , \tag{6.14a,b,c}$$

$$T_{ij} = -\, p\, \delta_{ij} + T'_{ij} \ , \quad p = -\, K\{e_{mm} - 3\alpha(\theta-\theta_0)\} \ , \quad T'_{ij} = 2\mu\, e'_{ij} \ . \tag{6.14d,e,f}$$

Internal energy (5.13)

$$\rho_0\varepsilon = \rho_0 C_v\, (\theta-\theta_0) + 3K\alpha\theta_0\, (\mathbf{e} \bullet \mathbf{I}) + \frac{1}{2} K\, (\mathbf{e} \bullet \mathbf{I})^2 + \mu\, \mathbf{e}' \bullet \mathbf{e}' \ . \tag{6.15}$$

SMALL TEMPERATURE VARIATIONS AND SMALL STRAINS

For small temperature variations and small strains it is possible to neglect quadratic terms in $\mathbf{e}$ and $(\theta-\theta_0)$ to obtain the following simplified constitutive equations

Entropy flux vector

$$\mathbf{p} = -\frac{\kappa}{\theta_0}\ \partial\theta/\partial\mathbf{x}\ , \qquad p_i = -\frac{\kappa}{\theta_0}\ \theta_{,i}\ . \tag{6.16a,b}$$

Entropy

$$\rho_0\theta_0\eta = \rho_0 C_v\,(\theta-\theta_0) + 3K\alpha\theta_0\,(\mathbf{e}\bullet\mathbf{I})\ , \tag{6.17a}$$

$$\rho_0\theta_0\eta = \rho_0 C_v\,(\theta-\theta_0) + 3K\alpha\theta_0\, e_{mm}\ , \tag{6.17b}$$

Internal rate of production of entropy (4.45), (5.7c)

$$\rho_0\theta_0\xi = 0\ , \tag{6.18}$$

Internal energy

$$\rho_0\varepsilon = \rho_0 C_v\,(\theta-\theta_0) + 3K\alpha\theta_0\,(\mathbf{e}\bullet\mathbf{I})\ . \tag{6.19}$$

LINEARIZED HEAT EQUATION

It then follows from (6.16)-(6.19), that the balance of entropy (6.6) and the balance of energy (6.8) reduce to the same linearized equation

$$\rho_0\,\dot{\varepsilon} = \rho_0\, r - \mathrm{div}\,\mathbf{q}\ , \quad \rho_0\,\dot{\varepsilon} = \rho_0\, r - q_{j,j}\ , \tag{6.20a,b}$$

where $\theta_0 s$ has been approximated by r, and the quadratic term $\mathbf{T}\bullet\dot{\mathbf{e}}$ has been neglected in the energy equation.

SUMMARY OF THE LINEARIZED THEORY

Using the above approximations the main balance laws and constitutive equation used to determine the mass density ρ, the displacements u_i, and the temperature can be summarized as follows.

<u>Balance laws</u>

$$\rho = \rho_0\,(1 - \mathbf{e}\bullet\mathbf{I})\ , \quad \rho = \rho_0\,(1 - e_{mm})\ , \tag{6.21a,b}$$

$$\rho_0\,\ddot{\mathbf{u}} = \rho_0\,\mathbf{b} + \mathrm{div}\,\mathbf{T}\ , \quad \rho_0\,\ddot{u}_i = \rho_0\, b_i + T_{ij,j}\ , \tag{6.21c,d}$$

$$\rho_0\,\dot{\varepsilon} = \rho_0\, r - \mathrm{div}\,\mathbf{q}\ , \quad \rho_0\,\dot{\varepsilon} = \rho_0\, r - q_{j,j}\ . \tag{6.21e,f}$$

<u>Constitutive equations</u>

$$\mathbf{T} = -\,p\,\mathbf{I} + \mathbf{T}'\ , \quad p = -\,K\{\mathbf{e}\bullet\mathbf{I} - 3\alpha(\theta-\theta_0)\}\ , \quad \mathbf{T}' = 2\mu\,\mathbf{e}'\ , \tag{6.22a,b,c}$$

$$T_{ij} = -\,p\,\delta_{ij} + T'_{ij} \;,\quad p = -\,K\{e_{mm} - 3\alpha(\theta-\theta_0)\}\;,\quad T'_{ij} = 2\mu\, e'_{ij} \;,\qquad (6.22d,e,f)$$

$$\mathbf{q} = -\,\kappa\,\partial\theta/\partial\mathbf{x} \;,\quad q_i = -\,\kappa\,\theta_{,i} \;,\qquad (6.22g,h)$$

$$\rho_0\varepsilon = \rho_0 C_v\,(\theta-\theta_0) + 3K\alpha\theta_0\,(\mathbf{e}\bullet\mathbf{I})\;,\quad \rho_0\varepsilon = \rho_0 C_v\,(\theta-\theta_0) + 3K\alpha\theta_0\,e_{mm}\;.\qquad (6.22i,j)$$

MATERIAL CONSTANTS

Table 6.1 lists material constants for a few materials. The values of $\{\rho_0\,, K\,, \mu\}$ were taken from p. 201 of Kolsky (1963), the values of $\{\alpha\,, C_v\,, \kappa\}$ were taken from p. D-185 of the CRC Handbook of Chemistry and Physics (1988), and the values of $\{Y\,, \sigma_T\}$ were taken from Ashby and Jones (1995). Here, σ_T is the tensile strength for brittle materials.

Material Property	Steel (Iron)	Aluminum	Silicon (Glass)
ρ_0 (Mg/m^3)	7.8	2.7	2.5
K (GPa)	167.0	73.0	47.0
μ (GPa)	81.0	26.0	28.0
E (GPa)*	209.0	69.7	70.1
ν *	0.291	0.341	0.251
Y (GPa)	0.220	0.27	—
σ_T(GPa)	—	—	7.20
α (K^{-1})	12.0×10^{-6}	25.0×10^{-6}	3.0×10^{-6}
C_v (kJ/kg/K)	0.452	0.900	0.712
κ (J/s/K/m)	80.3	237.0	83.5

Table 6.1 Material constants for steel, aluminum and silicon.

*Calculated using the formulas presented in Table 9.1.

7. Initial and boundary conditions, Saint Venant's principle

In general, the number of initial conditions and the type of boundary conditions required will depend on the specific type of material under consideration. However, for the thermoelastic material under consideration these initial and boundary conditions are quite clear.

To this end, it is recalled that the local forms of balance of linear momentum (6.21c,d) and the balance of energy (6.21e,f) are partial differential equations which require both initial and boundary conditions. Specifically, the balance of linear momentum (6.21c,d) is second order in time with respect to displacement **u** so that it is necessary to specify the initial value of **u** and the initial value of the velocity **v** at each point of the body

$$\mathbf{u}(\mathbf{x},0) = \bar{\mathbf{u}}(\mathbf{x}) \quad \text{on P for t =0} \ , \tag{7.1a}$$

$$\dot{\mathbf{u}}(\mathbf{x},0) = \bar{\mathbf{v}}(\mathbf{x}) \quad \text{on P for t = 0} \ , \tag{7.1b}$$

where $\bar{\mathbf{u}}(\mathbf{x})$ and $\bar{\mathbf{v}}(\mathbf{x})$ are specified function. Also, the balance of energy (6.21e,f) is first order in time with respect to the temperature θ and the displacement **u** so that it is necessary to specify the initial value of θ at each point of the body

$$\theta(\mathbf{x},0) = \bar{\theta}(\mathbf{x}) \quad \text{on P for t =0} \ , \tag{7.2}$$

where $\bar{\theta}(\mathbf{x})$ is a specified function.

Guidance for determining the appropriate form of the boundary conditions is usually obtained by considering the rate of work done by the stress vector and the rate of supply of heat in the balance of energy (4.31). From (4.6) and (4.29) it can be observed that $\mathbf{t} \bullet \dot{\mathbf{u}}$ is the rate of work per unit area ∂P done by the stress vector. Now, at each point of the surface ∂P it is possible to define a right-handed orthogonal coordinate system with base vectors $\{ \mathbf{s}_1 , \mathbf{s}_2 , \mathbf{n} \}$, where **n** is the unit outward normal to ∂P and $\mathbf{s}_1$ and $\mathbf{s}_2$ are orthogonal vectors tangent to ∂P. Then, with reference to this coordinate system it can be shown that

$$\mathbf{t} \bullet \dot{\mathbf{u}} = (\mathbf{t} \bullet \mathbf{s}_1)\,(\dot{\mathbf{u}} \bullet \mathbf{s}_1) + (\mathbf{t} \bullet \mathbf{s}_2)\,(\dot{\mathbf{u}} \bullet \mathbf{s}_2) + (\mathbf{t} \bullet \mathbf{n})\,(\dot{\mathbf{u}} \bullet \mathbf{n}) \ \text{ on } \ \partial P \ . \tag{7.3}$$

Thus, using this representation it is possible to define three types of boundary conditions

Kinematic: All three components of the velocity are specified

$$(\dot{\mathbf{u}} \bullet \mathbf{s}_1)\ ,\ (\dot{\mathbf{u}} \bullet \mathbf{s}_2)\ ,\ (\dot{\mathbf{u}} \bullet \mathbf{n}) \text{ specified on } \partial P \text{ for all } t \geq 0\ , \tag{7.4}$$

Kinetic: All three components of the stress vector are specified

$$(\mathbf{t} \bullet \mathbf{s}_1)\ ,\ (\mathbf{t} \bullet \mathbf{s}_2)\ ,\ (\mathbf{t} \bullet \mathbf{n}) \text{ specified on } \partial P \text{ for all } t \geq 0\ , \tag{7.5}$$

Mixed: Complementary components of both the velocity and the stress vector are specified

$$(\dot{\mathbf{u}} \bullet \mathbf{s}_1) \text{ or } (\mathbf{t} \bullet \mathbf{s}_1) \text{ specified on } \partial P \text{ for all } t \geq 0, \tag{7.6a}$$

$$(\dot{\mathbf{u}} \bullet \mathbf{s}_2) \text{ or } (\mathbf{t} \bullet \mathbf{s}_2) \text{ specified on } \partial P \text{ for all } t \geq 0\ , \tag{7.6b}$$

$$(\dot{\mathbf{u}} \bullet \mathbf{n}) \text{ or } (\mathbf{t} \bullet \mathbf{n}) \text{ specified on } \partial P \text{ for all } t \geq 0\ . \tag{7.6c}$$

Essentially, the complementary components $(\mathbf{t} \bullet \mathbf{s}_1)$,$(\mathbf{t} \bullet \mathbf{s}_2)$,$(\mathbf{t} \bullet \mathbf{n})$ are the responses to the motions $(\dot{\mathbf{u}} \bullet \mathbf{s}_1)$,$(\dot{\mathbf{u}} \bullet \mathbf{s}_2)$,$(\dot{\mathbf{u}} \bullet \mathbf{n})$, respectively. Therefore, it is important to emphasize that, for example, both $(\dot{\mathbf{u}} \bullet \mathbf{n})$ and $(\mathbf{t} \bullet \mathbf{n})$ cannot be specified at the same point of ∂P because this would mean that both the motion and the stress response can be specified independently of the material properties of the body. Notice also, that since the initial position of points on the boundary ∂P are specified by the initial condition (7.1a), the velocity boundary conditions (7.4) can be used to determine the position of the boundary for all time. This means that the kinematic boundary conditions (7.4) could also be characterized by specifying the position of points on the boundary for all time.

Next, it is observed from (4.30) that $(-\ \mathbf{q} \bullet \mathbf{n})$ is the rate of heat supplied to the body per unit area of ∂P. It then follows using the constitutive equation (5.10) that at each point of the surface ∂P two types of boundary conditions can be specified

Kinematic: The value of the temperature is specified

$$\theta \text{ specified on } \partial P \text{ for all } t \geq 0\ , \tag{7.7a}$$

Kinetic: The normal component of the heat flux vector is specified

$$\mathbf{q} \bullet \mathbf{n} \text{ specified on } \partial P \text{ for all } t \geq 0\ . \tag{7.7b}$$

STRESS TENSOR AND STRESS VECTOR

It is important to emphasize that the state of stress at a point in the body is characterized by the stress tensor $\mathbf{T}(\mathbf{x},t)$ which is a function of positions and time only. However, the boundary conditions (7.5) and (7.6) are specified in terms of components of the stress vector $\mathbf{t}(\mathbf{x},t;\mathbf{n})$, which is a function of position, time and the unit outward normal $\mathbf{n}$ to the surface ∂P at the point $\mathbf{x}$. Specifically, from (4.10) it follows that $\mathbf{t}$ is a linear function of and $\mathbf{n}$, such that

$$\mathbf{t}(\mathbf{x},t;\mathbf{n}) = \mathbf{T}(\mathbf{x},t)\,\mathbf{n} \quad , \quad t_i = T_{ij}\,n_j \quad . \tag{7.8}$$

This equation can also be written in the matrix form

$$\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{Bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} \quad , \tag{7.9}$$

where use has been made of the fact that $\mathbf{T}$ is a symmetric tensor. In particular, notice that if the values of t_i are given on a specific surface, then (7.9) represents only three equations in terms of the six stresses T_{ij}. This means that not all of the stress components can be determined from boundary conditions on a single surface. For example, consider the surface whose outward normal is $\mathbf{n} = \mathbf{e}_1$. It then follows that

$$t_i = T_{ij}n_j = T_{i1} = (T_{11}, T_{21}, T_{31}) \quad \text{for } n_i = (1,0,0) \quad . \tag{7.10}$$

Thus, no information can be obtained about the components (T_{22}, T_{23}, T_{33}) of the stress tensor from this boundary condition.

SAINT VENANT 'S PRINCIPLE

The global forms of the balance of linear momentum (4.5) and the balance of angular momentum (4.21) depend on the net effect of the tractions $\mathbf{t}$ (stress vector) on the boundary ∂P of the body. In particular, with reference to the surface S which is part of the boundary ∂P, the resultant force $\mathbf{F}$ and moment $\mathbf{M}_0$ (about the point $\mathbf{x}_0$) applied by $\mathbf{t}$ can be written in the forms

$$\mathbf{F} = \int_S \mathbf{t}\,da \quad , \quad \mathbf{M}_0 = \int_S (\mathbf{x} - \mathbf{x}_0) \times \mathbf{t}\,da \quad . \tag{7.11}$$

Consequently, any distribution of the traction vector **t** which produces the same values for **F** and $\mathbf{M}_0$ will have the same net effect on the response of the body. Such distributions of traction vectors are called equipollent.

Saint Venant's principle states that:

> The differences between the solutions associated with two equipollent tractions vectors diminishes with distance from the boundary at which they are applied.

Therefore, an approximate solution of a boundary value problem can be obtained by replacing the specified traction vector field with another simpler equipollent field. This principle is used often in approximating boundary conditions like those associated with a clamped edge of a beam.

8. Superposition

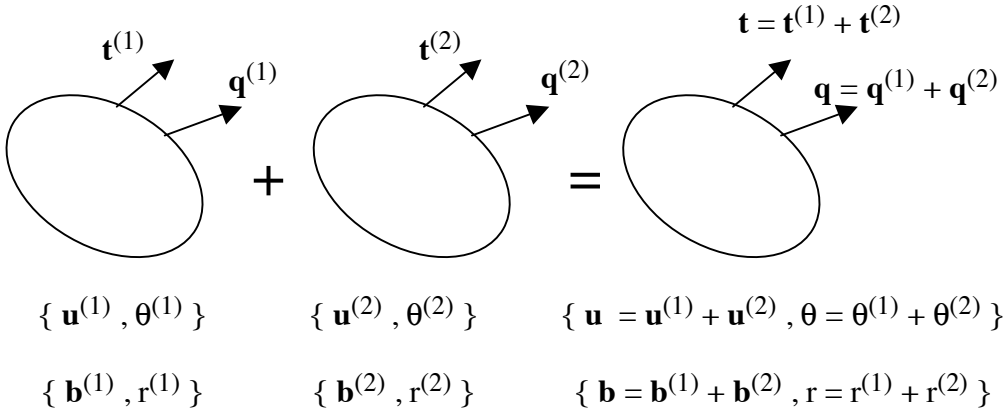

Fig. 8.1 Sketch of the principle of superposition.

A complete initial value and boundary value problem associated with thermoelasticity requires the determination of the displacement and temperature fields

$$\{ \mathbf{u} , \theta \} \quad , \tag{8.1}$$

and the kinetic quantities

$$\{ \rho , \mathbf{T} , \varepsilon , \mathbf{q} \} \quad , \tag{8.2}$$

which satisfy the balance laws (6.21), the constitutive equations (6.22), the initial conditions (7.1) and (7.2), and the boundary conditions (7.4)-(7.7) associated with a specific problem for specified values of the body force and the external heat supply

$$\{ \mathbf{b} , r \} \quad . \tag{8.3}$$

Since all of these equations are linear functions of the given variables, it follows that the principle of superposition holds (see Fig. 8.1). Specifically, the principle of superposition states that the sum of two solutions which satisfy the balance laws and the constitutive equations is also a solution. In particular, let the one solution be characterized by the displacement and temperature fields

$$\{ \mathbf{u}^{(1)} , \theta^{(1)} \} \quad , \tag{8.4}$$

the kinetic quantities

$$\{ \rho^{(1)} , \mathbf{T}^{(1)}, \varepsilon^{(1)} , \mathbf{q}^{(1)} \} \quad , \tag{8.5}$$

and the external fields

$$\{ \mathbf{b}^{(1)} , r^{(1)} \} \quad , \tag{8.6}$$

and let the second solution be characterized by the displacement and temperature fields

$$\{ \mathbf{u}^{(2)} \ , \theta^{(2)} \} \ , \tag{8.7}$$

the kinetic quantities

$$\{ \rho^{(2)} \ , \mathbf{T}^{(2)}, \varepsilon^{(2)} \ , \mathbf{q}^{(2)} \} \ , \tag{8.8}$$

and the external fields

$$\{ \mathbf{b}^{(2)} \ , r^{(2)} \} \ . \tag{8.9}$$

Then, the principle of superposition states that the displacement and temperature

$$\{ \mathbf{u} = \mathbf{u}^{(1)} + \mathbf{u}^{(2)} \ , \theta = \theta^{(1)} + \theta^{(2)} \} \ , \tag{8.10}$$

and the kinetic quantities

$$\{ \rho = \rho^{(1)} + \rho^{(2)} \ , \mathbf{T} = \mathbf{T}^{(1)} + \mathbf{T}^{(2)}, \varepsilon = \varepsilon^{(1)} + \varepsilon^{(2)}, \ \mathbf{q} = \mathbf{q}^{(1)} + \mathbf{q}^{(2)} \} \ , \tag{8.11}$$

satisfy the balance laws and the constitutive equations when the external fields are given by

$$\{ \mathbf{b} = \mathbf{b}^{(1)} + \mathbf{b}^{(2)} \ , r = r^{(1)} + r^{(2)} \} \ . \tag{8.12}$$

Moreover, the traction vector on the boundary ∂P of the body is given by

$$\mathbf{t} = \mathbf{t}^{(1)} + \mathbf{t}^{(2)} \ . \tag{8.13}$$

It will be shown through examples that the main use of the principle of superposition is to combine a number of known solutions to obtain a solution of a particular set boundary conditions.

9. Simple examples with quasi-static uniform fields: Free thermal expansion, uniaxial stress, uniaxial strain, simple shear, adiabatic processes, restrictions on material constants

In this section, attention is confined to quasi-static uniform fields for which the strain **e** and the temperature θ are independent of the position **x**. For quasi-static response the inertia is neglected. Consequently, in the absence of body forces (**b** = 0) the balance of linear momentum reduces to (6.21c)

$$\text{div}\,\mathbf{T} = 0 \;\;,\;\; T_{ij,j} = 0 \;\;. \tag{9.1}$$

Now, for uniform fields, the constitutive equations (6.22) indicate that the stress **T** is independent of position so that the balance law (9.1) is satisfied. Also, the constitutive equations indicate that the heat flux vector **q** vanishes so that the balance of energy (6.21e) reduces to

$$\dot{\varepsilon} = r \;\;. \tag{9.2}$$

When **T** is nonzero it is necessary to apply appropriate surface tractions **t** to the boundary ∂P of the body. Also, since **q** vanishes, no heat flows through the boundary ∂P. However, when ε is nonzero, it can be observed from the balance law (9.2) that heat must be supplied or extracted by r.

Since the strains e_{ij} are independent of position, it follows from (3.39) and (3.40) that the strain-displacement relations (6.1) can be integrated to deduce that

$$u_i = c_i + \omega_{ij}\, x_j + e_{ij}\, x_j \;\;,$$

$$u_1 = c_1 + e_{11}\, x_1 + (\omega_{12} + e_{12})\, x_2 + (\omega_{13} + e_{13})\, x_3 \;\;,$$

$$u_2 = c_2 + (-\omega_{12} + e_{12})\, x_1 + e_{22}\, x_2 + (\omega_{23} + e_{23})\, x_3 \;\;,$$

$$u_3 = c_3 + (-\omega_{13} + e_{13})\, x_1 + (-\omega_{23} + e_{23})\, x_2 + e_{33}\, x_3 \;\;, \tag{9.3}$$

where the constants c_i represents rigid-body translation and the constants ω_{ij} represents rigid-body rotation.

Before considering a number of special cases it is convenient to develop some results for the constitutive equations which are valid for general thermoelastic problems including dynamics and inhomogeneous deformations. Specifically, with the help of (6.2) the constitutive equation for stress becomes

$$T_{ij} = K\{e_{mm} - 3\alpha(\theta-\theta_0)\}\delta_{ij} + 2\mu\left[e_{ij} - \frac{1}{3}\, e_{mm}\,\delta_{ij}\right] ,$$

$$T_{ij} = K\left[(1 - \frac{2\mu}{3K})\, e_{mm} - 3\alpha(\theta-\theta_0)\right]\delta_{ij} + 2\mu\, e_{ij} \ . \tag{9.4}$$

This equation can be solved for the strain by first multiplying it by δ_{ij} to deduce that

$$T_{mm} = 3K\left[e_{mm} - 3\alpha(\theta-\theta_0)\right] \ , \ e_{mm} = 3\alpha(\theta-\theta_0) + \frac{T_{mm}}{3K} \ , \tag{9.5}$$

and then using (9.4) to obtain

$$e_{ij} = \frac{T_{ij}}{2\mu} - (1 - \frac{2\mu}{3K})\frac{T_{mm}}{6\mu}\,\delta_{ij} + \alpha(\theta-\theta_0)\,\delta_{ij} \ . \tag{9.6}$$

Also, equation (9.4) can then be expanded to yield

$$T_{11} = (K + \frac{4\mu}{3})\, e_{11} + (K - \frac{2\mu}{3})\, e_{22} + (K - \frac{2\mu}{3})\, e_{33} - 3K\alpha(\theta-\theta_0) \ ,$$

$$T_{22} = (K - \frac{2\mu}{3})\, e_{11} + (K + \frac{4\mu}{3})\, e_{22} + (K - \frac{2\mu}{3})\, e_{33} - 3K\alpha(\theta-\theta_0) \ ,$$

$$T_{33} = (K - \frac{2\mu}{3})\, e_{11} + (K - \frac{2\mu}{3})\, e_{22} + (K + \frac{4\mu}{3})\, e_{33} - 3K\alpha(\theta-\theta_0) \ ,$$

$$T_{12} = 2\mu\, e_{12} \ , \ T_{13} = 2\mu\, e_{13} \ , \ T_{23} = 2\mu\, e_{23} \ , \tag{9.7}$$

and equation (9.6) can be expanded to deduce that

$$e_{11} = (2 + \frac{2\mu}{3K})\frac{T_{11}}{6\mu} - (1 - \frac{2\mu}{3K})\frac{T_{22}}{6\mu} - (1 - \frac{2\mu}{3K})\frac{T_{33}}{6\mu} + \alpha(\theta-\theta_0) \ ,$$

$$e_{22} = -(1 - \frac{2\mu}{3K})\frac{T_{11}}{6\mu} + (2 + \frac{2\mu}{3K})\frac{T_{22}}{6\mu} - (1 - \frac{2\mu}{3K})\frac{T_{33}}{6\mu} + \alpha(\theta-\theta_0) \ ,$$

$$e_{33} = -(1 - \frac{2\mu}{3K})\frac{T_{11}}{6\mu} - (1 - \frac{2\mu}{3K})\frac{T_{22}}{6\mu} + (2 + \frac{2\mu}{3K})\frac{T_{33}}{6\mu} + \alpha(\theta-\theta_0) \ ,$$

$$e_{12} = \frac{T_{12}}{2\mu} \ , \ e_{13} = \frac{T_{13}}{2\mu} \ , \ e_{23} = \frac{T_{23}}{2\mu} \ . \tag{9.8}$$

However, it is usually more convenient to define Young's modulus of elasticity E and Poisson's ratio ν, such that (9.7) can be rewritten as

$$e_{11} = \frac{T_{11}}{E} - \frac{\nu T_{22}}{E} - \frac{\nu T_{33}}{E} + \alpha(\theta-\theta_0) \ ,$$

$$e_{22} = -\frac{\nu T_{11}}{E} + \frac{T_{22}}{E} - \frac{\nu T_{33}}{E} + \alpha(\theta-\theta_0) \ ,$$

$$e_{33} = -\frac{\nu T_{11}}{E} - \frac{\nu T_{22}}{E} + \frac{T_{33}}{E} + \alpha(\theta-\theta_0) \ ,$$

$$e_{12} = \frac{T_{12}}{2\mu} \ , \ e_{13} = \frac{T_{13}}{2\mu} \ , \ e_{23} = \frac{T_{23}}{2\mu} \ . \tag{9.9}$$

Thus, comparison of (9.8) and (9.9) yields relationships between the material constants of the forms

$$E = \frac{9K\mu}{3K+\mu} \ , \ \nu = \frac{(3K-2\mu)}{2(3K+\mu)} \ ,$$

$$E = 2(1+\nu)\mu \ , \quad K = \frac{E}{3(1-2\nu)} \ ,$$

$$K + \frac{4\mu}{3} = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \ , \ K - \frac{2\mu}{3} = \frac{\nu E}{(1+\nu)(1-2\nu)} \ . \tag{9.10}$$

Next, the constitutive equation (6.22j) for the energy is expanded to obtain

$$\rho_0\varepsilon = \rho_0 C_v (\theta-\theta_0) + 3K\alpha\theta_0 (e_{11} + e_{22} + e_{33}) \ ,$$

$$e_{11} + e_{22} + e_{33} = \frac{\rho_0\{\varepsilon - C_v(\theta-\theta_0)\}}{3K\alpha\theta_0} \ . \tag{9.11}$$

FREE THERMAL EXPANSION

For free thermal expansion, the stress T_{ij} vanishes

$$T_{ij} = 0 \ , \tag{9.12}$$

and (9.9) yields

$$e_{11} = \alpha(\theta-\theta_0) \ , \ e_{22} = \alpha(\theta-\theta_0) \ , \ e_{33} = \alpha(\theta-\theta_0) \ ,$$

$$e_{12} = e_{13} = e_{23} = 0 \ . \tag{9.13}$$

Moreover, it follows from (3.16) that the strain of all materials fibers are equal

$$E = e_{ij} N_i N_j = \alpha(\theta-\theta_0) \ . \tag{9.14}$$

For this reason α is called the coefficient of thermal expansion. Also, (9.11) yields

$$\rho_0\varepsilon = [\rho_0 C_v + 9K\alpha^2\theta_0] (\theta-\theta_0) \ . \tag{9.15}$$

UNIAXIAL STRESS

For uniaxial stress in the $\mathbf{e}_1$ direction, the only nonzero component of stress is T_{11}

$$T_{11} \neq 0 \ , \ \text{all other } T_{ij} = 0 \ , \tag{9.16}$$

so that (9.9) yields

$$e_{11} = \frac{T_{11}}{E} + \alpha(\theta-\theta_0) \ , \ e_{22} = e_{33} = -\frac{\nu T_{11}}{E} + \alpha(\theta-\theta_0) \ ,$$

$$e_{12} = e_{13} = e_{23} = 0 \ . \tag{9.17}$$

In particular, notice that the lateral strains e_{22} and e_{33} are equal and nonzero. Also, (9.11) yields

$$\rho_0\varepsilon = \left[\rho_0 C_v + 9K\alpha^2\theta_0\right](\theta-\theta_0) + \left[3K\alpha\theta_0\right]\frac{(1-2\nu)T_{11}}{E} \ . \tag{9.18}$$

For the simpler case when the temperature θ remains the reference temperature θ_0, these results reduce to

$$e_{11} = \frac{T_{11}}{E} \ , \ e_{22} = e_{33} = -\frac{\nu T_{11}}{E} \ , \ e_{12} = e_{13} = e_{23} = 0 \ ,$$

$$\rho_0\varepsilon = \left[3K\alpha\theta_0\right]\frac{(1-2\nu)T_{11}}{E} \ , \ \text{for } \theta=\theta_0 \ , \tag{9.19}$$

which show that tension ($T_{11}>0$) causes extension ($e_{11}>0$) in the axial direction, contraction ($e_{22}=e_{33}<0$) in the lateral direction, and increase in internal energy.

UNIAXIAL STRAIN

For uniaxial strain in the $\mathbf{e}_1$ direction, the only nonzero component of strain is e_{11}

$$e_{11} \neq 0 \ , \ \text{all other } e_{ij} = 0 \ , \tag{9.20}$$

so that (9.7) yields

$$T_{11} = (K + \frac{4\mu}{3})\, e_{11} - 3K\alpha(\theta-\theta_0) \ , \ T_{22} = T_{33} = (K - \frac{2\mu}{3})\, e_{11} - 3K\alpha(\theta-\theta_0) \ ,$$

$$T_{12} = T_{13} = T_{23} = 0 \ . \tag{9.21}$$

In particular, notice that the lateral stresses T_{22} and T_{33} are equal and nonzero. Also, (9.11) yields

$$\rho_0\varepsilon = \rho_0 C_v\,(\theta-\theta_0) + 3K\alpha\theta_0\, e_{11} \quad . \tag{9.22}$$

SIMPLE SHEAR

For simple shear in the $\mathbf{e}_1$–$\mathbf{e}_2$ plane, the only nonzero component of strain is e_{12}

$$e_{12} \neq 0 \quad , \text{ all other } e_{ij} = 0 \quad . \tag{9.23}$$

so that (9.7) yields

$$T_{11} = T_{22} = T_{33} = -\,3K\alpha(\theta-\theta_0) \quad , \quad T_{12} = 2\mu\, e_{12} \quad , \; T_{13} = T_{23} = 0 \quad . \tag{9.24}$$

Also, (9.11) yields

$$\rho_0\varepsilon = \rho_0 C_v\,(\theta-\theta_0) \quad . \tag{9.25}$$

In particular, notice that if the temperature θ remains the reference temperature θ_0, then T_{12} is the only nonzero component of stress

$$T_{12} = 2\mu\, e_{12} \quad , \text{ all other } T_{ij} = 0 \quad , \varepsilon = 0 \quad , \text{ for } \theta=\theta_0 \quad . \tag{9.26}$$

ADIABATIC PROCESSES

For adiabatic processes no external heat is supplied to the body so that r vanishes in the energy equation (9.2). Consequently, the internal energy ε remains zero, which means that the temperature is determined by the equation (9.11)

$$\varepsilon = 0 \quad , \; \theta = \theta_0 - \frac{3K\alpha\theta_0}{\rho_0 C_v}\,(e_{11} + e_{22} + e_{33}) \quad . \tag{9.27}$$

This result can then be substituted into the constitutive equation (9.7) for stress to deduce that

$$T_{11} = (\bar{K} + \frac{4\mu}{3})\, e_{11} + (\bar{K} - \frac{2\mu}{3})\, e_{22} + (\bar{K} - \frac{2\mu}{3})\, e_{33} \quad ,$$

$$T_{22} = (\bar{K} - \frac{2\mu}{3})\, e_{11} + (\bar{K} + \frac{4\mu}{3})\, e_{22} + (\bar{K} - \frac{2\mu}{3})\, e_{33} \quad ,$$

$$T_{33} = (\bar{K} - \frac{2\mu}{3})\, e_{11} + (\bar{K} - \frac{2\mu}{3})\, e_{22} + (\bar{K} + \frac{4\mu}{3})\, e_{33} \quad ,$$

$$T_{12} = 2\mu\, e_{12} \ , \ T_{13} = 2\mu\, e_{13} \ , \ T_{23} = 2\mu\, e_{23} \ , \tag{9.28}$$

where the constant $\bar{K}$ has been introduced for convenience

$$\bar{K} = K \left[1 + \frac{9K\alpha^2\theta_0}{\rho_0 C_v} \right] \ . \tag{9.29}$$

In particular, for an adiabatic process in uniaxial strain it follows that

$$T_{11} = (\bar{K} + \frac{4\mu}{3})\, e_{11} \ , \ T_{22} = (\bar{K} - \frac{2\mu}{3})\, e_{11} \ , \ T_{33} = (\bar{K} - \frac{2\mu}{3})\, e_{11} \ ,$$

$$T_{12} = T_{13} = T_{23} = 0 \ ,$$

$$\text{for } e_{11} \neq 0 \text{ , all other } e_{ij} = 0 \text{ , } \varepsilon = 0 \ . \tag{9.30}$$

RESTRICTIONS ON MATERIAL CONSTANTS

The material constants which characterize the response of a thermoelastic material

$$\{ \rho_0 \text{ , } K \text{ , } \mu \text{ , } \alpha \text{ , } C_v \} \ , \tag{9.31}$$

are specified at the reference temperature θ_0 which is usually specified by

$$\theta_0 = 300 \text{ K} \ . \tag{9.32}$$

(Here, the use of the symbol K for degrees Kelvin should not be confused with the use of the same symbol K for the bulk modulus.) Since these material constants model the response of real materials, they must satisfy certain physical restrictions. For example, for uniaxial stress at reference temperature (9.19), it is expected that the material fiber in the axial direction will extend (e_{11}>0) when the material is in tension (T_{11}>0), which requires Young's modulus to be positive

$$E > 0 \ . \tag{9.33}$$

Similarly, for simple shear (9.26), is expected that the material will shear in the direction of the shear stress so that the shear modulus must be positive

$$\mu > 0 \ . \tag{9.34}$$

These restrictions can be used together with the expressions (9.10) to show that the bulk modulus is positive

$$K > 0 \ , \tag{9.35}$$

and that Poisson's ratio is limited to the range

$$-1 < \nu < \frac{1}{2} \ . \tag{9.36}$$

In this regard, it is interesting to note that the restriction (9.35) can be alternatively obtained by requiring the part of the Helmholtz free energy (6.9) due to dilatational deformation to be positive definite

$$\frac{1}{2} K \, (\mathbf{e} \bullet \mathbf{I})^2 > 0 \quad \text{for} \ \ \mathbf{e} \neq 0 \ . \tag{9.37}$$

Also, the restriction (9.34) can be alternatively obtained by requiring the part of the Helmholtz free energy (6.9) due to distortional deformation to be positive definite

$$\mu \, \mathbf{e}' \bullet \mathbf{e}' > 0 \quad \text{for} \ \ \mathbf{e}' \neq 0 \ . \tag{9.38}$$

Moreover, it is expected that for zero strain (e_{ij}=0) the temperature of the material must increase if heat is added (r >0), which requires the specific heat at constant deformation to be positive

$$C_v > 0 \ . \tag{9.39}$$

RELATIONSHIPS BETWEEN ELASTIC CONSTANTS

Table 9.1 records the relationships between various pairs of elastic constants for isotropic elastic materials. In this table: λ is Lame's constant, μ is the shear modulus, E is Young's modulus, ν is Poisson's ratio, and K is the bulk modulus.

	λ	μ	E	ν	K
λ , μ			$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\frac{\lambda}{2(\lambda+\mu)}$	$\frac{3\lambda+2\mu}{3}$
λ , ν		$\frac{\lambda(1-2\nu)}{2\nu}$	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$		$\frac{\lambda(1+\nu)}{3\nu}$
λ , K		$\frac{3(K-\lambda)}{2}$	$\frac{9K(K-\lambda)}{3K-\lambda}$	$\frac{\lambda}{3K-\lambda}$	
μ , E	$\frac{\mu(2\mu-E)}{E-3\mu}$			$\frac{E-2\mu}{2\mu}$	$\frac{\mu E}{3(3\mu-E)}$
μ , ν	$\frac{2\mu\nu}{1-2\nu}$		$2\mu(1+\nu)$		$\frac{2\mu(1+\nu)}{3(1-2\nu)}$
μ , K	$\frac{3K-2\mu}{3}$		$\frac{9K\mu}{3K+\mu}$	$\frac{3K-2\mu}{2(3K+\mu)}$	
E, ν	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$			$\frac{E}{3(1-2\nu)}$
E , K	$\frac{3K(3K-E)}{9K-E}$	$\frac{3EK}{9K-E}$		$\frac{3K-E}{6K}$	
ν , K	$\frac{3K\nu}{1+\nu}$	$\frac{3K(1-2\nu)}{2(1+\nu)}$	$3K(1-2\nu)$		

$$\mu = \frac{(E-3\lambda)+\sqrt{(E-3\lambda)^2+8\lambda E}}{4} \quad , \qquad \nu = \frac{-(E+\lambda)+\sqrt{(E+\lambda)^2+8\lambda^2}}{4\lambda} \quad ,$$

$$K = \frac{(3\lambda+E)+\sqrt{(3\lambda+E)^2-4\lambda E}}{6}$$

Table 9.1 Relationships between various pairs of elastic constants.

10. Beltrami-Michell compatibility equations

For dynamic problem where inertia cannot be neglected it is necessary to solve the equations of linear momentum (6.21c,d) directly for the displacements u_i. Consequently, there is no need to check for compatibility since a displacement field exits. However, for static problems the equilibrium equation becomes

$$0 = \rho_0 b_i + T_{ij,j} \ , \tag{10.1}$$

which sometimes can be solved for the stresses without determining the displacements. Under these conditions it is necessary to limit the possible solutions for the stresses only to those stresses for which a displacement field exists. Specifically, it is necessary to ensure that the strain field associated with the proposed stress field satisfies the compatibility equations (3.58)

$$e_{ij,mm} + e_{mm,ij} - e_{im,mj} - e_{jm,im} = 0 \ . \tag{10.2}$$

To this end, it is noted that the expression (9.10) for E can be used to rewrite the constitutive equations (9.9) in the forms

$$e_{11} = \frac{T_{11}}{E} - \frac{\nu T_{22}}{E} - \frac{\nu T_{33}}{E} + \alpha(\theta - \theta_0) \ ,$$

$$e_{22} = -\frac{\nu T_{11}}{E} + \frac{T_{22}}{E} - \frac{\nu T_{33}}{E} + \alpha(\theta - \theta_0) \ ,$$

$$e_{33} = -\frac{\nu T_{11}}{E} - \frac{\nu T_{22}}{E} + \frac{T_{33}}{E} + \alpha(\theta - \theta_0) \ ,$$

$$e_{12} = \frac{(1+\nu)T_{12}}{E} \ , \ e_{13} = \frac{(1+\nu)T_{13}}{E} \ , \ e_{23} = \frac{(1+\nu)T_{23}}{E} \ , \tag{10.3}$$

Moreover, these equations can be written in the compact indicial form

$$e_{ij} = \frac{1}{E}\left[(1+\nu)\, T_{ij} - \nu\, T_{nn}\, \delta_{ij}\right] + \alpha(\theta - \theta_0)\, \delta_{ij} \ . \tag{10.4}$$

Next, with the help of this result, the compatibility equations (10.2) can be written in terms of the stresses and the temperature field in the forms

$$(1+\nu)\, T_{ij,mm} + T_{mm,ij} - (1+\nu)\left[T_{im,mj} + T_{jm,im}\right] - \nu\, T_{nn,mm}\, \delta_{ij}$$
$$+ E\alpha\left[\theta_{,ij} + \theta_{,mm}\, \delta_{ij}\right] = 0 \ . \tag{10.5}$$

Moreover, by contracting on ij it can be shown that

$$T_{nn,mm} = \frac{1+\nu}{1-\nu} T_{nm,mn} - \frac{2}{1-\nu} E\alpha\, \theta_{,mm} \quad , \tag{10.6}$$

which can be used to reduce (10.5) to the equation

$$(1+\nu)\, T_{ij,mm} + T_{mm,ij} - (1+\nu)\left[T_{im,mj} + T_{jm,im}\right] - \frac{\nu(1+\nu)}{(1-\nu)}\, T_{nm,mn}\, \delta_{ij}$$

$$+ E\alpha\left[\theta_{,ij} + \frac{1+\nu}{1-\nu}\, \theta_{,mm}\, \delta_{ij}\right] = 0 \quad . \tag{10.7}$$

Next, the equation (10.1) can be used to deduce the results

$$T_{im,mj} = -\rho_0\, b_{i,j} \quad , \quad T_{jm,im} = -\rho_0\, b_{j,i} \quad , \quad T_{nm,mn} = -\rho_0\, b_{m,m} \, , \tag{10.8}$$

so that (10.7) can be rewritten in the form

$$T_{ij,mm} + \left[\frac{1}{1+\nu}\right] T_{mm,ij} = -\rho_0 \left[b_{i,j} + b_{j,i} + \left\{\frac{\nu}{1-\nu}\right\} b_{m,m}\, \delta_{ij}\right]$$

$$- \left[\frac{E\alpha}{1+\nu}\right]\left[\theta_{,ij} + \frac{1+\nu}{1-\nu}\, \theta_{,mm}\, \delta_{ij}\right] \quad . \tag{10.9}$$

These equations are called the Beltrami-Michell compatibility equations.

For the special case when the body force is constant and the temperature gradient is constant

$$b_i = \text{constant} \quad , \quad \theta_{,i} = g_i = \text{constant} \quad , \tag{10.10}$$

the compatibility equations (10.9) reduce to

$$T_{ij,mm} + \left[\frac{1}{1+\nu}\right] T_{mm,ij} = 0 \quad , \tag{10.11}$$

which in expanded form become

$$T_{11,11} + T_{11,22} + T_{11,33} + \left[\frac{1}{1+\nu}\right]\left[T_{11,11} + T_{22,11} + T_{33,11}\right] = 0 \quad ,$$

$$T_{22,11} + T_{22,22} + T_{22,33} + \left[\frac{1}{1+\nu}\right]\left[T_{11,22} + T_{22,22} + T_{33,22}\right] = 0 \quad ,$$

$$T_{33,11} + T_{33,22} + T_{33,33} + \left[\frac{1}{1+\nu}\right]\left[T_{11,33} + T_{22,33} + T_{33,33}\right] = 0 \quad ,$$

$$T_{12,11} + T_{12,22} + T_{12,33} + \left[\frac{1}{1+\nu}\right]\left[T_{11,12} + T_{22,12} + T_{33,12}\right] = 0 \quad ,$$

$$T_{13,11} + T_{13,22} + T_{13,33} + [\frac{1}{1+\nu}][T_{11,13} + T_{22,13} + T_{33,13}] = 0 \ ,$$

$$T_{23,11} + T_{23,22} + T_{23,33} + [\frac{1}{1+\nu}][T_{11,23} + T_{22,23} + T_{33,23}] = 0 \ . \qquad (10.12)$$

11. Two-dimensional plane strain and generalized plane stress problems

Consider a thermoelastic body which has a right-cylindrical shape with a general lateral surface $\partial P'$, and flat bottom and top surfaces ∂P_1 and ∂P_2, respectively. If appropriate boundary conditions and body forces are applied then the response of this body can be purely planar. For these two-dimensional problems the displacements, temperature and stresses depend on only two space variables and time. More specifically, these two-dimensional problems can be either plane strain problems (which are special exact solutions of the three-dimensional equations) or they can be generalized stress problems (which often are only approximate solutions of the three-dimensional equations).

PLANE STRAIN PROBLEMS

For plane strain problems all field quantities are independent of one spatial coordinate which here is taken to be x_3. Specifically, the displacements u_i, the temperature θ, the stresses T_{ij}, and the body force b_i take the forms

$$u_\alpha = u_\alpha(x_\alpha,t) \ , \ u_3 = 0 \ , \ \theta = \theta(x_\alpha,t) \ ,$$
$$T_{\alpha\beta} = T_{\alpha\beta}(x_\alpha,t) \ , \ T_{3\alpha} = 0 \ , \ T_{33} = T_{33}(x_\alpha,t) \ ,$$
$$b_\alpha = b_\alpha(x_\alpha,t) \ , \ b_3 = 0 \ , \ \text{for } \alpha,\beta=1,2 \tag{11.1}$$

where for convenience, throughout the text Greek indices take only the values 1,2. Using the expressions (6.1), it then follows that the strain-displacement relations reduce to

$$e_{\alpha\beta} = e_{\alpha\beta}(x_\alpha,t) = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \ , \ e_{3i} = 0 \ . \tag{11.2a,b}$$

Moreover, the balance laws (6.21) become

$$\rho = \rho_0 (1 - e_{\sigma\sigma}) \ , \ \rho_0 \ddot{u}_\alpha = \rho_0 b_\alpha + T_{\alpha\beta,\beta} \ , \ \rho_0 \dot{\varepsilon} = \rho_0 r - q_{\sigma,\sigma} \ . \tag{11.3a,b,c}$$

Next, with the help of the expression for E in (9.10), the constitutive equation (10.4) can be rewritten in the form

$$e_{ij} = \frac{1}{2\mu}\left[T_{ij} - \frac{\nu}{1+\nu} T_{nn}\,\delta_{ij}\right] + \alpha(\theta-\theta_0)\,\delta_{ij} \ . \tag{11.4}$$

In particular, using (11.1) it follows that

$$e_{3\alpha} = 0 \ , \ e_{33} = \frac{1}{2\mu}\left[T_{33} - \frac{\nu}{1+\nu} T_{nn}\right] + \alpha(\theta-\theta_0) \ ,$$

$$e_{33} = \frac{1}{2\mu(1+\nu)}\left[T_{33} - \nu\, T_{\sigma\sigma}\right] + \alpha(\theta-\theta_0) \ , \tag{11.5}$$

where use has been made of the expression

$$T_{nn} = T_{\sigma\sigma} + T_{33} \ . \tag{11.6}$$

Thus, the strain e_{33} will vanish provided that T_{33} is given by

$$T_{33} = \nu\, T_{\sigma\sigma} - 2\mu(1+\nu)\alpha(\theta-\theta_0) \ , \tag{11.7}$$

so that (11.4) can be rewritten in the form

$$e_{\alpha\beta} = \frac{1}{2\mu}\left[T_{\alpha\beta} - \nu\, T_{\sigma\sigma}\,\delta_{\alpha\beta}\right] + (1+\nu)\alpha(\theta-\theta_0)\,\delta_{\alpha\beta} \ , \quad e_{3i} = 0 \ . \tag{11.8}$$

Moreover, this equation can be inverted and the constitutive equations for stress can be summarized as

$$T_{\alpha\beta} = 2\mu\left[e_{\alpha\beta} + \left\{\frac{\nu}{1-2\nu}\right\} e_{\sigma\sigma}\,\delta_{\alpha\beta}\right] - \frac{2\mu(1+\nu)}{(1-2\nu)}\,\alpha(\theta-\theta_0)\,\delta_{\alpha\beta} \ ,$$

$$T_{3\alpha} = 0 \ , \ T_{33} = \nu\, T_{\sigma\sigma} - 2\mu(1+\nu)\alpha(\theta-\theta_0) \ . \tag{11.9}$$

In particular, notice that for plane strain, a nonzero stress T_{33} is required to cause the strain e_{33} to vanish.

GENERALIZED PLANE STRESS

For generalized plane stress problems the displacements, temperature, stress and body force are independent of the variable x_3, such that

$$u_\alpha = u_\alpha(x_\alpha,t) \ , \ \theta = \theta(x_\alpha,t) \ , \ T_{\alpha\beta} = T_{\alpha\beta}(x_\alpha,t) \ , \ T_{3i} = 0 \ ,$$

$$b_\alpha = b_\alpha(x_\alpha,t) \ , \ b_3 = 0 \ , \ \text{for } \alpha,\beta=1,2 \ . \tag{11.10}$$

Next, the constitutive equations (11.4) can be written in the forms

$$e_{\alpha\beta} = \frac{1}{2\mu}\left[T_{\alpha\beta} - \frac{\nu}{1+\nu} T_{\sigma\sigma}\,\delta_{\alpha\beta}\right] + \alpha(\theta-\theta_0)\,\delta_{\alpha\beta} \ ,$$

$$e_{3\alpha} = 0 \ , \ \ e_{33} = -\frac{1}{2\mu}\left[\frac{\nu}{1+\nu}\right] T_{\sigma\sigma} + \alpha(\theta-\theta_0) \ , \tag{11.11}$$

which indicates that the strain e_{33} does not vanish. In this regard, it is important to emphasize that often generalized plane stress problems are only approximate solutions of the three-dimensional equations because a function for the displacement u_3 may not exist even though the displacements u_α do exit. Moreover, this equation can be inverted and the constitutive equations for stress can be summarized as

$$T_{\alpha\beta} = 2\mu\left[e_{\alpha\beta} + \left\{\frac{\nu}{1-\nu}\right\} e_{\sigma\sigma}\,\delta_{\alpha\beta}\right] - \frac{2\mu(1+\nu)}{(1-\nu)}\,\alpha(\theta-\theta_0)\,\delta_{\alpha\beta} \ , \ \ T_{3i} = 0 \ . \tag{11.12}$$

GENERAL TWO-DIMENSIONAL CONSTITUTIVE EQUATIONS

Comparison of the constitutive equations (11.8) and (11.9) for plane strain with (11.11) and (11.12) for generalized plane stress indicates the constitutive equations for both plane strain and generalized plane stress can be brought into a one-to-one correspondence by introducing a modified value $\bar{\nu}$ for Poisson's ratio and the modified value $\bar{\alpha}$ for the coefficient of thermal expansion, such that

$$\bar{\nu} = \nu \ \text{ and } \ \bar{\alpha} = \alpha \ \text{ for plane strain} \ ,$$

$$\bar{\nu} = \frac{\nu}{1+\nu} \ \text{ and } \ \bar{\alpha} = \left[\frac{1+\nu}{1+2\nu}\right]\alpha \ \text{ for generalized plane stress} \ , \tag{11.13}$$

where ν and α are the actual value of Poisson's ratio and the thermal coefficient of expansion of the three-dimensional material. More specifically, it follows from this definition that

$$(1+\bar{\nu})\bar{\alpha} = \alpha \ , \ \ \frac{\bar{\nu}}{1-2\bar{\nu}} = \frac{\nu}{1-\nu} \ , \ \ \frac{(1+\bar{\nu})}{(1-2\bar{\nu})}\,\bar{\alpha} = \frac{(1+\nu)}{(1-\nu)}\,\alpha \ . \tag{11.14}$$

Thus, with the help of the equations (11.8), (11.9), (11.11), (11.12) and the definitions (11.13), it can be shown that the general two-dimensional constitutive equations can be written in the forms

$$e_{\alpha\beta} = \frac{1}{2\mu}\left[T_{\alpha\beta} - \bar{\nu}\,T_{\sigma\sigma}\,\delta_{\alpha\beta}\right] + (1+\bar{\nu})\bar{\alpha}(\theta-\theta_0)\,\delta_{\alpha\beta} \ , \ \ e_{3\alpha} = 0 \ ,$$

$$T_{\alpha\beta} = 2\mu \left[e_{\alpha\beta} + \left\{\frac{\bar{\nu}}{1-2\bar{\nu}}\right\} e_{\sigma\sigma}\, \delta_{\alpha\beta}\right] - \frac{2\mu(1+\bar{\nu})}{(1-2\bar{\nu})}\, \bar{\alpha}(\theta-\theta_0)\, \delta_{\alpha\beta} \ , \ T_{3\alpha} = 0 \ . \quad (11.15)$$

Also, for plane strain

$$\bar{\nu} = \nu \ , \ \bar{\alpha} = \alpha \ ,$$

$$e_{33} = 0 \ , \ T_{33} = \nu\, T_{\sigma\sigma} - 2\mu(1+\nu)\alpha(\theta-\theta_0) \ , \quad (11.16)$$

whereas for generalized plane stress

$$\bar{\nu} = \frac{\nu}{1+\nu} \ , \ \bar{\alpha} = \left[\frac{1+\nu}{1+2\nu}\right] \alpha \ ,$$

$$e_{33} = -\frac{1}{2\mu}\left[\frac{\nu}{1+\nu}\right] T_{\sigma\sigma} + \alpha(\theta-\theta_0) \ , \ T_{33} = 0 \ . \quad (11.17)$$

Next, with the help of (6.22) it follows that the constitutive equations for the heat flux q_i and the internal energy ε for planar problems can be written as

$$q_\alpha = -\kappa\, \theta_{,\alpha} \ , \ q_3 = 0 \ , \quad (11.18)$$

and

$$\rho_0\varepsilon = \rho_0 C_v\, (\theta-\theta_0) + 3K\alpha\theta_0 \left[e_{\sigma\sigma} + e_{33}\right] . \quad (11.19)$$

Thus, for plane strain (11.19) reduces to

$$\rho_0\varepsilon = \rho_0 C_v\, (\theta-\theta_0) + 3K\alpha\theta_0 \left[e_{\sigma\sigma}\right] , \quad (11.20)$$

whereas for generalized plane stress it follows from (11.12) that

$$T_{\sigma\sigma} = 2\mu \left[\frac{1+\nu}{1-\nu}\right] \left[e_{\sigma\sigma} - 2\alpha(\theta-\theta_0)\right] \ , \quad (11.21)$$

so that (11.17) yields

$$e_{33} = -\left[\frac{\nu}{1-\nu}\right] \left[e_{\sigma\sigma} - 2\alpha(\theta-\theta_0)\right] + \alpha(\theta-\theta_0)$$

$$e_{\sigma\sigma} + e_{33} = \left[\frac{1-2\nu}{1-\nu}\right] e_{\sigma\sigma} + \left[\frac{1+\nu}{1-\nu}\right] \alpha(\theta-\theta_0) \ . \quad (11.22)$$

Consequently, substituting (11.22) into (11.19) yields the internal energy for generalized plane stress in the form

$$\rho_0\varepsilon = \left[\rho_0 C_v + 3K\alpha^2\theta_0\left\{\frac{1+\nu}{1-\nu}\right\}\right](\theta-\theta_0) + 3K\alpha\theta_0\left[\frac{1-2\nu}{1-\nu}\right] e_{\sigma\sigma} \ . \qquad (11.23)$$

INITIAL AND BOUNDARY CONDITIONS

The balance of linear momentum (11.3b) and the balance of energy (11.3c) are partial differential equations which are second order in time for the displacements u_α, first order in time for the temperature θ, and second order in space for both the displacements and the temperature. It therefore, follows that the initial conditions in the region P occupied by the body are specified by (7.1) and (7.2), whereas the boundary conditions used to solve the equations (11.3b) and (11.3c) are specified in forms similar to (7.4)-(7.7) on the lateral surface $\partial P'$ of P. More specifically, for two-dimensional problems the traction vector on the boundary of the body becomes

$$t_\alpha = T_{\alpha\beta}\, n_\beta \ , \ t_3 = 0 \ \text{ on } \partial P' \ ,$$

$$t_\alpha = 0 \ , \ t_3 = -T_{33} \ \text{ on } \partial P_1 \ , \ \ t_\alpha = 0 \ , \ t_3 = T_{33} \ \text{ on } \partial P_2 \ . \qquad (11.24)$$

12. Compatibility equations and Airy's stress function for two-dimensional problems

For two-dimensional dynamic problem where inertia cannot be neglected it is necessary to solve the equations of linear momentum (11.3b) directly for the displacements u_α. Consequently, there is no need to check for compatibility since a displacement field exits. However, for static problems the equilibrium equation becomes

$$\rho_0\, b_\alpha + T_{\alpha\beta,\beta} = 0 \quad , \tag{12.1}$$

which sometimes can be solved for the stresses without determining the displacements. Under these conditions it is necessary to limit the possible solutions for the stresses only to those stresses for which a displacement field exists. Specifically, it is necessary to ensure that the strain field $e_{\alpha\beta}$ associated with the proposed stress field satisfies the compatibility equations (3.58).

COMPATIBILITY EQUATIONS

Specifically, with the help of the expressions (11.2) it follows that the compatibility equations (3.58) reduce to

$$e_{\alpha\beta,\sigma\sigma} + e_{\sigma\sigma,\alpha\beta} - e_{\alpha\sigma,\sigma\beta} - e_{\beta\sigma,\sigma\alpha} = 0 \quad , \tag{12.2}$$

which restrict the inplane components of strain so that the displacements u_α exit. However, these three equations represent only one nontrivial equation since one of them (α=1,β=2) automatically vanishes and the other two equations (α=β=1; and α=β=2) are identical and require

$$e_{11,22} + e_{22,11} - 2e_{12,12} = 0 \quad , \tag{12.3}$$

which is the same as the first of (3.57).

Next, with the help of the constitutive equation (11.15) it can be shown that

$$e_{\alpha\beta} = \frac{1}{2\mu}\left[T_{\alpha\beta} - \bar{\nu}\, T_{\sigma\sigma}\, \delta_{\alpha\beta}\right] + (1+\bar{\nu})\bar{\alpha}(\theta-\theta_0)\,\delta_{\alpha\beta} \ , \tag{12.4}$$

where the variables $\bar{\nu}$ and $\bar{\alpha}$ have been defined in (11.13). Thus, it follows that

$$e_{11} = \frac{1}{2\mu}\left[(1-\bar{\nu})\, T_{11} - \bar{\nu}\, T_{22}\right] + (1+\bar{\nu})\bar{\alpha}(\theta-\theta_0) \quad ,$$

$$e_{22} = \frac{1}{2\mu}\left[-\bar{\nu}\, T_{11} + (1-\bar{\nu})\, T_{22}\right] + (1+\bar{\nu})\bar{\alpha}(\theta-\theta_0) \ ,$$

$$e_{12} = \frac{1}{2\mu}\left[T_{12}\right] \ , \tag{12.5}$$

so that the compatibility equation (12.3) can be rewritten in the form

$$\left[(1-\bar{\nu})\, T_{11,22} - \bar{\nu}\, T_{22,22}\right] + \left[-\bar{\nu}\, T_{11,11} + (1-\bar{\nu})\, T_{22,11}\right] - 2T_{12,12}$$

$$= -2\mu(1+\bar{\nu})\bar{\alpha}\theta_{,\sigma\sigma} \ . \tag{12.6}$$

Rearranging this equation it is possible to deduce that

$$\left[T_{11,22} + T_{22,11} - 2T_{12,12}\right] - \bar{\nu}\left[T_{11,11} + T_{11,22} + T_{22,11} + T_{22,22}\right]$$

$$= -2\mu(1+\bar{\nu})\bar{\alpha}\theta_{,\sigma\sigma} \ . \tag{12.7}$$

Thus, the stress field must satisfy this compatibility equation in order to a strain field u_α to exist.

AIRY'S STRESS FUNCTION

For the simple case when the body force is derivable from a potential V

$$\rho_0\, b_\alpha = -V_{,\alpha} \ , \tag{12.8}$$

the equations of equilibrium (12.1) reduce to

$$T_{\alpha\beta,\beta} - V_{,\alpha} = 0 \ . \tag{12.9}$$

Now, it can easily be seen that these equilibrium equations are automatically satisfied if the stress field $T_{\alpha\beta}$ is determined by derivatives of the Airy's stress function $\phi(x_\alpha)$, such that

$$T_{\alpha\beta} = \left[\phi_{,\sigma\sigma} + V\right]\delta_{\alpha\beta} - \phi_{,\alpha\beta} \ ,$$

$$T_{11} = \phi_{,22} + V \ , \ T_{22} = \phi_{,11} + V \ , \ T_{12} = -\phi_{,12} \ . \tag{12.10}$$

Moreover, it can be shown that

$$T_{11,22} + T_{22,11} - 2T_{12,12} = \phi_{,\alpha\alpha\beta\beta} + V_{,\sigma\sigma} = \nabla^2\nabla^2\phi + \nabla^2 V \ ,$$

$$T_{11,11} + T_{11,22} + T_{22,11} + T_{22,22} = \phi_{,\alpha\alpha\beta\beta} + 2V_{,\sigma\sigma} = \nabla^2\nabla^2\phi + 2\nabla^2 V \ , \tag{12.11}$$

so that for the body force (12.8), the compatibility equation (12.7) reduces to

$$\nabla^2\nabla^2\,\phi = -\left[\frac{1-2\bar{\nu}}{1-\bar{\nu}}\right]\nabla^2 V - 2\mu\left[\frac{1+\bar{\nu}}{1-\bar{\nu}}\right]\bar{\alpha}\,\nabla^2\theta \quad . \tag{12.12}$$

Alternatively, it follows from (12.10) that

$$\nabla^2\phi = T_{\sigma\sigma} - 2V \quad , \tag{12.13}$$

so that the compatibility condition (12.12) can be rewritten in the form

$$\nabla^2 T_{\sigma\sigma} = \left[\frac{1}{1-\bar{\nu}}\right]\nabla^2 V - 2\mu\left[\frac{1+\bar{\nu}}{1-\bar{\nu}}\right]\bar{\alpha}\,\nabla^2\theta \quad . \tag{12.14}$$

Thus, when the body force potential V is a harmonic function ($\nabla^2 V=0$), the temperature field θ is steady (independent of time) and there is no heat supply ($\nabla^2\theta=0$), the compatibility equations require the Airy's stress function to be a biharmonic function

$$\nabla^2\nabla^2\,\phi = \nabla^2 T_{\sigma\sigma} = 0 \quad . \tag{12.15}$$

Using the Airy's stress function it is relatively easy to find stress fields which satisfy the equations of equilibrium and the compatibility equations for two-dimensional problems. Consequently, the main effort in finding the solution of a two-dimensional problem is shifted to the problem of satisfying the boundary conditions.

13. Two-dimensional problems in rectangular Cartesian coordinates

In this section attention is confined to the solution of two-dimensional problems in rectangular Cartesian coordinates for which the temperature is constant

$$\theta = \theta_0 \ , \tag{13.1}$$

and the body force is constant

$$b_\alpha = \text{constant} \ . \tag{13.2}$$

It then follows that the body force is determined by a potential V such that

$$\rho_0 \, b_\alpha = - V_{,\alpha} \ , \quad V = - \rho_0 \, b_\alpha x_\alpha \ . \tag{13.3}$$

Moreover, the stresses for equilibrium are determined by the equations (12.10) in terms of V and the Airy's stress function ϕ, such that

$$T_{11} = \phi_{,22} + V \ , \ T_{22} = \phi_{,11} + V \ , \ T_{12} = - \phi_{,12} \ , \tag{13.3}$$

and the compatibility equation (12.12) then requires

$$\nabla^2\nabla^2 \phi = \nabla^2 T_{\sigma\sigma} = 0 \ . \tag{13.4}$$

Also, the expressions (12.5) for the strains reduce to

$$e_{11} = (1-\bar{\nu}) \frac{T_{11}}{2\mu} - \bar{\nu} \frac{T_{22}}{2\mu} \ , \ e_{22} = - \bar{\nu} \frac{T_{11}}{2\mu} + (1-\bar{\nu}) \frac{T_{22}}{2\mu} \ , \ e_{12} = \frac{T_{12}}{2\mu} \ , \tag{13.5}$$

where $\bar{\nu}$ is defined by (11.13). Furthermore, with the help of (11.20) and (11.23), the internal energy becomes

$$\rho_0 \varepsilon = 3K\alpha\theta_0 \, [e_{\sigma\sigma}] \ , \tag{13.6}$$

for plane strain and becomes

$$\rho_0 \varepsilon = 3K\alpha\theta_0 \, [\frac{1-2\nu}{1-\nu}] \, e_{\sigma\sigma} \ . \tag{13.7}$$

for generalized plane stress.

A number of problems for beams can be solved by considering polynomial solutions of the biharmonic equation (13.4). Specifically, let $\phi^{(m)}$ be a polynomial of order m defined by

$$\phi^{(m)} = \phi^{(m)}(x_1,x_2) = \sum_{n=0}^{m} C_{m-n,n} \ x_1^{m-n} \ x_2^n \quad , \tag{13.8}$$

where $C_{m-n,n}$ are constant coefficients. This function will be biharmonic if it satisfies the equation

$$\nabla^2\nabla^2\,\phi^{(m)} = \phi^{(m)}{}_{,\alpha\alpha\beta\beta} = \frac{\partial^4\phi^{(m)}}{\partial x_1^4} + 2\,\frac{\partial^4\phi^{(m)}}{\partial x_1^2\partial x_2^2} + \frac{\partial^4\phi^{(m)}}{\partial x_2^4} = 0 \quad . \tag{13.9}$$

For polynomials of order ($m < 4$), the function (13.8) satisfies the biharmonic equation (13.9) for all values of the coefficients C_{ij}. However, for polynomials of order ($m \geq 4$) the coefficients C_{ij} must satisfy certain restrictions in order for $\phi^{(m)}$ to be a biharmonic function. These restrictions can be developed by substituting (13.8) into (13.9) to obtain

$$\begin{aligned}
&\sum_{n=0}^{m-4}\big[(m-n)(m-n-1)(m-n-2)(m-n-3)\big]\,C_{m-n,n}\;x_1^{m-n-4}\,x_2^{n}\\
&+\sum_{n=2}^{m-2}2\,\big[(m-n)(m-n-1)(n)(n-1)\big]\,C_{m-n,n}\;x_1^{m-n-2}\,x_2^{n-2}\\
&+\sum_{n=4}^{m}\big[(n)(n-1)(n-2)(n-3)\big]\,C_{m-n,n}\;x_1^{m-n}\,x_2^{n-4} = 0 \quad .
\end{aligned} \tag{13.10}$$

Then, the indices can be changed so that

$$\begin{aligned}
&\sum_{n=0}^{m-4}\{\big[(m-n)(m-n-1)(m-n-2)(m-n-3)\big]\,C_{m-n,n}\\
&\quad+2\,\big[(m-n-2)(m-n-3)(n+2)(n+1)\big]\,C_{m-n-2,n+2}\\
&+\big[(n+4)(n+3)(n+2)(n+1)\big]\,C_{m-n-4,n+4}\}\;x_1^{m-n-4}\,x_2^{n} = 0 \quad .
\end{aligned} \tag{13.11}$$

Now, since the coefficient of each of the (m–3) terms must vanish, it follows that coefficients are restricted by the (m–3) equations

$$\begin{aligned}
&\big[(m-n)(m-n-1)(m-n-2)(m-n-3)\big]\,C_{m-n,n}\\
&+2\,\big[(m-n-2)(m-n-3)(n+2)(n+1)\big]\,C_{m-n-2,n+2}\\
&+\big[(n+4)(n+3)(n+2)(n+1)\big]\,C_{m-n-4,n+4} = 0\\
&\qquad\text{for } n = 0,1,2,\ldots,m-4 \quad \text{and} \quad m \geq 4 \quad .
\end{aligned} \tag{13.12}$$

The stresses vanish for the polynomials (m=1 and m=2). Moreover, the following give the polynomials of orders 2-5 as well as the associated stresses. The specific values of the constant coefficients can be determined by using superposition to combine the solutions to match specified boundary conditions.

<u>For m = 2</u>

$$\phi^{(2)} = C_{20}\, x_1^2 + C_{11}\, x_1\, x_2 + C_{02}\, x_2^2 \ ,$$

$$T_{11}^{(2)} = \phi^{(2)}{}_{,22} = 2\, C_{02} \ , \ T_{22}^{(2)} = \phi^{(2)}{}_{,11} = 2\, C_{20} \ , \ T_{12}^{(2)} = -\, \phi^{(2)}{}_{,12} = -\, C_{11} \ , \qquad (13.13)$$

<u>For m = 3</u>

$$\phi^{(3)} = C_{30}\, x_1^3 + C_{21}\, x_1^2\, x_2 + C_{12}\, x_1\, x_2^2 + C_{03}\, x_2^3 \ ,$$

$$T_{11}^{(3)} = \phi^{(3)}{}_{,22} = 6\, C_{03}\, x_2 + 2\, C_{12}\, x_1 \ , \ T_{22}^{(3)} = \phi^{(3)}{}_{,11} = 6\, C_{30}\, x_1 + 2\, C_{21}\, x_2 \ ,$$

$$T_{12}^{(3)} = -\, \phi^{(3)}{}_{,12} = -\, 2\, C_{21}\, x_1 - 2\, C_{12}\, x_2 \ , \qquad (13.14)$$

<u>For m = 4</u>

$$\phi^{(4)} = C_{40}\, x_1^4 + C_{31}\, x_1^3\, x_2 + C_{22}\, x_1^2\, x_2^2 + C_{13}\, x_1\, x_2^3 + C_{04}\, x_2^4 \ ,$$

$$3\, C_{40} + C_{22} + 3\, C_{04} = 0 \ ,$$

$$T_{11}^{(4)} = \phi^{(4)}{}_{,22} = 2\, C_{22}\, x_1^2 + 6\, C_{13}\, x_1\, x_2 + 12\, C_{04}\, x_2^2 \ ,$$

$$T_{22}^{(4)} = \phi^{(4)}{}_{,11} = 12\, C_{40}\, x_1^2 + 6\, C_{31}\, x_1\, x_2 + 2\, C_{22}\, x_2^2 \ ,$$

$$T_{12}^{(4)} = -\, \phi^{(4)}{}_{,12} = -\, 3\, C_{31}\, x_1^2 - 4\, C_{22}\, x_1\, x_2 - 3\, C_{13}\, x_1\, x_2^2 \ , \qquad (13.15)$$

<u>For m = 5</u>

$$\phi^{(5)} = C_{50}\, x_1^5 + C_{41}\, x_1^4\, x_2 + C_{32}\, x_1^3\, x_2^2 + C_{23}\, x_1^2\, x_2^3 + C_{14}\, x_1\, x_2^4 + C_{05}\, x_2^5 \ ,$$

$$5\, C_{50} + C_{32} + C_{04} = 0 \ , \ C_{41} + C_{23} + 5\, C_{05} = 0 \ ,$$

$$T_{11}^{(5)} = \phi^{(5)}{}_{,22} = 2\, C_{32}\, x_1^3 + 6\, C_{23}\, x_1^2\, x_2 + 12\, C_{14}\, x_1\, x_2^2 + 20\, C_{05}\, x_2^3 \ ,$$

$$T_{22}^{(5)} = \phi^{(5)}{}_{,11} = 20\, C_{50}\, x_1^3 + 12\, C_{41}\, x_1^2\, x_2 + 6\, C_{32}\, x_1\, x_2^2 + 2\, C_{23}\, x_2^3 \ ,$$

$$T_{12}^{(5)} = -\, \phi^{(5)}{}_{,12} = -\, 4\, C_{41}\, x_1^3 - 6\, C_{32}\, x_1^2\, x_2 - 6\, C_{23}\, x_1\, x_2^2 - 4\, C_{14}\, x_2^3 \ , \qquad (13.16)$$

14. Two-dimensional beam problems

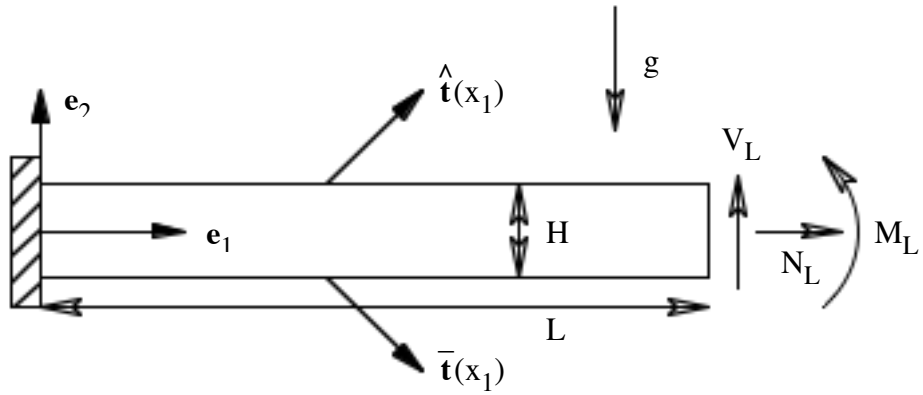

Fig. 14.1 Sketch of a cantilever beam

This section considers a class of two-dimensional beam problems. Specifically, consider a rectangular cantilever beam of length L, height H, and depth W which occupies the region of space such that (see Fig. 14.1)

$$0 \le x_1 \le L \ , \ -\frac{H}{2} \le x_2 \le \frac{H}{2} \ , \ -\frac{W}{2} \le x_3 \le \frac{W}{2} \ . \tag{14.1}$$

The beam is subjected to surface tractions $\hat{\mathbf{t}}$ on its top surface (x_2=H/2) and surface tractions $\bar{\mathbf{t}}$ on its bottom surface (x_2=–H/2)

$$\mathbf{t}(x_1,H/2;\mathbf{e}_2) = \hat{\mathbf{t}}(x_1) \ , \ \mathbf{t}(x_1,-H/2;-\mathbf{e}_2) = \bar{\mathbf{t}}(x_1) \ , \tag{14.2}$$

and axial force N_L, shear force V_L and bending moment M_L (about the centroid of the cross-section) at its end (x_1=L). Also, the force of gravity g (per unit mass) acts in the negative $\mathbf{e}_2$ direction so that

$$b_1 = 0 \ , \ b_2 = -g \ ,$$

$$\rho_0 \, b_\alpha = -V_{,\alpha} \ , \ V = \rho_0 \, g \, x_2 \ . \tag{14.3}$$

In addition, the beam remains at constant temperature

$$\theta = \theta_0 \ . \tag{14.4}$$

In order to compare the elasticity results with those of standard beam theory it is convenient to define the axial force $n(x_1)$, the shear force $v(x_1)$ and the bending moment $m(x_1)$ by the expressions

$$n(x_1) = W \int_{-H/2}^{H/2} T_{11}\, dx_2 \ , \quad v(x_1) = W \int_{-H/2}^{H/2} T_{12}\, dx_2 \ ,$$

$$m(x_1) = -\, W \int_{-H/2}^{H/2} x_2\, T_{11}\, dx_2 \ . \tag{14.5}$$

Moreover, it is convenient to define the average displacement $\mathbf{w}(x_1)$ and the average rotation $\boldsymbol{\delta}(x_1)$ by the formulas

$$\mathbf{w}(x_1) = \frac{1}{H} \int_{-H/2}^{H/2} \mathbf{u}\, dx_2 \ , \ \boldsymbol{\delta}(x_1) = \frac{1}{H} \int_{-H/2}^{H/2} \frac{\partial \mathbf{u}}{\partial x_2}\, dx_2 = \frac{1}{H} \left[\mathbf{u}(x_1, H/2) - \mathbf{u}(x_1, -H/2) \right] \ . \tag{14.6}$$

Next, the equations of equilibrium of beam theory can be obtained by averaging the equilibrium equation of two-dimensional elasticity

$$T_{11,1} + T_{12,2} + \rho_0\, b_1 = 0 \ ,$$

$$T_{21,1} + T_{22,2} + \rho_0\, b_2 = 0 \ . \tag{14.7}$$

Specifically, since for two-dimensional problems there is no dependence of quantities on x_3, these equations can be integrated over the cross-section of the beam to obtain the averaged equations

$$W \int_{-H/2}^{H/2} \left[T_{11,1} + T_{12,2} + \rho_0\, b_1 \right] dx_2 = 0 \ ,$$

$$W \int_{-H/2}^{H/2} \left[T_{21,1} + T_{22,2} + \rho_0\, b_2 \right] dx_2 = 0 \ . \tag{14.8}$$

Now, using the definitions (14.5) these equations can be rewritten in the simpler forms

$$\frac{dn}{dx_1} + q_1 = 0 \ , \quad \frac{dv}{dx_1} + q_2 = 0 \ , \tag{14.9}$$

where q_1 and q_2 are assigned fields which represent forces per unit length of the beam applied in the $\mathbf{e}_1$ and $\mathbf{e}_2$ directions, respectively,

$$q_1(x_1) = \rho_0\, HWb_1 + W \left[T_{12}(x_1, H/2) - T_{12}(x_1, -H/2) \right] \ ,$$

$$q_2(x_1) = \rho_0\, HWb_2 + W \left[T_{22}(x_1, H/2) - T_{22}(x_1, -H/2) \right] \ . \tag{14.10}$$

In particular, notice that q_α include both the effects of body forces and the loads on the top and bottom surfaces of the beam.

The equations (14.9) represent the averages of the equations of equilibrium (14.7). In order to derive the equation for the moment in standard beam theory, it is convenient to consider a weighted average of these equilibrium equations. Specifically, the first of (14.7) is multiplied by the weighting function x_2 to obtain

$$(x_2\, T_{11})_{,1} + (x_2\, T_{12})_{,2} - T_{12} + x_2\, \rho_0\, b_1 = 0 \quad . \tag{14.11}$$

Now, integrating this equation over the cross-section of the beam yields the result

$$-\frac{dm}{dx_1} - v + \chi(x_1) = 0 \quad , \tag{14.12}$$

where $\chi(x_1)$ is an assigned field which represents a couple per unit length that is generated by the shear stresses that are applied to the top and bottom surfaces of the beam

$$\chi(x_1) = \frac{HW}{2}\left[T_{12}(x_1,H/2) + T_{12}(x_1,-H/2)\right] \quad . \tag{14.13}$$

The simplest standard beam theory usually does not consider these shear forces. It also does not consider the additional equation of equilibrium which is obtained by taking a similar weighted average of the second equation of equilibrium in (14.7).

In order to solve the problem sketched in Fig. 14.1 it is convenient to consider superposition of the following simpler problems. For each case it is necessary to determine the stresses, strains, displacements, average displacements and rotations, and the resultant forces and moment.

Case I: Rigid body displacements

Boundary conditions on the top and bottom surfaces

$$\hat{t}_1^I(x_1) = 0 \quad , \quad \hat{t}_2^I(x_1) = 0 \quad ,$$

$$\bar{t}_1^I(x_1) = 0 \quad , \bar{t}_2^I(x_1) = 0 \quad , \tag{14.14}$$

Body force

$$b_\alpha^I = 0 \quad , \tag{14.15}$$

Stresses

$$T_{\alpha\beta}^I = 0 \quad , \tag{14.16}$$

Strains

$$e^{I}_{\alpha\beta} = 0 \quad , \tag{14.17}$$

Displacements

$$u^{I}_{1} = \alpha\, x_2 + c_1 \quad , \quad u^{I}_{2} = -\,\alpha\, x_1 + c_2 \quad , \tag{14.18}$$

Average displacements and rotations

$$w^{I}_{1} = c_1 \quad , \; w^{I}_{2} = -\,\alpha\, x_1 + c_2 \quad ,$$

$$\delta^{I}_{1} = \alpha \quad , \; \delta^{I}_{2} = 0 \quad , \tag{14.19}$$

Boundary conditions on the average displacements and rotations

$$w^{I}_{1}(0) = c_1 \quad , \; w^{I}_{2}(0) = c_2 \quad ,$$

$$\delta^{I}_{1}(0) = \alpha \quad , \; \delta^{I}_{2}(0) = 0 \quad , \tag{14.20}$$

Resultant forces and moment

$$n^{I} = 0 \quad , \quad v^{I} = 0 \quad , \; m^{I} = 0 \quad , \tag{14.21}$$

Boundary conditions on the resultant forces and moment

$$n^{I}(L) = N_I = 0 \quad , \quad v^{I}(L) = V_I = 0 \; , \quad m^{I}(L) = M_I = 0 \quad . \tag{14.22}$$

Case II: End forces and moment

Boundary conditions on the top and bottom surfaces

$$\hat{t}^{II}_{1}(x_1) = 0 \quad , \quad \hat{t}^{II}_{2}(x_1) = 0 \quad ,$$

$$\bar{t}^{II}_{1}(x_1) = 0 \quad , \bar{t}^{II}_{2}(x_1) = 0 \; , \tag{14.23}$$

Body force

$$b^{II}_{\alpha} = 0 \quad , \tag{14.24}$$

Stresses

$$T^{II}_{11} = \{\frac{N_{II}}{HW}\} - \{\frac{12M_{II}}{H^3W}\}x_2 - \{\frac{12V_{II}}{H^3W}\}(L-x_1)\, x_2 \; ,$$

$$T^{II}_{22} = 0 \; ,$$

$$T^{II}_{12} = \{\frac{6V_{II}}{H^3W}\}\{\frac{H^2}{4} - x_2^2\} \; , \tag{14.25}$$

Strains

$$e_{11}^{II} = (1-\bar{\nu})\left[\left\{\frac{N_{II}}{2\mu HW}\right\} - \left\{\frac{12M_{II}}{2\mu H^3W}\right\}x_2 - \left\{\frac{12V_{II}}{2\mu H^3W}\right\}(L-x_1)\,x_2\right] ,$$

$$e_{22}^{II} = -\,\bar{\nu}\left[\left\{\frac{N_{II}}{2\mu HW}\right\} - \left\{\frac{12M_{II}}{2\mu H^3W}\right\}x_2 - \left\{\frac{12V_{II}}{2\mu H^3W}\right\}(L-x_1)\,x_2\right] ,$$

$$e_{12}^{II} = \left\{\frac{3V_{II}}{\mu H^3W}\right\}\left\{\frac{H^2}{4} - x_2^2\right\} , \tag{14.26}$$

Displacements

$$u_1^{II} = (1-\bar{\nu})\left[\left\{\frac{N_{II}}{2\mu HW}\right\}x_1 - \left\{\frac{6M_{II}}{\mu H^3W}\right\}x_1\,x_2 - \left\{\frac{3V_{II}}{\mu H^3W}\right\}\left\{2Lx_1 - x_1^2\right\}x_2\right]$$

$$+\left\{\frac{2V_{II}}{\mu H^3W}\right\}\left\{\frac{3H^2}{4}x_2 - x_2^3\right\} + \bar{\nu}\left\{\frac{V_{II}}{\mu H^3W}\right\}x_2^3 ,$$

$$u_2^{II} = -\,\bar{\nu}\left[\left\{\frac{N_{II}}{2\mu HW}\right\}x_2 - \left\{\frac{3M_{II}}{\mu H^3W}\right\}x_2^2 - \left\{\frac{3V_{II}}{\mu H^3W}\right\}(L-x_1)\,x_2^2\right]$$

$$+(1-\bar{\nu})\left[\left\{\frac{3M_{II}}{\mu H^3W}\right\}x_1^2 + \left\{\frac{V_{II}}{\mu H^3W}\right\}\left\{3Lx_1^2 - x_1^3\right\}\right] , \tag{14.27}$$

Average displacements and rotations

$$w_1^{II} = (1-\bar{\nu})\left[\left\{\frac{N_{II}}{2\mu HW}\right\}x_1\right] ,$$

$$w_2^{II} = \bar{\nu}\left[\left\{\frac{M_{II}}{4\mu HW}\right\} + \left\{\frac{V_{II}}{4\mu HW}\right\}(L-x_1)\right]$$

$$+(1-\bar{\nu})\left[\left\{\frac{3M_{II}}{\mu H^3W}\right\}x_1^2 + \left\{\frac{V_{II}}{\mu H^3W}\right\}\left\{3Lx_1^2 - x_1^3\right\}\right] ,$$

$$\delta_1^{II} = -\,(1-\bar{\nu})\left[\left\{\frac{6M_{II}}{\mu H^3W}\right\}x_1 + \left\{\frac{3V_{II}}{\mu H^3W}\right\}\left\{2Lx_1 - x_1^2\right\}\right] + \left\{\frac{V_{II}}{\mu HW}\right\} + \bar{\nu}\left\{\frac{V_{II}}{4\mu HW}\right\} ,$$

$$\delta_2^{II} = -\,\bar{\nu}\left\{\frac{N_{II}}{2\mu HW}\right\} , \tag{14.28}$$

Boundary conditions on the average displacements and rotations

$$w_1^{II}(0) = 0 ,$$

$$w_2^{II}(0) = \bar{\nu}\,[\{\frac{M_{II}}{4\mu HW}\} + \{\frac{V_{II}}{4\mu HW}\}L] \ ,$$

$$\delta_1^{II}(0) = \{\frac{V_{II}}{\mu HW}\} + \bar{\nu}\{\frac{V_{II}}{4\mu HW}\} \ ,$$

$$\delta_2^{II}(0) = -\,\bar{\nu}\,\{\frac{N_{II}}{2\mu HW}\} \ , \tag{14.29}$$

Resultant forces and moment

$$n^{II} = N_{II} \ , \quad v^{II} = V_{II} \ , \ m^{II} = M_{II} + V_{II}\,(L - x_1) \ , \tag{14.30}$$

Boundary conditions on the resultant forces and moment

$$n^{II}(L) = N_{II} \ , \quad v^{II}(L) = V_{II} \ , \quad m^{II}(L) = M_{II} \ . \tag{14.31}$$

Case III: Uniform loads on the top and bottom surfaces with gravity

Boundary conditions on the top and bottom surfaces

$$\hat{t}_1^{III}(x_1) = 0 \ , \quad \hat{t}_2^{III}(x_1) = \hat{Q} \ ,$$

$$\bar{t}_1^{III}(x_1) = 0 \ , \bar{t}_2^{III}(x_1) = \bar{Q} \ , \tag{14.32}$$

Body force

$$b_1^{III} = 0 \ , \quad b_2^{III} = -\,g \ , \tag{14.33}$$

Stresses

$$T_{11}^{III} = -\{\frac{6Q}{H^3}\}x_1^2\,x_2 + \{\frac{4Q}{H^3}\}x_2^3 \ ,$$

$$T_{22}^{III} = \frac{Q}{4}\,[2 + \{\frac{6}{H}\}x_2 - \{\frac{8}{H^3}\}x_2^3\,] + \rho_0 g[x_2 + \frac{H}{2}\,] - \bar{Q} \ ,$$

$$T_{12}^{III} = -\{\frac{6Q}{H^3}\}\{\frac{H^2}{4} - x_2^2\}x_1 \ ,$$

$$Q = \bar{Q} + \hat{Q} - \rho_0 gH \ , \tag{14.34}$$

Strains

$$e_{11}^{III} = (1-\bar{\nu})\,[-\{\frac{3Q}{\mu H^3}\}x_1^2\,x_2 + \{\frac{2Q}{\mu H^3}\}x_2^3\,]$$

$$-\bar{\nu}\left[\frac{Q}{8\mu}\left(2+\left\{\frac{6}{H}\right\}x_2-\left\{\frac{8}{H^3}\right\}x_2^3\right)+\frac{\rho_0 g}{2\mu}\left\{x_2+\frac{H}{2}\right\}-\frac{\bar{Q}}{2\mu}\right],$$

$$e_{22}^{III}=-\bar{\nu}\left[-\left\{\frac{3Q}{\mu H^3}\right\}x_1^2\,x_2+\left\{\frac{2Q}{\mu H^3}\right\}x_2^3\right]$$

$$+(1-\bar{\nu})\left[\frac{Q}{8\mu}\left(2+\left\{\frac{6}{H}\right\}x_2-\left\{\frac{8}{H^3}\right\}x_2^3\right)+\frac{\rho_0 g}{2\mu}\left\{x_2+\frac{H}{2}\right\}-\frac{\bar{Q}}{2\mu}\right],$$

$$e_{12}^{III}=-\left\{\frac{3Q}{\mu H^3}\right\}\left\{\frac{H^2}{4}-x_2^2\right\}x_1\ , \tag{14.35}$$

Displacements

$$u_1^{III}=(1-\bar{\nu})\left[-\left\{\frac{Q}{\mu H^3}\right\}x_1^3\,x_2+\left\{\frac{2Q}{\mu H^3}\right\}x_1x_2^3\right]$$

$$-\bar{\nu}\left[\frac{Q}{8\mu}\left\{2+\left(\frac{6}{H}\right)x_2-\left(\frac{8}{H^3}\right)x_2^3\right\}x_1+\frac{\rho_0 g}{2\mu}\left\{x_2+\frac{H}{2}\right\}x_1-\left\{\frac{\bar{Q}}{2\mu}\right\}x_1\right],$$

$$u_2^{III}=-\bar{\nu}\left[-\left\{\frac{3Q}{2\mu H^3}\right\}x_1^2\,x_2^2+\left\{\frac{Q}{2\mu H^3}\right\}x_2^4\right]$$

$$+(1-\bar{\nu})\left[\frac{Q}{8\mu}\left\{2\,x_2+\left(\frac{3}{H}\right)x_2^2-\left(\frac{2}{H^3}\right)x_2^4\right\}+\frac{\rho_0 g}{4\mu}\left\{x_2^2+H\,x_2\right\}-\left\{\frac{\bar{Q}}{2\mu}\right\}x_2\right]$$

$$-\left\{\frac{3Q}{4\mu H}\right\}x_1^2+(1-\bar{\nu})\left\{\frac{Q}{4\mu H^3}\right\}x_1^4+\bar{\nu}\left\{\frac{3Q}{8\mu H}+\frac{\rho g}{4\mu}\right\}x_1^2\ , \tag{14.36}$$

Average displacements and rotations

$$w_1^{III}=-\bar{\nu}\left[\frac{Q}{4\mu}+\frac{\rho_0 gH}{4\mu}-\left\{\frac{\bar{Q}}{2\mu}\right\}\right]x_1\ ,$$

$$w_2^{III}=-\bar{\nu}\left[-\left\{\frac{Q}{8\mu H}\right\}x_1^2+\left\{\frac{QH}{160\mu}\right\}\right]$$

$$+(1-\bar{\nu})\left[\frac{Q}{8\mu}\left\{\left(\frac{H}{4}\right)-\left(\frac{H}{40}\right)\right\}+\frac{\rho_0 gH^2}{48\mu}\right]$$

$$-\left\{\frac{3Q}{4\mu H}\right\}x_1^2+(1-\bar{\nu})\left\{\frac{Q}{4\mu H^3}\right\}x_1^4+\bar{\nu}\left\{\frac{3Q}{8\mu H}+\frac{\rho_0 g}{4\mu}\right\}x_1^2\ ,$$

$$\delta_1^{III} = (1-\bar{\nu})\left[-\{\frac{Q}{\mu H^3}\}x_1^3 + \{\frac{Q}{2\mu H}\}x_1\right] - \bar{\nu}\left[\frac{3Q}{4\mu H}\ x_1 - \frac{Q}{4\mu H}\ x_1 + \frac{\rho_0 g}{2\mu}\ x_1\right],$$

$$\delta_2^{III} = (1-\bar{\nu})\left[\frac{Q}{4\mu} + \frac{\rho_0 gH}{4\mu} - \{\frac{\bar{Q}}{2\mu}\}\right] \ . \tag{14.37}$$

Boundary conditions on the average displacements and rotations

$$w_1^{III}(0) = 0 \ ,$$

$$w_2^{III}(0) = -\bar{\nu}\ \{\frac{QH}{160\mu}\} + (1-\bar{\nu})\left[\frac{Q}{8\mu}\{(\frac{H}{4}) - (\frac{H}{40})\} + \frac{\rho_0 gH^2}{48\mu}\right] \ ,$$

$$\delta_1^{III}(0) = 0 \ ,$$

$$\delta_2^{III}(0) = (1-\bar{\nu})\left[\frac{Q}{4\mu} + \frac{\rho_0 gH}{4\mu} - \{\frac{\bar{Q}}{2\mu}\}\right] \ , \tag{14.38}$$

Resultant forces and moment

$$n^{III} = 0 \ , \ \ v^{III} = -QW\ x_1 \ , \ \ m^{III} = \frac{QW}{2}\left[x_1^2 - \frac{H^2}{10}\right] \ , \tag{14.39}$$

Boundary conditions on the resultant forces and moment

$$n^{III}(L) = N_{III} = 0 \ , \ v^{III}(L) = V_{III} = -QWL \ ,$$

$$m^{III}(L) = M_{III} = \frac{QWL^2}{2}\left[1 - \frac{H^2}{10L^2}\right] \ . \tag{14.40}$$

Case IV: Linear loads on the top and bottom surfaces

Boundary conditions on the top and bottom surfaces

$$\hat{t}_1^{IV}(x_1) = 0 \ , \ \ \hat{t}_2^{IV}(x_1) = \hat{S}\ x_1 \ ,$$

$$\bar{t}_1^{IV}(x_1) = 0 \ , \bar{t}_2^{IV}(x_1) = \bar{S}\ x_1 \ , \tag{14.41}$$

Body force

$$b_\alpha^{IV} = 0 \ , \tag{14.42}$$

Stresses

$$T_{11}^{IV} = -\{\frac{2S}{H^3}\}x_1^3\ x_2 + \{\frac{4S}{H^3}\}\ x_1\ x_2^3 \ ,$$

$$T_{22}^{IV} = \{\frac{S}{2}\}[1 + \frac{3}{H} x_2 - \frac{4}{H^3} x_2^3] x_1 - \bar{S} x_1 ,$$

$$T_{12}^{IV} = \{\frac{SH}{16}\}[1 - \frac{16}{H^4} x_2^4] - \{\frac{3S}{4H}\}[1 - \frac{4}{H^2} x_2^2]x_1^2 \quad ,$$

$$S = \bar{S} + \hat{S} , \tag{14.43}$$

Strains

$$e_{11}^{IV} = (1-\bar{\nu}) [-\{\frac{S}{\mu H^3}\}x_1^3 x_2 + \{\frac{2S}{\mu H^3}\} x_1 x_2^3]$$

$$- \bar{\nu} [\{\frac{S}{4\mu}\}[1 + \frac{3}{H} x_2 - \frac{4}{H^3} x_2^3] x_1 - \{\frac{\bar{S}}{2\mu}\} x_1] ,$$

$$e_{22}^{IV} = - \bar{\nu} [-\{\frac{S}{\mu H^3}\}x_1^3 x_2 + \{\frac{2S}{\mu H^3}\} x_1 x_2^3]$$

$$+ (1-\bar{\nu}) [\{\frac{S}{4\mu}\}[1 + \frac{3}{H} x_2 - \frac{4}{H^3} x_2^3] x_1 - \{\frac{\bar{S}}{2\mu}\} x_1] \quad ,$$

$$e_{12}^{IV} = \{\frac{SH}{32\mu}\}[1 - \frac{16}{H^4} x_2^4] - \{\frac{3S}{8\mu H}\}[1 - \frac{4}{H^2} x_2^2] x_1^2 \quad , \tag{14.44}$$

Displacements

$$u_1^{IV} = (1-\bar{\nu}) [-\{\frac{S}{4\mu H^3}\}x_1^4 x_2 + \{\frac{S}{\mu H^3}\} x_1^2 x_2^3]$$

$$- \bar{\nu} [\{\frac{S}{8\mu}\}\{1 + \frac{3}{H} x_2 - \frac{4}{H^3} x_2^3 \}x_1^2 - \frac{\bar{S}}{4\mu} x_1^2]$$

$$+ \{\frac{SH}{16\mu}\}[x_2 - \frac{8}{5H^4} x_2^5] - (1-\bar{\nu}) [\{\frac{S}{8\mu}\}\{x_2^2 + \frac{1}{H} x_2^3 + \frac{2}{5H^3} x_2^5 \} - \{\frac{\bar{S}}{4\mu}\} x_2^2] \quad ,$$

$$u_2^{IV} = - \bar{\nu} [-\{\frac{S}{2\mu H^3}\}x_1^3 x_2^2 + \{\frac{S}{2\mu H^3}\} x_1 x_2^4]$$

$$+ (1-\bar{\nu}) [\{\frac{S}{4\mu}\}\{x_2 + \frac{3}{2H} x_2^2 - \frac{1}{H^3} x_2^4 \} x_1 - \frac{\bar{S}}{2\mu} x_1 x_2]$$

$$- \{\frac{S}{8\mu H}\}[x_1^3 + (1-\bar{\nu})\{x_1^3 - \frac{2}{5H^2} x_1^5\}] \quad . \tag{14.45}$$

Average displacements and rotations

$$w_1^{IV} = -\bar{\nu}\left[\frac{S}{8\mu} - \frac{\bar{S}}{4\mu}\right] x_1^2 - (1-\bar{\nu})\left[\frac{SH^2}{96\mu} - \frac{\bar{S}H^2}{48\mu}\right] ,$$

$$w_2^{IV} = -\bar{\nu}\left[-\left\{\frac{S}{24\mu H}\right\} x_1^3 + \left\{\frac{SH}{160\mu}\right\} x_1\right]$$

$$+ (1-\bar{\nu})\left\{\frac{9SH}{320\mu}\right\} x_1 - \left\{\frac{S}{8\mu H}\right\}\left[x_1^3 + (1-\bar{\nu})\left\{x_1^3 - \frac{2}{5H^2} x_1^5\right\}\right] ,$$

$$\delta_1^{IV} = (1-\bar{\nu})\left[-\left\{\frac{S}{4\mu H^3}\right\} x_1^4 + \left\{\frac{S}{4\mu H}\right\} x_1^2\right]$$

$$+ \bar{\nu}\left\{\frac{S}{8\mu H}\right\} x_1^2 + \left\{\frac{9SH}{160\mu}\right\} - (1-\bar{\nu})\left\{\frac{11SH}{320\mu}\right\} ,$$

$$\delta_2^{IV} = (1-\bar{\nu})\left[\frac{S}{4\mu} - \frac{\bar{S}}{2\mu}\right] x_1 . \tag{14.46}$$

Boundary conditions on the average displacements and rotations

$$w_1^{IV}(0) = -(1-\bar{\nu})\left[\frac{SH^2}{96\mu} - \frac{\bar{S}H^2}{48\mu}\right] ,$$

$$w_2^{IV}(0) = 0 ,$$

$$\delta_1^{IV}(0) = \bar{\nu}\left\{\frac{9SH}{160\mu}\right\} - (1-\bar{\nu})\left\{\frac{11SH}{320\mu}\right\} ,$$

$$\delta_2^{IV}(0) = 0 , \tag{14.47}$$

Resultant forces and moment

$$n^{IV} = 0 , \quad v^{IV} = \frac{SH^2W}{20} - \frac{SW}{2} x_1^2 ,$$

$$m^{IV} = \frac{SW}{6} x_1^3 - \frac{SH^2W}{20} x_1 , \tag{14.48}$$

Boundary conditions on the resultant forces and moment

$$n^{IV}(L) = N_{IV} = 0 , \quad v^{IV}(L) = V_{IV} = -\frac{SWL^2}{2}\left[1 - \frac{H^2}{10L^2}\right] ,$$

$$m^{IV}(L) = M_{IV} = \frac{SWL^3}{6}\left[1 - \frac{3H^2}{10L^2}\right] . \tag{14.49}$$

Case V: Uniform shear loads on the top and bottom surfaces

Boundary conditions on the top and bottom surfaces

$$\hat{t}_1^V(x_1) = \hat{\tau} \ , \ \hat{t}_2^V(x_1) = 0 \ ,$$

$$\bar{t}_1^V(x_1) = \bar{\tau} \ , \bar{t}_2^V(x_1) = 0 \ , \tag{14.50}$$

Body force

$$b_\alpha^V = 0 \ , \tag{14.51}$$

Stresses

$$T_{11}^V = -\{\frac{\tau}{H}\}x_1 \ , \ T_{22}^V = 0 \ ,$$

$$T_{12}^V = \{\frac{\tau}{2}\}[1 + \frac{2}{H}x_2] - \bar{\tau} \ , \quad \tau = \hat{\tau} + \bar{\tau} \ , \tag{14.52}$$

Strains

$$e_{11}^V = (1-\bar{\nu})\ [-\{\frac{\tau}{2\mu H}\}x_1] \ , \ e_{22}^V = -\bar{\nu}\ [-\{\frac{\tau}{2\mu H}\}x_1] \ ,$$

$$e_{12}^V = \{\frac{\tau}{4\mu}\}[1 + \frac{2}{H}x_2] - \frac{\bar{\tau}}{2\mu} \ , \tag{14.53}$$

Displacements

$$u_1^V = -(1-\bar{\nu})\{\frac{\tau}{4\mu H}\}x_1^2 - \bar{\nu}\ \{\frac{\tau}{4\mu H}\}\ x_2^2 + \{\frac{\tau}{2\mu}\}[x_2 + \frac{1}{H}x_2^2] - \frac{\bar{\tau}}{\mu}x_2 \ ,$$

$$u_2^V = \bar{\nu}\ \{\frac{\tau}{2\mu H}\}\ x_1\ x_2 \ . \tag{14.54}$$

Average displacements and rotations

$$w_1^V = -(1-\bar{\nu})\{\frac{\tau}{4\mu H}\}x_1^2 \ - \bar{\nu}\ \{\frac{\tau H}{48\mu}\} + \{\frac{\tau H}{24\mu}\} \ ,$$

$$w_2^V = 0 \ ,$$

$$\delta_1^V = \frac{\tau}{2\mu} - \frac{\bar{\tau}}{\mu} \ , \ \delta_2^V = \bar{\nu}\ \{\frac{\tau}{2\mu H}\}\ x_1 \ . \tag{14.55}$$

Boundary conditions on the average displacements and rotations

$$w_1^V(0) = -\bar{\nu}\left\{\frac{\tau H}{48\mu}\right\} + \left\{\frac{\tau H}{24\mu}\right\} \ , \ w_2^V(0) = 0 \ ,$$

$$\delta_1^V(0) = \frac{\tau}{2\mu} - \frac{\bar{\tau}}{\mu} \ , \ \delta_2^V(0) = 0 \ , \tag{14.56}$$

Resultant forces and moment

$$n^V = -\tau W x_1 \ , \ v^V = HW\left[\frac{\hat{\tau} - \bar{\tau}}{2}\right] \ , \ m^V = 0 \ , \tag{14.57}$$

Boundary conditions on the resultant forces and moment

$$n^V(L) = N_V = -\tau WL \ , \ v^V(L) = V_V = HW\left[\frac{\hat{\tau} - \bar{\tau}}{2}\right] \ , \ m^V(L) = M_V = 0 \ . \tag{14.58}$$

Superposition

It is important to note that the solutions for Cases I-V generate average displacements and rotation at the clamped end (x_1=0) and forces and moment at the end (x_1=L). However, the constants c_1, c_2 and α associated with average displacements and rotation in the rigid body solution (Case I), and the constants N_{II}, V_{II} and M_{II} associated with the forces and moment in the solution for Case II, can be specified arbitrarily. Thus, the solution for the general boundary conditions and body force associated with all of the solutions (Cases I-V) can be obtained by superposition.

Boundary conditions on the top and bottom surfaces

$$\hat{t}_\alpha(x_1) = \hat{t}_\alpha^I + \hat{t}_\alpha^{II} + \hat{t}_\alpha^{III} + \hat{t}_\alpha^{IV} + \hat{t}_\alpha^V \ ,$$

$$\bar{t}_\alpha(x_1) = \bar{t}_\alpha^I + \bar{t}_\alpha^{II} + \bar{t}_\alpha^{III} + \bar{t}_\alpha^{IV} + \bar{t}_\alpha^V \ , \tag{14.59}$$

Body force

$$b_\alpha = b_\alpha^I + b_\alpha^{II} + b_\alpha^{III} + b_\alpha^{IV} + b_\alpha^V \ , \tag{14.60}$$

Stresses

$$T_{\alpha\beta}(x_1,x_2) = T_{\alpha\beta}^I + T_{\alpha\beta}^{II} + T_{\alpha\beta}^{III} + T_{\alpha\beta}^{IV} + T_{\alpha\beta}^V \ , \tag{14.61}$$

Strains

$$e_{\alpha\beta}(x_1,x_2) = e_{\alpha\beta}^I + e_{\alpha\beta}^{II} + e_{\alpha\beta}^{III} + e_{\alpha\beta}^{IV} + e_{\alpha\beta}^V \ , \tag{14.62}$$

Displacements

$$u_\alpha(x_1,x_2) = u_\alpha^{I} + u_\alpha^{II} + u_\alpha^{III} + u_\alpha^{IV} + u_\alpha^{V} \quad , \tag{14.63}$$

Average displacements and rotations

$$w_\alpha(x_1) = w_\alpha^{I} + w_\alpha^{II} + w_\alpha^{III} + w_\alpha^{IV} + w_\alpha^{V} \quad ,$$

$$\delta_\alpha(x_1) = \delta_\alpha^{I} + \delta_\alpha^{II} + \delta_\alpha^{III} + \delta_\alpha^{IV} + \delta_\alpha^{V} \quad , \tag{14.64}$$

Boundary conditions on the average displacements and rotations

$$w_\alpha(0) = c_\alpha + w_\alpha^{II}(0) + w_\alpha^{III}(0) + w_\alpha^{IV}(0) + w_\alpha^{V}(0) \quad ,$$

$$\delta_1(0) = -\,\alpha + \delta_1^{II}(0) + \delta_1^{III}(0) + \delta_1^{IV}(0) + \delta_1^{V}(0) \quad , \tag{14.65}$$

Resultant forces and moment

$$n(x_1) = n^{I} + n^{II} + n^{III} + n^{IV} + n^{V} \quad ,$$

$$v(x_1) = v^{I} + v^{II} + v^{III} + v^{IV} + v^{V} \quad ,$$

$$m(x_1) = m^{I} + m^{II} + m^{III} + m^{IV} + m^{V} \quad , \tag{14.66}$$

Boundary conditions on the resultant forces and moment

$$N_L = n(L) = N_I + N_{II} + N_{III} + N_{IV} + N_V \quad ,$$

$$V_L = v(L) = V_I + V_{II} + V_{III} + V_{IV} + V_V \quad ,$$

$$M_L = m(L) = M_I + M_{II} + M_{III} + M_{IV} + M_V \quad . \tag{14.67}$$

Now, for a clamped end the average displacements and rotation are specified by

$$w_\alpha(0) = 0 \;\; , \;\; \delta_1(0) = 0 \quad . \tag{14.68}$$

Also, general forces and moment can be specified at the end (x_1=L), such that

$$n(L) = N_L \;\; , \;\; v(L) = V_L \;\; , \;\; m(L) = M_L \quad . \tag{14.69}$$

Thus, with the help of (14.65) and (14.67) it follows that these boundary conditions can be satisfied by the specifications

$$c_\alpha = -\left[w_\alpha^{II}(0) + w_\alpha^{III}(0) + w_\alpha^{IV}(0) + w_\alpha^{V}(0)\right] \quad ,$$

$$\alpha = \delta_1^{II}(0) + \delta_1^{III}(0) + \delta_1^{IV}(0) + \delta_1^{V}(0) \quad ,$$

$$N_{II} = N_L - \left[N_I + N_{III} + N_{IV} + N_V\right] \quad ,$$

$$V_{II} = V_L - \left[V_I + V_{III} + V_{IV} + V_V\right] \quad ,$$

$$M_{II} = M_L - \left[M_I + M_{III} + M_{IV} + M_V\right] \quad . \tag{14.70}$$

15. Cylindrical polar coordinates

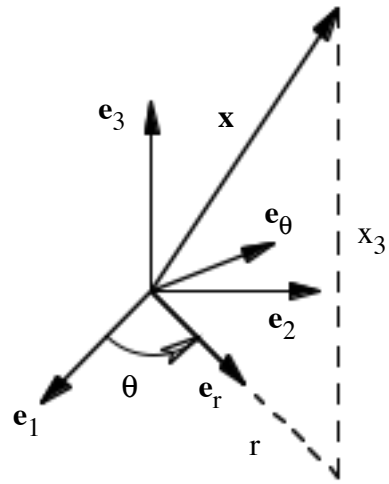

Fig. 15.1 Definition of cylindrical polar coordinates and base vectors.

The position vector $\mathbf{x}$ of a material point can be expressed in terms of cylindrical polar coordinates in the form

$$\mathbf{x} = r\,\mathbf{e}_r(\theta) + x_3\,\mathbf{e}_3 \quad , \tag{15.1}$$

where

$$\{\, r\,,\theta\,,x_3\,\} \quad , \tag{15.2}$$

are the coordinates and

$$\{\, \mathbf{e}_r\,,\mathbf{e}_\theta\,,\mathbf{e}_3\,\} \quad , \tag{15.3}$$

are the unit base vectors. Moreover, the base vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ can be related to those of the rectangular Cartesian coordinate system by the formulas

$$\mathbf{e}_r(\theta) = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2 \quad ,$$

$$\mathbf{e}_\theta(\theta) = -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2 \quad . \tag{15.4}$$

Also, by substituting the expression for $\mathbf{e}_r$ into (15.1) it can be seen that the coordinates of the cylindrical polar coordinate system can be related to those of the rectangular Cartesian coordinate system by the formulas

$$x_1 = r\cos\theta \quad , \quad x_2 = r\sin\theta \quad ,$$

$$r = \sqrt{x_1^2 + x_2^2} \quad , \quad \theta = \tan^{-1}(x_2/x_1) \quad . \tag{15.5}$$

Here, the coordinate θ should not be confused with the same symbol that is used for the temperature.

Now, the displacement vector $\mathbf{u}$, the body force $\mathbf{b}$, the stress tensor $\mathbf{T}$, the strain tensor $\mathbf{e}$ and the heat flux vector $\mathbf{q}$ can be expressed in terms of their cylindrical polar components in the forms

$$\mathbf{u} = u_r\,\mathbf{e}_r + u_\theta\,\mathbf{e}_\theta + u_3\,\mathbf{e}_3 \quad ,$$

$$\mathbf{b} = b_r\,\mathbf{e}_r + b_\theta\,\mathbf{e}_\theta + b_3\,\mathbf{e}_3 \quad ,$$

$$\begin{aligned}\mathbf{T} = {} & T_{rr}\,(\mathbf{e}_r\otimes\mathbf{e}_r) + T_{r\theta}\,(\mathbf{e}_r\otimes\mathbf{e}_\theta) + T_{r3}\,(\mathbf{e}_r\otimes\mathbf{e}_3) \\ & + T_{r\theta}\,(\mathbf{e}_\theta\otimes\mathbf{e}_r) + T_{\theta\theta}\,(\mathbf{e}_\theta\otimes\mathbf{e}_\theta) + T_{\theta 3}\,(\mathbf{e}_\theta\otimes\mathbf{e}_3) \\ & + T_{r3}\,(\mathbf{e}_3\otimes\mathbf{e}_r) + T_{\theta 3}\,(\mathbf{e}_3\otimes\mathbf{e}_\theta) + T_{33}\,(\mathbf{e}_3\otimes\mathbf{e}_3) \quad ,\end{aligned}$$

$$\begin{aligned}\mathbf{e} = {} & e_{rr}\,(\mathbf{e}_r\otimes\mathbf{e}_r) + e_{r\theta}\,(\mathbf{e}_r\otimes\mathbf{e}_\theta) + e_{r3}\,(\mathbf{e}_r\otimes\mathbf{e}_3) \\ & + e_{r\theta}\,(\mathbf{e}_\theta\otimes\mathbf{e}_r) + e_{\theta\theta}\,(\mathbf{e}_\theta\otimes\mathbf{e}_\theta) + e_{\theta 3}\,(\mathbf{e}_\theta\otimes\mathbf{e}_3) \\ & + e_{r3}\,(\mathbf{e}_3\otimes\mathbf{e}_r) + e_{\theta 3}\,(\mathbf{e}_3\otimes\mathbf{e}_\theta) + e_{33}\,(\mathbf{e}_3\otimes\mathbf{e}_3) \quad ,\end{aligned}$$

$$\mathbf{q} = q_r\,\mathbf{e}_r + q_\theta\,\mathbf{e}_\theta + q_3\,\mathbf{e}_3 \quad , \tag{15.6}$$

where no summation is implied here for repeated values of the indices r and θ. Since the balance laws (6.21a,c,e) are expressed in coordinate free form they can be easily translated to any set of coordinates. In particular, it is necessary to emphasize that the base vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ depend on the coordinate θ, so the expressions for the gradient and the divergence operators are more complicated than those associated with rectangular Cartesian coordinates. Specifically, it can be shown that

$$\nabla V = \frac{\partial V}{\partial r}\,\mathbf{e}_r + \frac{\partial V}{\partial \theta}\,\frac{1}{r}\,\mathbf{e}_\theta + \frac{\partial V}{\partial x_3}\,\mathbf{e}_3 \quad ,$$

$$\nabla\mathbf{u} = \frac{\partial \mathbf{u}}{\partial r}\otimes\mathbf{e}_r + \frac{\partial \mathbf{u}}{\partial \theta}\otimes\frac{1}{r}\,\mathbf{e}_\theta + \frac{\partial \mathbf{u}}{\partial x_3}\otimes\mathbf{e}_3 \quad ,$$

$$\operatorname{div}\mathbf{q} = \frac{\partial \mathbf{q}}{\partial r}\bullet\mathbf{e}_r + \frac{\partial \mathbf{q}}{\partial \theta}\bullet\frac{1}{r}\,\mathbf{e}_\theta + \frac{\partial \mathbf{q}}{\partial x_3}\bullet\mathbf{e}_3 \quad ,$$

$$\operatorname{div}\mathbf{T} = \frac{\partial \mathbf{T}}{\partial r}\bullet\mathbf{e}_r + \frac{\partial \mathbf{T}}{\partial \theta}\bullet\frac{1}{r}\,\mathbf{e}_\theta + \frac{\partial \mathbf{T}}{\partial x_3}\bullet\mathbf{e}_3 \quad ,$$

$$\nabla^2 V = \text{div}\,(\nabla V) = \frac{\partial(\nabla V)}{\partial r} \bullet \mathbf{e}_r + \frac{\partial(\nabla V)}{\partial \theta} \bullet \frac{1}{r}\,\mathbf{e}_\theta + \frac{\partial(\nabla V)}{\partial x_3} \bullet \mathbf{e}_3 \ . \tag{15.7}$$

Next, using these formulas it follows that the gradient of the displacement vector is given by

$$\nabla \mathbf{u} = \frac{\partial u_r}{\partial r}(\mathbf{e}_r \otimes \mathbf{e}_r) + \left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right](\mathbf{e}_r \otimes \mathbf{e}_\theta) + \frac{\partial u_r}{\partial x_3}(\mathbf{e}_r \otimes \mathbf{e}_3)$$

$$+ \frac{\partial u_\theta}{\partial r}(\mathbf{e}_\theta \otimes \mathbf{e}_r) + \left[\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right](\mathbf{e}_\theta \otimes \mathbf{e}_\theta) + \frac{\partial u_\theta}{\partial x_3}(\mathbf{e}_\theta \otimes \mathbf{e}_3)$$

$$+ \frac{\partial u_3}{\partial r}(\mathbf{e}_3 \otimes \mathbf{e}_r) + \left[\frac{1}{r}\frac{\partial u_3}{\partial \theta}\right](\mathbf{e}_3 \otimes \mathbf{e}_\theta) + \frac{\partial u_3}{\partial x_3}(\mathbf{e}_3 \otimes \mathbf{e}_3) \ , \tag{15.8}$$

so the strain components become

$$e_{rr} = \frac{\partial u_r}{\partial r} \ , \ e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \ , \ e_{33} = \frac{\partial u_3}{\partial x_3} \ ,$$

$$e_{r\theta} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right] \ , \ e_{r3} = \frac{1}{2}\left[\frac{\partial u_r}{\partial x_3} + \frac{\partial u_3}{\partial r}\right] \ , \ e_{\theta 3} = \frac{1}{2}\left[\frac{\partial u_\theta}{\partial x_3} + \frac{1}{r}\frac{\partial u_3}{\partial \theta}\right] . \tag{15.9}$$

Also, the divergence of the heat flux vector becomes

$$\text{div}\,\mathbf{q} = \frac{\partial q_r}{\partial r} + \frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{\partial q_3}{\partial x_3} \ , \tag{15.10}$$

the divergence of the stress tensor becomes

$$\text{div}\,\mathbf{T} = \left[\frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{r3}}{\partial x_3}\right]\mathbf{e}_r$$

$$+ \left[\frac{\partial T_{r\theta}}{\partial r} + \frac{2T_{r\theta}}{r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta 3}}{\partial x_3}\right]\mathbf{e}_\theta$$

$$+ \left[\frac{\partial T_{r3}}{\partial r} + \frac{T_{r3}}{r} + \frac{1}{r}\frac{\partial T_{\theta 3}}{\partial \theta} + \frac{\partial T_{33}}{\partial x_3}\right]\mathbf{e}_3 \ , \tag{15.11}$$

and the Laplacian of the scalar V becomes

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial x_3^2} \ . \tag{15.12}$$

Thus, the balance laws (6.21a,c,e) can be written in the forms

$$\rho = \rho_0 (1 - e_{rr} - e_{\theta\theta} - e_{33}) \ ,$$

$$\rho_0 \ddot{u}_r = \rho_0 b_r + \frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{r3}}{\partial x_3} \ ,$$

$$\rho_0 \ddot{u}_\theta = \rho_0 b_\theta + \frac{\partial T_{r\theta}}{\partial r} + \frac{2T_{r\theta}}{r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta 3}}{\partial x_3} \ ,$$

$$\rho_0 \ddot{u}_3 = \rho_0 b_3 + \frac{\partial T_{r3}}{\partial r} + \frac{T_{r3}}{r} + \frac{1}{r}\frac{\partial T_{\theta 3}}{\partial \theta} + \frac{\partial T_{33}}{\partial x_3} \ ,$$

$$\rho_0 \dot{\varepsilon} = \rho_0 r - \left[\frac{\partial q_r}{\partial r} + \frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{\partial q_3}{\partial x_3} \right] \ . \tag{15.13}$$

Also, the strain-stress relations can be written in forms similar to (10.3)

$$e_{rr} = \frac{T_{rr}}{E} - \frac{\nu T_{\theta\theta}}{E} - \frac{\nu T_{33}}{E} + \alpha(\theta^* - \theta_0^*) \ ,$$

$$e_{\theta\theta} = -\frac{\nu T_{rr}}{E} + \frac{T_{\theta\theta}}{E} - \frac{\nu T_{33}}{E} + \alpha(\theta^* - \theta_0^*) \ ,$$

$$e_{33} = -\frac{\nu T_{rr}}{E} - \frac{\nu T_{\theta\theta}}{E} + \frac{T_{33}}{E} + \alpha(\theta^* - \theta_0^*) \ ,$$

$$e_{r\theta} = \frac{(1+\nu)T_{r\theta}}{E} \ , \ e_{r3} = \frac{(1+\nu)T_{r3}}{E} \ , \ e_{\theta 3} = \frac{(1+\nu)T_{\theta 3}}{E} \ , \tag{15.14}$$

and the constitutive equation for the heat flux vector (6.22g) becomes

$$q_r = -\kappa \frac{\partial \theta^*}{\partial r} \ , \ q_\theta = -\kappa \frac{1}{r}\frac{\partial \theta^*}{\partial \theta} \ , \ q_3 = -\kappa \frac{\partial \theta^*}{\partial x_3} \ , \tag{15.15}$$

where, for clarity, the temperature has been denoted as θ^* to avoid confusion with the coordinate θ. Also, using (6.22i) the internal energy can be expressed in the form

$$\rho_0 \varepsilon = \rho_0 C_v (\theta^* - \theta_0^*) + 3K\alpha\theta_0^*(e_{rr} + e_{\theta\theta} + e_{33}) \ . \tag{15.16}$$

Next, it is of interest to determine the form of rigid body displacements in cylindrical polar coordinates. To this end, it is recalled from (3.43) that rigid body displacements can be expressed in terms of rectangular Cartesian coordinates in the forms

$$\mathbf{u} = u_i \mathbf{e}_i \ , \ u_i = c_i + H_{ij} x_j \ , \ H_{ij} = -H_{ji} \ ,$$

$$\mathbf{u} = [c_1 + H_{12} x_2 + H_{13} x_3] \mathbf{e}_1 + [c_2 - H_{12} x_1 + H_{23} x_3] \mathbf{e}_2$$

$$+ \left[c_3 - H_{13}\, x_1 - H_{23}\, x_2\right] \mathbf{e}_3 \quad , \tag{15.17}$$

where c_i and H_{ij} are independent of position. Now, the cylindrical polar components of these rigid body displacements can be obtained by using (15.4) and (15.5) to deduce that

$$u_r = \mathbf{u} \bullet \mathbf{e}_r = \left[c_1 + H_{12}\, r \sin\theta + H_{13}\, x_3\right] \cos\theta + \left[c_2 - H_{12}\, r \cos\theta + H_{23}\, x_3\right] \sin\theta \quad ,$$

$$u_\theta = -\left[c_1 + H_{12}\, r \sin\theta + H_{13}\, x_3\right] \sin\theta + \left[c_2 - H_{12}\, r \cos\theta + H_{23}\, x_3\right] \cos\theta \quad ,$$

$$u_3 = \left[c_3 - H_{13}\, r \cos\theta - H_{23}\, r \sin\theta\right] \quad , \tag{15.18}$$

which can be rewritten in the forms

$$u_r = \left[c_1 \cos\theta + c_2 \sin\theta\right] + \left[H_{13} \cos\theta + H_{23} \sin\theta\right] x_3 \quad ,$$

$$u_\theta = \left[- c_1 \sin\theta + c_2 \cos\theta\right] - H_{12}\, r + \left[- H_{13} \sin\theta + H_{23} \cos\theta\right] x_3 \quad ,$$

$$u_3 = c_3 - \left[H_{13} \cos\theta + H_{23} \sin\theta\right] r \quad . \tag{15.19}$$

16. Two-dimensional problems in polar coordinates

The equations for polar coordinates can be obtained by considering the equations for cylindrical polar coordinates section 15 and neglecting dependence on the coordinate x_3. Thus, the position vector $\mathbf{x}$ of a material point can be expressed in terms of polar coordinates in the form

$$\mathbf{x} = r\,\mathbf{e}_r(\theta) \quad , \tag{16.1}$$

where

$$\{\, r\,,\theta \,\} \quad , \tag{16.2}$$

are the coordinates and

$$\{\, \mathbf{e}_r\,,\mathbf{e}_\theta \,\} \quad , \tag{16.3}$$

are the unit base vectors. Moreover, the base vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ can be related to those of the rectangular Cartesian coordinate system by the formulas

$$\mathbf{e}_r(\theta) = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2 \quad ,$$

$$\mathbf{e}_\theta(\theta) = -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2 \quad . \tag{16.4}$$

Also, by substituting the expression for $\mathbf{e}_r$ into (15.1) it can be seen that the coordinates of the cylindrical polar coordinate system can be related to those of the rectangular Cartesian coordinate system by the formulas

$$x_1 = r\cos\theta \quad , \quad x_2 = r\sin\theta \quad ,$$

$$r = \sqrt{x_1^2 + x_2^2} \quad , \quad \theta = \tan^{-1}(x_2/x_1) \quad . \tag{16.5}$$

Here, the coordinate θ should not be confused with the same symbol that is used for the temperature.

Now, the displacement vector $\mathbf{u}$, the body force $\mathbf{b}$, the stress tensor $\mathbf{T}$, the strain tensor $\mathbf{e}$ and the heat flux vector $\mathbf{q}$ can be expressed in terms of their cylindrical polar components in the forms

$$\mathbf{u} = u_r\,\mathbf{e}_r + u_\theta\,\mathbf{e}_\theta \quad ,$$

$$\mathbf{b} = b_r\,\mathbf{e}_r + b_\theta\,\mathbf{e}_\theta \quad ,$$

$$\mathbf{T} = T_{rr}\,(\mathbf{e}_r\otimes\mathbf{e}_r) + T_{r\theta}\,(\mathbf{e}_r\otimes\mathbf{e}_\theta + \mathbf{e}_\theta\otimes\mathbf{e}_r) + T_{\theta\theta}\,(\mathbf{e}_\theta\otimes\mathbf{e}_\theta) + T_{33}\,(\mathbf{e}_3\otimes\mathbf{e}_3) \quad ,$$

$$\mathbf{e} = e_{rr}\,(\mathbf{e}_r \otimes \mathbf{e}_r) + e_{r\theta}\,(\mathbf{e}_r \otimes \mathbf{e}_\theta) + e_{\theta r}\,(\mathbf{e}_\theta \otimes \mathbf{e}_r) + e_{\theta\theta}\,(\mathbf{e}_\theta \otimes \mathbf{e}_\theta) \quad ,$$

$$\mathbf{q} = q_r\,\mathbf{e}_r + q_\theta\,\mathbf{e}_\theta \quad , \tag{16.6}$$

where no summation is implied here for repeated values of the indices r and θ. For generalized plane stress, the strain e_{33} is determined so that T_{33} vanishes. Since the balance laws (6.21a,c,e) are expressed in coordinate free form they can be easily translated to any set of coordinates. In particular, it is necessary to emphasize that the base vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ depend on the coordinate θ, so the expressions for the gradient and the divergence operators are more complicated than those associated with rectangular Cartesian coordinates. Specifically, it can be shown that

$$\nabla V = \frac{\partial V}{\partial r}\,\mathbf{e}_r + \frac{\partial V}{\partial \theta}\,\frac{1}{r}\,\mathbf{e}_\theta \quad ,$$

$$\nabla \mathbf{u} = \frac{\partial \mathbf{u}}{\partial r} \otimes \mathbf{e}_r + \frac{\partial \mathbf{u}}{\partial \theta} \otimes \frac{1}{r}\,\mathbf{e}_\theta \quad ,$$

$$\text{div}\,\mathbf{q} = \frac{\partial \mathbf{q}}{\partial r} \bullet \mathbf{e}_r + \frac{\partial \mathbf{q}}{\partial \theta} \bullet \frac{1}{r}\,\mathbf{e}_\theta \quad ,$$

$$\text{div}\,\mathbf{T} = \frac{\partial \mathbf{T}}{\partial r} \bullet \mathbf{e}_r + \frac{\partial \mathbf{T}}{\partial \theta} \bullet \frac{1}{r}\,\mathbf{e}_\theta \quad ,$$

$$\nabla^2 V = \text{div}\,(\nabla V) = \frac{\partial(\nabla V)}{\partial r} \bullet \mathbf{e}_r + \frac{\partial(\nabla V)}{\partial \theta} \bullet \frac{1}{r}\,\mathbf{e}_\theta \quad . \tag{16.7}$$

Next, using these formulas it follows that the gradient of the displacement vector is given by

$$\nabla \mathbf{u} = \frac{\partial u_r}{\partial r}\,(\mathbf{e}_r \otimes \mathbf{e}_r) + \left[\,\frac{1}{r}\,\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\,\right](\mathbf{e}_r \otimes \mathbf{e}_\theta)$$

$$+ \frac{\partial u_\theta}{\partial r}\,(\mathbf{e}_\theta \otimes \mathbf{e}_r) + \left[\,\frac{1}{r}\,\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\,\right](\mathbf{e}_\theta \otimes \mathbf{e}_\theta) \quad , \tag{16.8}$$

so the strain components become

$$e_{rr} = \frac{\partial u_r}{\partial r} \;,\; e_{\theta\theta} = \frac{1}{r}\,\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \;,\; e_{r\theta} = \frac{1}{2}\left[\,\frac{1}{r}\,\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\,\right] \quad . \tag{16.9}$$

Also, the divergence of the heat flux vector becomes

$$\text{div}\,\mathbf{q} = \frac{\partial q_r}{\partial r} + \frac{1}{r}\frac{\partial q_\theta}{\partial \theta} \quad , \tag{16.10}$$

the divergence of the stress tensor becomes

$$\text{div}\,\mathbf{T} = \left[\frac{\partial T_{rr}}{\partial r} + \frac{T_{rr}-T_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta}\right]\mathbf{e}_r$$

$$+\left[\frac{\partial T_{r\theta}}{\partial r} + \frac{2T_{r\theta}}{r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta}\right]\mathbf{e}_\theta \quad , \tag{16.11}$$

and the Laplacian of the scalar V becomes

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} \quad . \tag{16.12}$$

Thus, the balance laws (6.21a,c,e) can be written in the forms

$$\rho = \rho_0\,(1 - e_{rr} - e_{\theta\theta} - e_{33}) \quad ,$$

$$\rho_0\,\ddot{u}_r = \rho_0\,b_r + \frac{\partial T_{rr}}{\partial r} + \frac{T_{rr}-T_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} \quad ,$$

$$\rho_0\,\ddot{u}_\theta = \rho_0\,b_\theta + \frac{\partial T_{r\theta}}{\partial r} + \frac{2T_{r\theta}}{r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} \quad ,$$

$$\rho_0\,\dot{\varepsilon} = \rho_0\,r - \left[\frac{\partial q_r}{\partial r} + \frac{1}{r}\frac{\partial q_\theta}{\partial \theta}\right] \quad , \tag{16.13}$$

where e_{33} vanishes for plane strain and is given by (11.22) for generalized plane stress. Also, the strain-stress relations can be written in forms similar to (11.15), such that

$$e_{rr} = \frac{1}{2\mu}\left[(1-\bar{\nu})\,T_{rr} - \bar{\nu}\,T_{\theta\theta}\right] + (1+\bar{\nu})\bar{\alpha}(\theta^* - \theta_0^*) \quad ,$$

$$e_{\theta\theta} = \frac{1}{2\mu}\left[-\bar{\nu}\,T_{rr} + (1-\bar{\nu})\,T_{\theta\theta}\right] + (1+\bar{\nu})\bar{\alpha}(\theta^* - \theta_0^*) \quad ,$$

$$e_{r\theta} = \frac{T_{r\theta}}{2\mu} \quad , \quad e_{r3} = e_{\theta 3} = 0 \quad , \tag{16.14}$$

where, for clarity, the temperature has been denoted as θ^* to avoid confusion with the coordinate θ. Also, the stress-strain relations can be written as

$$T_{rr} = \frac{2\mu}{(1-2\bar{\nu})}\left[(1-\bar{\nu})\,e_{rr} + \bar{\nu}\,e_{\theta\theta}\right] - \frac{2\mu(1+\bar{\nu})}{(1-2\bar{\nu})}\bar{\alpha}(\theta^* - \theta_0^*) \quad ,$$

$$T_{\theta\theta} = \frac{2\mu}{(1-2\bar{\nu})}\left[\ \bar{\nu}\, e_{rr} + (1-\bar{\nu})\, e_{\theta\theta}\ \right] - \frac{2\mu(1+\bar{\nu})}{(1-2\bar{\nu})}\,\bar{\alpha}(\theta^* - \theta_0^*)\ ,$$

$$T_{r\theta} = 2\mu\, e_{r\theta}\ ,\ \ T_{r3} = T_{\theta 3} = 0\ , \tag{16.15}$$

where $\bar{\nu}$ and $\bar{\alpha}$ are defined by (11.13). In addition, the constitutive equation for the heat flux vector (6.22g) becomes

$$q_r = -\,\kappa\frac{\partial\theta^*}{\partial r}\ ,\ q_\theta = -\,\kappa\frac{1}{r}\frac{\partial\theta^*}{\partial\theta}\ , \tag{16.16}$$

Furthermore, with the help of (11.20) and (11.23), the internal energy becomes

$$\rho_0\varepsilon = \rho_0 C_v\,(\theta^* - \theta_0) + 3K\alpha\theta_0\,(e_{rr} + e_{\theta\theta})\ , \tag{16.17}$$

for plane strain and becomes

$$\rho_0\varepsilon = \left[\rho_0 C_v\ + 3K\alpha^2\theta_0\left\{\frac{1+\nu}{1-\nu}\right\}\right](\theta^* - \theta_0^*) +\ 3K\alpha\theta_0\left[\frac{1-2\nu}{1-\nu}\right](e_{rr} + e_{\theta\theta})\ . \tag{16.18}$$

for generalized plane stress.

If the body force is derivable from a potential V, then

$$\rho_0\,\mathbf{b} = -\,\nabla V\ ,\ \ \rho_0\, b_r = -\frac{\partial V}{\partial r}\ ,\ \rho_0 b_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta}\ , \tag{16.19}$$

and the equations of equilibrium can be satisfied using the Airy's stress function ϕ, which from the first of (12.10) can be generalized to yield

$$\mathbf{T} = \left[\ V + \nabla^2\phi\ \right]\mathbf{I} - \nabla(\nabla\phi)\ , \tag{16.20}$$

where $\mathbf{I}$ is the two-dimensional identity tensor. Thus, using expressions (16.7) and (16.12) it follows that

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\ ,$$

$$\nabla(\nabla\phi) = \nabla\left[\frac{\partial\phi}{\partial r}\,\mathbf{e}_r + \frac{\partial\phi}{\partial\theta}\frac{1}{r}\,\mathbf{e}_\theta\right],$$

$$\nabla(\nabla\phi) = \left[\frac{\partial^2\phi}{\partial r^2}\right](\mathbf{e}_r\otimes\mathbf{e}_r) + \left[\frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right](\mathbf{e}_\theta\otimes\mathbf{e}_\theta)$$

$$+\left[\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial\phi}{\partial\theta}\right](\mathbf{e}_r\otimes\mathbf{e}_\theta + (\mathbf{e}_\theta\otimes\mathbf{e}_r)\ . \tag{16.21}$$

Then, the stresses can be written in the forms

$$T_{rr} = V + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} \ , \ T_{\theta\theta} = V + \frac{\partial^2\phi}{\partial r^2} \ ,$$

$$T_{r\theta} = -\left[\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial\phi}{\partial\theta}\right] = -\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right] \ . \tag{16.22}$$

Next, the compatibility equation can be written in the form (12.12) or (12.14)

$$\nabla^2\nabla^2\phi = -\left[\frac{1-2\bar{\nu}}{1-\bar{\nu}}\right]\nabla^2 V - 2\mu\left[\frac{1+\bar{\nu}}{1-\bar{\nu}}\right]\bar{\alpha}\,\nabla^2\theta^* \ ,$$

$$\nabla^2(T_{rr} + T_{\theta\theta}) = \left[\frac{1}{1-\bar{\nu}}\right]\nabla^2 V - 2\mu\left[\frac{1+\bar{\nu}}{1-\bar{\nu}}\right]\bar{\alpha}\,\nabla^2\theta^* \ . \tag{16.23}$$

Thus, when the body force potential V is a harmonic function ($\nabla^2 V=0$), the temperature field θ^* is steady (independent of time) and there is no heat supply ($\nabla^2\theta^*=0$), the compatibility equations require the Airy's stress function to be a biharmonic function

$$\nabla^2\nabla^2\phi = \nabla^2(T_{rr} + T_{\theta\theta}) = 0 \ , \tag{16.24}$$

where

$$\nabla^2\nabla^2\phi = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]\left[\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right] \ . \tag{16.25}$$

Michell developed a solution of the biharmonic equation (16.24) which is quite useful for problems in cylindrical polar coordinates. The main features of this solution are summarized in the following page which has been taken from Little (1973).

Next, it is of interest to determine the form of rigid body displacements in polar coordinates. To this end, it is only necessary to eliminate dependence of the results (15.19) on the $\mathbf{e}_3$ direction so that the rigid body displacements in the plane become

$$u_r = \left[c_1\cos\theta + c_2\sin\theta\right] \ ,$$

$$u_\theta = \left[-c_1\sin\theta + c_2\cos\theta\right] - H_{12}\,r \ , \tag{16.26}$$

where c_α and H_{12} are independent of position.

The next four pages are copied from the book
Elasticity by R.Wm. Little, Prentice-Hall 1973
which is out of print.

The solution is

$$
\begin{aligned}
\varphi = {} & a_0 + b_0 \ln r + c_0 r^2 + d_0 r^2 \ln r \\
& + (A_0 + B_0 \ln r + C_0 r^2 + D_0 r^2 \ln r)\theta \\
& + \left(a_1 r + b_1 r \ln r + \frac{c_1}{r} + d_1 r^3\right)_{\cos\theta}^{\sin\theta} \\
& + (A_1 r + B_1 r \ln r)\theta \,{}^{\sin\theta}_{\cos\theta} \\
& + \sum_{n=2}^{\infty} (a_n r^n + b_n r^{2+n} + c_n r^{-n} + d_n r^{2-n}) {}^{\sin n\theta}_{\cos n\theta}.
\end{aligned}
\tag{8-4.9}
$$

The notation for $\sin n\theta$ and $\cos n\theta$ is meant to imply that $b_n r^{2+n} \sin n\theta$ is one term and $b_n r^{2+n} \cos n\theta$ is another independent term available to satisfy boundary conditions. Later, b_n and b'_n are distinguished by "upper" and "lower" superscripts.

This is essentially the solution obtained by J. H. Michell, except that he omitted some of the repeated roots. It should be pointed out that this is not a general solution because a may take any noninteger value. These, however, do not lead to periodic solutions or allow the use of orthogonal functions on the boundaries and are not of practical value in many problems. Particular examples using these will be shown later.

Using the stress definitions in terms of the Airy stress function produces the following stresses:

$$
\begin{aligned}
\sigma_{rr} = {} & \frac{b_0}{r^2} + 2c_0 + d_0(2\ln r + 1) + B_0 \frac{\theta}{r^2} + 2C_0\theta + D_0(2\ln r + 1)\theta \\
& + \left(\frac{b_1}{r} - \frac{2c_1}{r^3} + 2d_1 r\right)_{\cos\theta}^{\sin\theta} + \left(\frac{2A_1}{r}\right)_{-\sin\theta}^{\cos\theta} \\
& + \left(\frac{B_1}{r}\theta\right)_{\cos\theta}^{\sin\theta} + \left(\frac{2B_1}{r}\ln r\right)_{-\sin\theta}^{\cos\theta} \\
& - \sum_{n=2}^{\infty} \{a_n n(n-1) r^{n-2} + b_n (n+1)(n-2) r^n + c_n n(n+1) r^{-(n+2)} \\
& + d_n (n-1)(n+2) r^{-n}\} {}^{\sin n\theta}_{\cos n\theta},
\end{aligned}
\tag{8-4.10}
$$

$$
\begin{aligned}
\sigma_{\theta\theta} = {} & -\frac{b_0}{r^2} + 2c_0 + d_0(2\ln r + 3) - \frac{B_0\theta}{r^2} + 2C_0\theta + D_0(2\ln r + 3)\theta \\
& + \left(\frac{b_1}{r} + \frac{2c_1}{r^3} + 6d_1 r\right)_{\cos\theta}^{\sin\theta} + \left(\frac{B_1\theta}{r}\right)_{\cos\theta}^{\sin\theta} \\
& + \sum_{n=2}^{\infty} \{a_n n(n-1) r^{n-2} + b_n (n+1)(n+2) r^n + c_n n(n+1) r^{-(n+2)} \\
& + d_n (n-2)(n-1) r^{-n}\} {}^{\sin n\theta}_{\cos n\theta},
\end{aligned}
\tag{8-4.11}
$$

$$
\begin{aligned}
\sigma_{r\theta} = {} & \frac{A_0}{r^2} + B_0 \frac{\ln r - 1}{r^2} - C_0 - D_0(\ln r + 1) + \left(-\frac{b_1}{r} + \frac{2c_1}{r} - 2d_1 r\right)_{-\sin\theta}^{\cos\theta} \\
& - \left(\frac{B_1}{r}\right)_{\cos\theta}^{\sin\theta} - \left(\frac{B_1}{r}\theta\right)_{-\sin\theta}^{\cos\theta} \\
& - \sum_{n=2}^{\infty} \{a_n n(n-1) r^{n-2} + b_n n(n+1) r^n - c_n n(n+1) r^{-(n+2)} \\
& - d_n n(n-1) r^{-n}\} {}^{\cos n\theta}_{-\sin n\theta}.
\end{aligned}
\tag{8-4.12}
$$

The displacements may be obtained from the following relations:

$$\frac{\partial u_r}{\partial r} = \frac{1}{E}\{\sigma_{rr} - \nu\sigma_{\theta\theta}\}, \tag{8-4.13}$$

$$\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = \frac{1}{E}\{\sigma_{\theta\theta} - \nu\sigma_{rr}\} - \frac{u_r}{r}, \tag{8-4.14}$$

$$\frac{1+\nu}{E}\sigma_{r\theta} = \frac{1}{2}\left\{\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right\}. \tag{8-4.15}$$

Substituting equations 8-4.10 and 8-4.11 into 8-4.13 and integrating yields

$$\begin{aligned} u_r = \frac{1}{E}\Big\{ & -\frac{b_0}{r}(1+\nu) + 2c_0(1-\nu)r + d_0(1-\nu)(2r\ln r - r) - 2d_0\nu r \\ & + \left[-\frac{B_0}{r}(1+\nu) + 2C_0(1-\nu)r + D_0(1-\nu)(2r\ln r - r) - 2D_0\nu r\right]\theta \\ & + \left[b_1(1-\nu)\ln r + \frac{c_1}{r^2}(1+\nu) + d_1 r^2(1-\nu) - 2d_1\nu r^2\right]_{\cos\theta}^{\sin\theta} \\ & + [2A_1\ln r]_{-\sin\theta}^{\cos\theta} + [B_1\ln r(1-\nu)]_{\theta\cos\theta}^{\theta\sin\theta} + [B_1\ln^2 r]_{-\sin\theta}^{\cos\theta} \\ & + \sum_{n=2}^{\infty}[a_n n(1+\nu)r^{n-1} + b_n\{(n-2)+\nu(n+2)\}r^{n+1} \\ & - c_n n(1+\nu)r^{-n+1} - d_n\{(n+2)+\nu(n-2)\}r^{-(n-1)}]_{\cos n\theta}^{\sin n\theta} + g(\theta)\Big\}, \end{aligned} \tag{8-4.16}$$

where $g(\theta)$ is an arbitrary function of integration. From equation 8-4.14, we may now obtain

$$\begin{aligned} u_\theta = \frac{1}{E}\Big\{ & 4d_0\theta r + 2D_0\theta^2 r + \Big[b_1(1-\nu)(1-\ln r) + \frac{c_1(1+\nu)}{r^2} \\ & + d_1(5+\nu)r^2\Big]_{\sin\theta}^{-\cos\theta} - [2A_1(\ln r + \nu)]_{\cos\theta}^{\sin\theta} \\ & + B_1[(1-\nu) - (1+\nu)\ln r - \ln^2 r]_{\cos\theta}^{\sin\theta} \\ & - B_1[(1-\nu) - \ln r(1-\nu)]_{-\theta\sin\theta}^{\theta\cos\theta} + \sum_{n=2}^{\infty}[a_n n(1+\nu)r^{n-1} \\ & + b_n\{n(1+\nu)+4\}r^{n+1} + c_n n(1+\nu)r^{-(n+1)} \\ & + d_n\{n(1+\nu)-4\}r^{-(n-1)}]_{\sin n\theta}^{-\cos n\theta} - \int_\theta g(\theta)\,d\theta + f(r)\Big\}. \end{aligned} \tag{8-4.17}$$

Substituting equations 8-4.12, 8-4.16, and 8-4.17 into equation 8-4.15 yields

$$\begin{aligned} & -2(1+\nu)\frac{A_0}{r^2} + (1-\nu)B_0\left[\frac{-2\ln r + 1}{r^2}\right] + 4C_0 + D_0[4\ln r + 1 + 3\nu] \\ & + \left[\frac{4b_1}{r}\right]_{-\sin\theta}^{\cos\theta} - \left[\frac{2A_1}{r}(1-\nu)\right]_{\cos\theta}^{\sin\theta} + B_1\left[\frac{4}{r}\right]_{-\theta\sin\theta}^{\theta\cos\theta} + \left[B_1\frac{2\nu}{r}\right]_{\cos\theta}^{\sin\theta} \\ & + f'(r) + \frac{1}{r}\left\{g'(\theta) + \int_\theta g(\theta)\,d\theta\right\} - \frac{f(r)}{r} = 0, \end{aligned} \tag{8-4.18}$$

This leads to the following differential equations for the functions of integra-

tion:

$$rf'(r) - f(r) = 2(1+\nu)\frac{A_0}{r} + (1+\nu)B_0\left[\frac{2\ln r - 1}{r}\right] - 4C_0 r \qquad (8\text{-}4.19)$$

$$- D_0[4r\ln r + (1+\nu)r] + K \qquad (8\text{-}4.20)$$

$$g'(\theta) + \int_\theta g(\theta)\,d\theta = -[4b_1]_{-\sin\theta}^{\cos\theta} + [2A_1(1-\nu) - B_1 2\nu]_{\cos\theta}^{\sin\theta}$$
$$- [B_1 4]_{-\theta\sin\theta}^{\theta\cos\theta} - K.$$

To solve equation 8-4.19, we introduce the change of variable $\xi = \ln r$,

$$\frac{df}{dr} = \frac{1}{r}\frac{df}{d\xi}$$

Therefore, equation 8-4.19 becomes

$$f'(\xi) - f(\xi) = 2(1+\nu)A_0 e^{-\xi} + (1+\nu)B_0(2\xi - 1)e^{-\xi}$$
$$- (4C_0 + (1+\nu)D_0)e^{\xi} - 4D_0\xi e^{\xi} + K.$$

Solving yields

$$f(r) = R_1 r - (1+\nu)B_0\frac{\ln r}{r^2} - 4C_0 r\ln r - (1+\nu)D_0 r\ln r$$
$$- \frac{(1+\nu)A_0}{r} - 2D_0 r\ln^2 r - K, \qquad (8\text{-}4.21)$$

where R_1 is an arbitrary constant.

Treating $\int_\theta g(\theta)\,d\theta$ as the unknown function, the solution of 8-4.20 is

$$\int_\theta g(\theta)\,d\theta = -[2b_1]_{\theta\cos\theta}^{\theta\sin\theta} - [B_1]_{\theta^2\cos\theta}^{\theta^2\sin\theta} - [A_1(1-\nu)$$
$$+ B_1(1+\nu)]_{-\theta\sin\theta}^{\theta\cos\theta} + S_1\sin\theta + S_2\cos\theta - K, \qquad (8\text{-}4.22)$$

where S_1 and S_2 are arbitrary constants.

Differentiating equation 8-4.22 yields

$$g(\theta) = -[2b_1]_{-\theta\sin\theta}^{\theta\cos\theta} - [2b_1]_{\cos\theta}^{\sin\theta} - [B_1]_{(-\theta^2\sin\theta+2\theta\cos\theta)}^{(\theta^2\cos\theta+2\theta\sin\theta)}$$
$$- [A_1(1-\nu) + B_1(1+\nu)]_{(-\sin\theta-\theta\cos\theta)}^{(\cos\theta-\theta\sin\theta)} + S_1\cos\theta - S_2\sin\theta. \qquad (8\text{-}4.23)$$

We now examine terms that lead to multivalued displacements or stresses. Examining equations 8-4.10, 8-4.11, and 8-4.12, we note that the coefficients B_0, C_0, D_0, and B_1 must be selected to be zero if we wish single-valued stresses, that is, for cases where the origin of the system lies within the body under consideration. In a simply connected region, these terms are single valued if the origin is placed outside the body or on its boundary. In this case they should be included for a complete set of integer solutions. Examining equations 8-4.16, 8-4.17, 8-4.21, 8-4.22, and 8-4.23 indicates that the coefficients d_0, B_0, C_0, D_0, A_1, B_1, and b_1 multiply multivalued displacement terms. For a problem requiring single-valued stresses and displacements,

the following conditions must hold:

$$d_0 = B_0 = C_0 = D_0 = B_1 = 0, \tag{8-4.24}$$

and b_1 and A_1 must be related in the following manner:

$$b_1^{\text{upper}} = \frac{(1-\nu)}{2} A_1^{\text{lower}}, \tag{8-4.25}$$

$$b_1^{\text{lower}} = -\frac{(1-\nu)}{2} A_1^{\text{upper}}, \tag{8-4.26}$$

where "upper" and "lower" indicate the upper or lower multiplier of the coefficient, as is given in the solution shown in equation 8-4.9.

The physical significance of some of the terms may be better understood if we consider a small circle of radius a, including the origin (Figure 8-5). The resultant of the traction from the material inside a acting on the

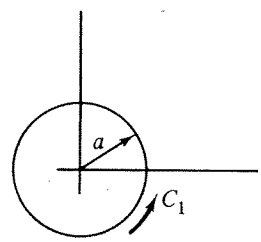

Figure 8-5

material exterior to the contour C_1 is

$$\oint_{C_1} \hat{n} \cdot \bar{\bar{\sigma}}\, ds = \bar{R}, \tag{8-4.27}$$

where $\hat{n} = -\hat{i}_{(r)}, \qquad ds = a\, d\theta,$

$$\bar{\bar{\sigma}} = \hat{i}_{(r)}\sigma_{rr}\hat{i}_{(r)} + \hat{i}_{(r)}\sigma_{r\theta}\hat{i}_{(\theta)} + \hat{i}_{(\theta)}\sigma_{\theta\theta}\hat{i}_{(\theta)} + \hat{i}_{(\theta)}\sigma_{\theta r}\hat{i}_{(r)}.$$

Therefore, the integral becomes

$$-\oint_{C_1} (\sigma_{rr}\hat{i}_{(r)} + \sigma_{r\theta}\hat{i}_{(\theta)}) a\, d\theta = \bar{R}. \tag{8-4.28}$$

Since $\hat{i}_{(r)}$ and $\hat{i}_{(\theta)}$ are functions of θ, it is easier to convert these to Cartesian coordinates:

$$\hat{i}_{(r)} = \hat{i}_{(x)} \cos\theta + \hat{i}_{(y)} \sin\theta,$$
$$\hat{i}_{(\theta)} = -\hat{i}_{(x)} \sin\theta + \hat{i}_{(y)} \cos\theta.$$

Therefore,

$$\bar{R} = -\oint_{C_1} [\{\sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta\}\hat{i}_{(x)} + \{\sigma_{rr}\sin\theta + \sigma_{r\theta}\cos\theta\}\hat{i}_{(y)}] a\, d\theta. \tag{8-4.29}$$

Using equations 8-4.10 and 8-4.11 evaluated at $r = a$, and omitting terms that

17. Lame's problem: Internal and external pressure on a cylindrical tube

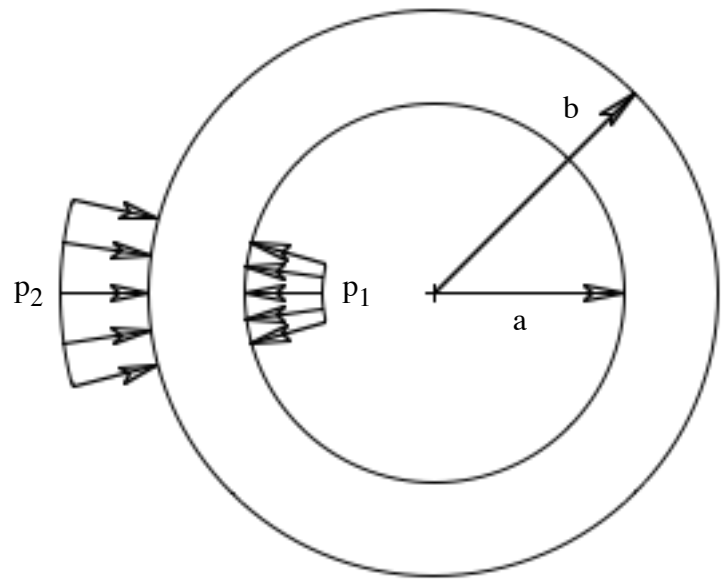

Fig. 17.1 Sketch of a cylindrical tube subjected to internal pressure p_1 and external pressure p_2.

Lame's problem considers two-dimensional deformation of a cylindrical tube that is subjected to an internal pressure p_1 and an external pressure p_2. The internal radius of the cylinder is take to be a and its external radius is take to be b. Also, the body force vanishes

$$b_r = b_\theta = 0 \quad , \tag{17.1}$$

and the temperature θ is taken to be uniform but not necessarily equal to θ_0.

For this problem it is sufficient to assume that the Airy's stress function $\phi(r)$ is a function of r only so that the biharmonic equation (16.24) reduces to

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \right] \left[\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} \right] = \frac{\partial^4 \phi}{\partial r^4} + \frac{2}{r}\frac{\partial^3 \phi}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^3}\frac{\partial \phi}{\partial r} = 0 \quad . \tag{17.2}$$

The relevant solution for Lame's problem is

$$\phi = \frac{A}{2} r^2 + B \ln(r) \quad , \tag{17.3}$$

where A and B are constants to be determined. Now, using (16.22), with the potential V set to zero, it follows that the stresses are given by

$$T_{rr} = A + \frac{B}{r^2} \ , \ T_{\theta\theta} = A - \frac{B}{r^2} \ , \ T_{r\theta} = 0 \ . \tag{17.4}$$

However, the boundary conditions can be expressed in the forms

On r = b, $\mathbf{n} = \mathbf{e}_r$,

$$\mathbf{t} = -\, p_2\, \mathbf{e}_r = \mathbf{T}(b)\, \mathbf{e}_r = T_{rr}(b)\, \mathbf{e}_r \ , \ \ T_{rr}(b) = A + \frac{B}{b^2} = -\, p_2 \ , \tag{17.5}$$

On r = a, $\mathbf{n} = -\, \mathbf{e}_r$,

$$\mathbf{t} = p_1\, \mathbf{e}_r = \mathbf{T}(a)\, (-\, \mathbf{e}_r) = -\, T_{rr}(a)\, \mathbf{e}_r \ , \ \ T_{rr}(a) = A + \frac{B}{a^2} = -\, p_1 \ . \tag{17.6}$$

These equations can be solved to deduce that

$$A = \frac{a^2 p_1 - b^2 p_2}{b^2 - a^2} \ , \ B = \frac{a^2 b^2 (p_2 - p_1)}{b^2 - a^2} \ . \tag{17.7}$$

Thus, using these results it follows that

$$T_{rr}(a) = -\, p_1 \ , \ T_{\theta\theta}(a) = \frac{(a^2+b^2)p_1 - 2b^2 p_2}{b^2 - a^2} \ ,$$

$$T_{rr}(b) = -\, p_2 \ , \ T_{\theta\theta}(b) = \frac{2a^2 p_1 - (a^2+b^2)p_2}{b^2 - a^2} \ . \tag{17.8}$$

Next, the constitutive equation (16.14) is used to determine the strains

$$e_{rr} = \frac{1}{2\mu} \left[(1-2\bar{\nu})\, A + \frac{B}{r^2} \right] + (1+\bar{\nu})\bar{\alpha}(\theta^* - \theta_0^*) \ ,$$

$$e_{\theta\theta} = \frac{1}{2\mu} \left[(1-2\bar{\nu})\, A - \frac{B}{r^2} \right] + (1+\bar{\nu})\bar{\alpha}(\theta^* - \theta_0^*) \ , \ \ e_{r\theta} = 0 \ . \tag{17.9}$$

Consequently, for axisymmetric deformation

$$u_r = u_r(r) \ , \ u_\theta = 0 \ , \tag{17.10}$$

so the strain-displacement relations (16.9) yield

$$u_r = \frac{1}{2\mu} \left[(1-2\bar{\nu})\, A\, r - \frac{B}{r} \right] + (1+\bar{\nu})\bar{\alpha}(\theta^* - \theta_0^*)\, r \ . \tag{17.11}$$

Notice that if the cylinder is solid (a=0), then B vanishes and the solution reduces to

$$a = 0 \ , \ A = -p_2 \ , \ B = 0 \ ,$$

$$T_{rr} = T_{\theta\theta} = -\, p_2 \ , \ T_{r\theta} = 0 \ ,$$

$$u_r = \left[-\frac{(1-2\bar{\nu})p_2}{2\mu} + (1+\bar{\nu})\bar{\alpha}(\theta^*-\theta_0^*)\right] r \ , \ u_\theta = 0 \ , \tag{17.12}$$

which corresponds to homogeneous deformation in the plane.

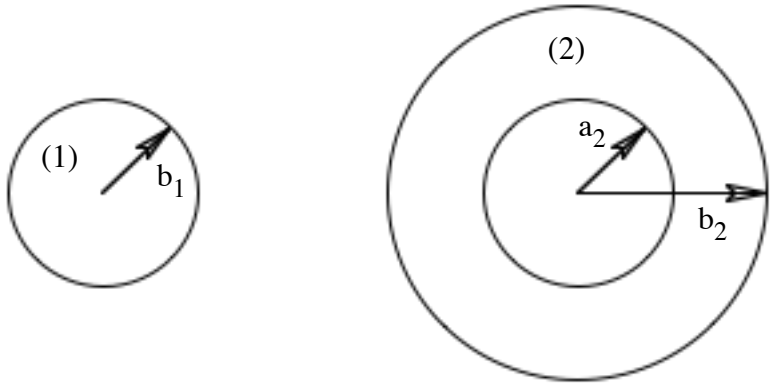

Fig. 17.2 Shrink fitting of a cylindrical tube onto a solid cylinder.

As another example consider the case when the cylinder is subjected to only an exterior pressure ($p_2 > 0$ and $p_1=0$). Then, the stresses (17.8) simplify to

$$T_{rr}(a) = 0 \ , \ T_{\theta\theta}(a) = -\frac{2b^2p_2}{b^2-a^2} \ ,$$

$$T_{rr}(b) = -p_2 \ , \ T_{\theta\theta}(b) = -\frac{(a^2+b^2)p_2}{b^2-a^2} \ . \tag{17.13}$$

Now, if the inner radius is much smaller than the outer radius ($a << b$), then these expressions further simplify to

$$T_{rr}(a) = 0 \ , \ T_{\theta\theta}(a) = -2p_2 \ ,$$

$$T_{rr}(b) = -p_2 \ , \ T_{\theta\theta}(b) = -p_2 \ , \tag{17.14}$$

which shows that there is a stress concentration of a factor of 2 at the inner boundary. Moreover, by taking the thickness of the tube to be H it follows that

$$a = b - H \ ,$$

$$T_{\theta\theta}(a) = -\frac{bp_2}{H\left[1-\frac{H}{2b}\right]} \ , \ T_{\theta\theta}(b) = -\frac{bp_2\left[1-\frac{H}{b}+\frac{H^2}{2b^2}\right]}{H\left[1-\frac{H}{2b}\right]} \ , \tag{17.15}$$

which yields the simple strength of materials solution for a thin tube ($a=b-H$, $H/b<<1$)

$$T_{\theta\theta} \approx -\frac{bP_2}{H} \quad . \tag{17.16}$$

As another example, consider a solid cylinder (1) of outer radius b_1, and a hollow tube (2) of inner radius a_2 and outer radius b_2 [see Fig. 17.2]. Both bodies are made of the same material, but b_1 is slightly larger than a_2

$$b_1 > a_2 \quad . \tag{17.17}$$

In order to fit the hollow tube over the solid cylinder, the hollow tube is heated to the temperature θ_2^* which is the minimum temperature required to have the hollow tube just fit over the solid cylinder (which remains at temperature θ_0^*). It therefore, follows that the heated location of the inner radius of the hollow tube is given by

$$b_1 = a_2 + u_r^{(2)}(a_2) \quad . \tag{17.18}$$

Since the stresses in the hollow tube vanish, it follows (17.4), (17.7) and (17.11) that

$$u_r^{(2)}(a_2) = (1+\bar{\nu})\bar{\alpha}(\theta_2^* - \theta_0^*)\, a_2 \quad , \tag{17.19}$$

so that the temperature θ_2^* is given by

$$\theta_2^* = \theta_0^* + \frac{b_1 - a_2}{a_2(1+\bar{\nu})\bar{\alpha}} \quad . \tag{17.20}$$

After placing the hollow tube over the solid cylinder, the hollow tube is allowed to cool down to room temperature θ_0^* and the contact pressure p develops at the interface of the solid cylinder and the hollow tube. To determine this pressure it is necessary to specify both a kinematic and a kinetic boundary condition at this interface. Specifically, the kinematic boundary condition requires the deformed location of the outer radius of the solid cylinder to be the same as the deformed inner radius of the hollow tube

$$b_1 + u_r^{(1)}(b_1) = a_2 + u_r^{(2)}(a_2) \quad , \tag{17.21}$$

and the kinetic boundary condition requires the radial stress to be continuous at this interface

$$T_{rr}^{(1)}(b_1) = T_{rr}^{(2)}(a_2) = -\,p \quad . \tag{17.22}$$

Now, using (17.12) for the solid cylinder it follows that

$$u_r^{(1)}(b_1) = -\left[\frac{(1-2\bar{\nu})b_1}{2\mu}\right] p \ , \tag{17.23}$$

and using (17.7) and (17.11) for the cooled down hollow tube it follows that

$$A^{(2)} = \left[\frac{a_2^2}{b_2^2-a_2^2}\right] p \ , \quad B^{(2)} = -\left[\frac{a_2^2 b_2^2}{b_2^2-a_2^2}\right] p \ ,$$

$$u_r^{(2)}(a_2) = \frac{1}{2\mu}\left[(1-2\bar{\nu})\left\{\frac{a_2^3}{b_2^2-a_2^2}\right\} + \left\{\frac{a_2 b_2^2}{b_2^2-a_2^2}\right\}\right] p \ . \tag{17.24}$$

Then, (17.23) and (17.24) can be substituted into the kinematic condition (17.21) to determine the contact pressure

$$b_1 - \left[\frac{(1-2\bar{\nu})b_1}{2\mu}\right] p = a_2 + \frac{1}{2\mu}\left[(1-2\bar{\nu})\left\{\frac{a_2^3}{b_2^2-a_2^2}\right\} + \left\{\frac{a_2 b_2^2}{b_2^2-a_2^2}\right\}\right] p \ ,$$

$$p = \frac{2\mu(b_1-a_2)}{\left[(1-2\bar{\nu})b_1 + \frac{(1-2\bar{\nu})a_2^3}{b_2^2-a_2^2} + \frac{a_2 b_2^2}{b_2^2-a_2^2} \right]} \ , \tag{17.25}$$

which then can be used in (17.21) to determine the deformed radius of the interface.

18. Kirsch's problem: Loading of a plate with a circular hole

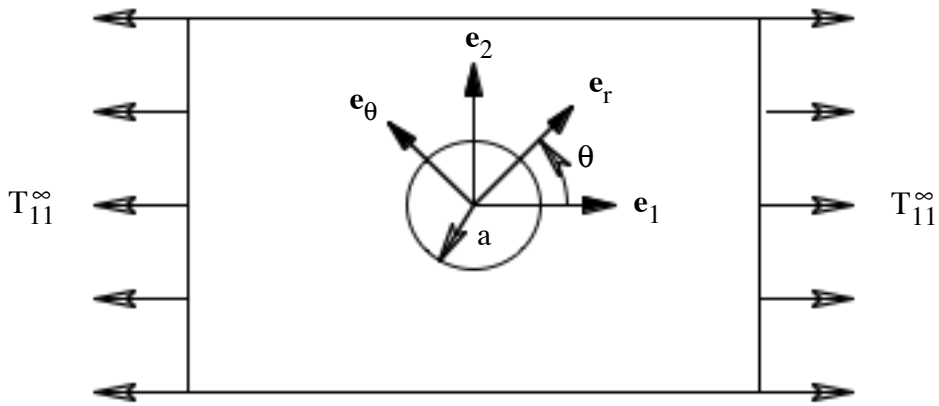

Fig. 18.1 Sketch of a plate with a circular hole of radius a, which is subjected to a tension T_{11}^∞ in the $\mathbf{e}_1$ direction far away from the hole.

Kirsch's problem considers two-dimensional deformation of a plate with a circular hole of radius a, which is subjected to a tension T_{11}^∞ in the $\mathbf{e}_1$ direction far away from the hole. The surface of the hole remains stress-free, the body force vanishes

$$b_r = b_\theta = 0 \quad , \tag{18.1}$$

and the temperature θ^* is taken to be uniform but not necessarily equal to θ_0^*.

LOADING IN THE $\mathbf{e}_1$ DIRECTION

The boundary conditions for the hole are most easily stated in terms of cylindrical polar coordinates, whereas those for the loading far away from the hole (at infinity) are most easily stated in terms of rectangular Cartesian coordinates

$$\mathbf{t} = \mathbf{T}(a)\,(-\mathbf{e}_r) = 0 \quad ,$$

$$\Rightarrow T_{rr}(a) = 0 \ , \ T_{r\theta}(a) = 0 \ , \tag{18.2a,b}$$

$$\lim_{b\to\infty} \mathbf{t} = \lim_{b\to\infty} \mathbf{T}(\pm b, x_2)\,(\pm\,\mathbf{e}_1) = \pm T_{11}^\infty\,\mathbf{e}_1$$

$$\Rightarrow \lim_{b\to\infty} T_{11}(\pm b, x_2) = T_{11}^\infty \ , \ \lim_{b\to\infty} T_{12}(\pm b, x_2) = 0 \ , \tag{18.2c,d}$$

$$\lim_{b\to\infty} \mathbf{t} = \lim_{b\to\infty} \mathbf{T}(x_1, \pm b)\,(\pm\,\mathbf{e}_2) = 0 \ ,$$

$$\Longrightarrow \lim_{b\to\infty} T_{12}(x_1,\pm b) = 0 \ , \ \lim_{b\to\infty} T_{22}(x_1,\pm b) = 0 \ . \tag{18.2e,f}$$

Moreover, to solve the problem it is most convenient to use cylindrical polar coordinates and to transform the boundary conditions at r=∞ from rectangular Cartesian components to cylindrical polar coordinates. In particular, it follows from (18.2c-f) that the stress tensor at infinity is given by

$$\lim_{r\to\infty} \mathbf{T} = T_{11}^{\infty}\,(\mathbf{e}_1\otimes\mathbf{e}_1) \ . \tag{18.3}$$

Consequently, using the definitions (16.4) for the base vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ it follows that

$$\lim_{r\to\infty} T_{rr} = T_{11}^{\infty}\,(\mathbf{e}_1\otimes\mathbf{e}_1)\bullet(\mathbf{e}_r\otimes\mathbf{e}_r) = T_{11}^{\infty}\cos^2\theta = \frac{T_{11}^{\infty}}{2}\left[1+\cos(2\theta)\right] \ ,$$

$$\lim_{r\to\infty} T_{\theta\theta} = T_{11}^{\infty}\,(\mathbf{e}_1\otimes\mathbf{e}_1)\bullet(\mathbf{e}_\theta\otimes\mathbf{e}_\theta) = T_{11}^{\infty}\sin^2\theta = \frac{T_{11}^{\infty}}{2}\left[1-\cos(2\theta)\right] \ ,$$

$$\lim_{r\to\infty} T_{r\theta} = T_{11}^{\infty}\,(\mathbf{e}_1\otimes\mathbf{e}_1)\bullet(\mathbf{e}_r\otimes\mathbf{e}_\theta) = -\frac{T_{11}^{\infty}}{2}\sin(2\theta) \ . \tag{18.4}$$

Now, with the help of the general solution of Michell and recognizing that T_{rr} and $T_{r\theta}$ must vanish at r=a, it follows that the stresses should take the forms

$$T_{rr} = \frac{b_0}{r^2} + 2c_0 - \left[2a_2 + \frac{6c_2}{r^4} + \frac{4d_2}{r^2}\right]\cos(2\theta) \ ,$$

$$T_{\theta\theta} = -\frac{b_0}{r^2} + 2c_0 + \left[2a_2 + \frac{6c_2}{r^4}\right]\cos(2\theta) \ ,$$

$$T_{r\theta} = \frac{A_0}{r^2} + \left[2a_2 - \frac{6c_2}{r^4} - \frac{2d_2}{r^2}\right]\sin(2\theta) \ , \tag{18.5}$$

where $\{b_0,\ c_0,\ a_2,\ c_2,\ A_0\}$ are constants to be determined. To make sure that no typographical error in the Michell solution affects this stress field, it is necessary to check that (18.5) satisfies the equilibrium equations (16.13)

$$\frac{\partial T_{rr}}{\partial r} + \frac{T_{rr}-T_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial\theta} = 0 \ ,$$

$$\frac{\partial T_{r\theta}}{\partial r} + \frac{2T_{r\theta}}{r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} = 0 \ , \tag{18.6}$$

and the compatibility equations (16.23)

$$\nabla^2(T_{rr} + T_{\theta\theta}) = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right] (T_{rr} + T_{\theta\theta}) = 0 \ . \tag{18.7}$$

Next, substituting (18.5) into the boundary conditions (18.2a,b) and (18.4) yields the conditions

$$\frac{b_0}{a^2} + 2c_0 - \left[2a_2 + \frac{6c_2}{a^4} + \frac{4d_2}{a^2}\right] \cos(2\theta) = 0 \ ,$$

$$\frac{A_0}{a^2} + \left[2a_2 - \frac{6c_2}{a^4} - \frac{2d_2}{a^2}\right] \sin(2\theta) = 0 \ ,$$

$$2c_0 - 2a_2 \cos(2\theta) = \frac{T_{11}^{\infty}}{2}\left[1 + \cos(2\theta)\right] \ ,$$

$$2c_0 + 2a_2 \cos(2\theta) = \frac{T_{11}^{\infty}}{2}\left[1 - \cos(2\theta)\right] \ ,$$

$$2a_2 \sin(2\theta) = -\frac{T_{11}^{\infty}}{2} \sin(2\theta) \ , \tag{18.8}$$

which can be solved to deduce that

$$b_0 = -\frac{a^2 T_{11}^{\infty}}{2} \ , c_0 = \frac{T_{11}^{\infty}}{4} \ , a_2 = -\frac{T_{11}^{\infty}}{4} \ , c_2 = -\frac{a^4 T_{11}^{\infty}}{4} \ , \ d_2 = \frac{a^2 T_{11}^{\infty}}{2} \ , \ A_0 = 0 \ . \tag{18.9}$$

Thus, the stress field (18.5) can be rewritten in the form

$$T_{rr} = \frac{T_{11}^{\infty}}{2}\left[1 - \frac{a^2}{r^2}\right] + \frac{T_{11}^{\infty}}{2}\left[1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right] \cos(2\theta) \ ,$$

$$T_{\theta\theta} = \frac{T_{11}^{\infty}}{2}\left[1 + \frac{a^2}{r^2}\right] - \frac{T_{11}^{\infty}}{2}\left[1 + \frac{3a^4}{r^4}\right] \cos(2\theta) \ ,$$

$$T_{r\theta} = -\frac{T_{11}^{\infty}}{2}\left[1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right] \sin(2\theta) \ . \tag{18.10}$$

It is important to emphasize that this solution predicts a stress concentration at the boundary of the hole [see Fig. 17.2]. Specifically, it follows from (18.10) that

$$T_{\theta\theta}(a,\theta) = T_{11}^{\infty}\left[1 - 2\cos(2\theta)\right] ,$$

$$T_{\theta\theta}(a,0) = -T_{11}^{\infty} , \; T_{\theta\theta}(a,\pi/2) = 3\,T_{11}^{\infty} . \tag{18.11}$$

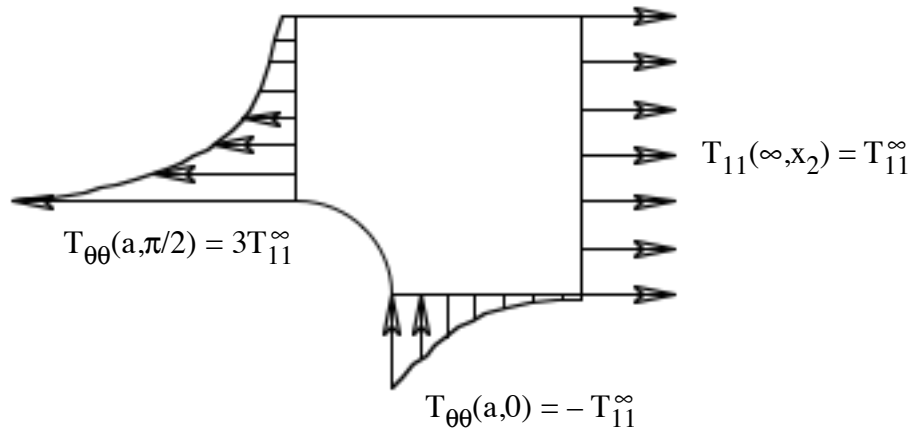

Fig. 18.2 Stress concentration at the boundary of a hole.

In order to determine the displacement field associated with the solution (18.10), use is made of the constitutive equations (16.14) with the temperature set to the constant value θ_1^*

$$e_{rr} = \frac{T_{11}^{\infty}}{4\mu}\left[(1-2\bar{\nu}) - \frac{a^2}{r^2}\right] + \frac{T_{11}^{\infty}}{4\mu}\left[1 + \frac{3a^4}{r^4} - \frac{4(1-\bar{\nu})a^2}{r^2}\right]\cos(2\theta) + (1+\bar{\nu})\bar{\alpha}(\theta_1^*-\theta_0^*) ,$$

$$e_{\theta\theta} = \frac{T_{11}^{\infty}}{4\mu}\left[(1-2\bar{\nu}) + \frac{a^2}{r^2}\right] - \frac{T_{11}^{\infty}}{4\mu}\left[1 + \frac{3a^4}{r^4} - \frac{4\bar{\nu}a^2}{r^2}\right]\cos(2\theta) + (1+\bar{\nu})\bar{\alpha}(\theta_1^*-\theta_0^*) ,$$

$$e_{r\theta} = -\frac{T_{11}^{\infty}}{4\mu}\left[1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right]\sin(2\theta) . \tag{18.12}$$

Next, integration of the strain-displacement relations (16.9) yields

$$u_r = \frac{T_{11}^{\infty}}{4\mu}\left[(1-2\bar{\nu})\,r + \frac{a^2}{r}\right] + \frac{T_{11}^{\infty}}{4\mu}\left[r - \frac{a^4}{r^3} + \frac{4(1-\bar{\nu})a^2}{r}\right]\cos(2\theta)$$

$$+ (1+\bar{\nu})\bar{\alpha}(\theta_1^*-\theta_0^*)\,r + \frac{df(\theta)}{d\theta} ,$$

$$u_\theta = -\frac{T_{11}^{\infty}}{4\mu}\left[r + \frac{a^4}{r^3} + \frac{2(1-2\bar{\nu})a^2}{r}\right]\sin(2\theta) - f(\theta) + g(r) , \tag{18.13}$$

where $f(\theta)$ and $g(r)$ are functions of integration. Moreover, substituting these results into the expression (16.9) for the strain $e_{r\theta}$ and use of (18.12) yields the equation

$$\frac{1}{r}\left[\frac{d^2f(\theta)}{d\theta^2}+f(\theta)\right]+\left[\frac{dg(r)}{dr}-\frac{1}{r}\,g(r)\right]=0\ . \tag{18.14}$$

Thus, in view of the expressions (16.8) for rigid body displacements it follows that the solution of (18.14) can be expressed in the form

$$f(\theta)=c_1\sin\theta-c_2\cos\theta\ ,\ g(r)=-H_{12}\,r\ , \tag{18.15}$$

where c_α and H_{12} are constants, which for the present purposes can be set equal to zero so that

$$f(\theta)=0\ ,\ g(r)=0\ . \tag{18.16}$$

Then, the displacements reduce to

$$u_r=\frac{T_{11}^\infty}{4\mu}\left[(1-2\bar{\nu})\,r+\frac{a^2}{r}\right]+\frac{T_{11}^\infty}{4\mu}\left[r-\frac{a^4}{r^3}+\frac{4(1-\bar{\nu})a^2}{r}\right]\cos(2\theta)+(1+\bar{\nu})\bar{\alpha}(\theta_1^*-\theta_0^*)\,r\ ,$$

$$u_\theta=-\frac{T_{11}^\infty}{4\mu}\left[r+\frac{a^4}{r^3}+\frac{2(1-2\bar{\nu})a^2}{r}\right]\sin(2\theta)\ . \tag{18.17}$$

LOADING IN THE $\mathbf{e}_2$ DIRECTION

This previous solution can be used to obtain the solution for tension T_{22}^∞ in the $\mathbf{e}_2$ direction by making the replacements

$$\theta\rightarrow(\theta-\frac{\pi}{2})\ ,\ T_{11}^\infty\rightarrow T_{22}^\infty\ , \tag{18.18}$$

in (18.10) for the stresses

$$T_{rr}=\frac{T_{22}^\infty}{2}\left[1-\frac{a^2}{r^2}\right]-\frac{T_{22}^\infty}{2}\left[1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right]\cos(2\theta)\ ,$$

$$T_{\theta\theta}=\frac{T_{22}^\infty}{2}\left[1+\frac{a^2}{r^2}\right]+\frac{T_{22}^\infty}{2}\left[1+\frac{3a^4}{r^4}\right]\cos(2\theta)\ ,$$

$$T_{r\theta}=\frac{T_{22}^\infty}{2}\left[1-\frac{3a^4}{r^4}+\frac{2a^2}{r^2}\right]\sin(2\theta)\ , \tag{18.19}$$

and in (18.17) for the displacements

$$u_r = \frac{T_{22}^{\infty}}{4\mu}\left[\,(1-2\bar{\nu})\,r + \frac{a^2}{r}\,\right] - \frac{T_{22}^{\infty}}{4\mu}\left[\,r - \frac{a^4}{r^3} + \frac{4(1-\bar{\nu})a^2}{r}\,\right]\cos(2\theta) + (1+\bar{\nu})\bar{\alpha}(\theta_1^* - \theta_0^*)\,r \;,$$

$$u_\theta = \frac{T_{22}^{\infty}}{4\mu}\left[\,r + \frac{a^4}{r^3} + \frac{2(1-2\bar{\nu})a^2}{r}\,\right]\sin(2\theta) \quad . \tag{18.20}$$

SHEAR LOADING

In order to develop the solution for shear loading it is convenient to consider the axes $\mathbf{e}_1'$ and $\mathbf{e}_2'$, which are rotated by $\pi/4$ relative to the axes $\mathbf{e}_1$ and $\mathbf{e}_2$, such that

$$\mathbf{e}_1' = \frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2) \;,\; \mathbf{e}_2' = \frac{1}{\sqrt{2}}(-\mathbf{e}_1 + \mathbf{e}_2) \quad . \tag{18.21}$$

Now, pure shear relative to the $\mathbf{e}_i$ axes can be expressed in the form

$$\mathbf{T} = S\,(\mathbf{e}_1'\otimes\mathbf{e}_1' - \mathbf{e}_2'\otimes\mathbf{e}_2') = S\,(\mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1) \;, \tag{18.22}$$

which corresponds to tensor ($T_{11}^{\infty} = S$) in the $\mathbf{e}_1'$ direction and compression ($T_{22}^{\infty} = -S$) in the $\mathbf{e}_2'$ direction. Thus, the solution for pure shear relative to the $\mathbf{e}_1$ and $\mathbf{e}_2$ axes can be obtained adding the previous two solutions after making the replacements

$$\theta \to (\theta - \frac{\pi}{4}) \;,\; T_{11}^{\infty} \to T_{12}^{\infty} \;,\; T_{22}^{\infty} \to -T_{12}^{\infty} \;, \tag{18.23}$$

in (18.10) and (18.19) for the stresses

$$T_{rr} = T_{12}^{\infty}\left[\,1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\,\right]\sin(2\theta) \;,\quad T_{\theta\theta} = -T_{12}^{\infty}\left[\,1 + \frac{3a^4}{r^4}\,\right]\sin(2\theta) \;,$$

$$T_{r\theta} = T_{12}^{\infty}\left[\,1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\,\right]\cos(2\theta) \quad . \tag{18.24}$$

and in (18.17) and (18.20) [with $\theta_1^* = \theta_0^*$] for the displacements

$$u_r = \frac{T_{12}^{\infty}}{2\mu}\left[\,r - \frac{a^4}{r^3} + \frac{4(1-\bar{\nu})a^2}{r}\,\right]\sin(2\theta) + (1+\bar{\nu})\bar{\alpha}(\theta_1^* - \theta_0^*)\,r \;,$$

$$u_\theta = \frac{T_{12}^{\infty}}{2\mu}\left[\,r + \frac{a^4}{r^3} + \frac{2(1-2\bar{\nu})a^2}{r}\,\right]\cos(2\theta) \quad . \tag{18.25}$$

Here, it is important to emphasize that the influence of temperature should only be included in one of the two solutions that are being superposed.

GENERAL LOADING

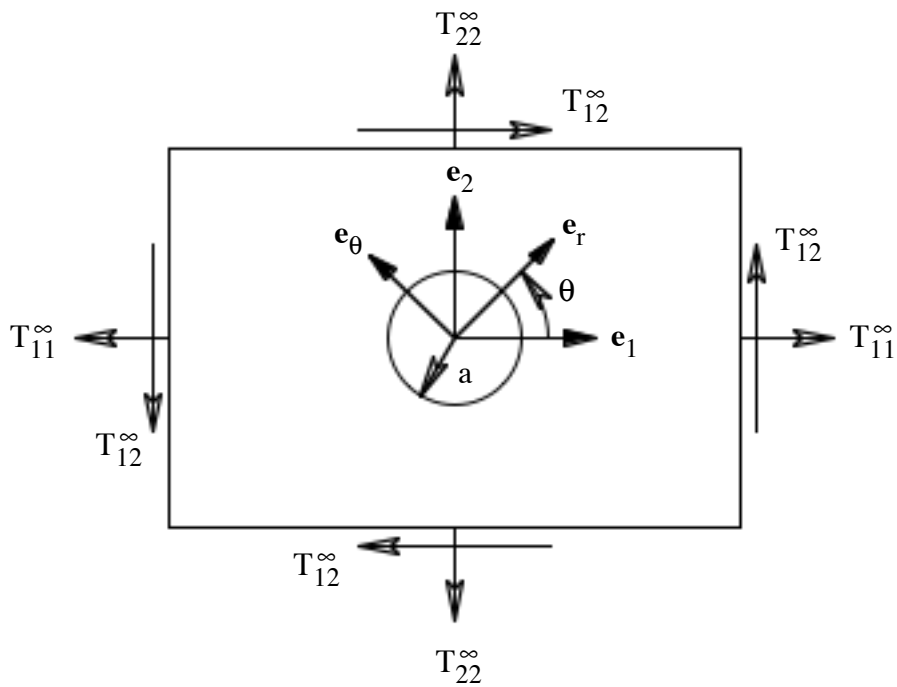

Fig. 18.3 Sketch of a plate with a circular hole of radius a, which is subjected to general loading far away from the hole.

The solution to the problem of general loading far away from the hole [Fig. 18.3] can be obtained by superposing (18.10), (18.19) and (18.24) for the stresses

$$T_{rr} = \frac{(T_{11}^{\infty}+T_{22}^{\infty})}{2}\left[\,1-\frac{a^2}{r^2}\,\right]+\frac{(T_{11}^{\infty}-T_{22}^{\infty})}{2}\left[\,1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\,\right]\cos(2\theta)$$

$$+\,T_{12}^{\infty}\left[\,1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\,\right]\sin(2\theta)\ ,$$

$$T_{\theta\theta} = \frac{(T_{11}^{\infty}+T_{22}^{\infty})}{2}\left[\,1+\frac{a^2}{r^2}\,\right]-\frac{(T_{11}^{\infty}-T_{22}^{\infty})}{2}\left[\,1+\frac{3a^4}{r^4}\,\right]\cos(2\theta)$$

$$-\,T_{12}^{\infty}\left[\,1+\frac{3a^4}{r^4}\,\right]\sin(2\theta),$$

$$T_{r\theta} = -\frac{(T_{11}^{\infty}-T_{22}^{\infty})}{2}\left[\,1-\frac{3a^4}{r^4}+\frac{2a^2}{r^2}\,\right]\sin(2\theta)+T_{12}^{\infty}\left[\,1-\frac{3a^4}{r^4}+\frac{2a^2}{r^2}\,\right]\cos(2\theta)\ ,\quad(18.26)$$

and by superposing (18.17) [with $\theta_1^*=\theta_0^*$], (18.20) [with $\theta_1^*=\theta_0^*$] and (18.25) for the displacements

$$u_r = \frac{(T_{11}^{\infty}+T_{22}^{\infty})}{4\mu}\left[\,(1-2\bar{\nu})\,r + \frac{a^2}{r}\,\right] + \frac{(T_{11}^{\infty}-T_{22}^{\infty})}{4\mu}\left[\,r - \frac{a^4}{r^3} + \frac{4(1-\bar{\nu})a^2}{r}\,\right]\cos(2\theta)$$

$$+ \frac{T_{12}^{\infty}}{2\mu}\left[\,r - \frac{a^4}{r^3} + \frac{4(1-\bar{\nu})a^2}{r}\,\right]\sin(2\theta) + (1+\bar{\nu})\bar{\alpha}(\theta_1^{*}-\theta_0^{*})\,r\ ,$$

$$u_\theta = -\frac{(T_{11}^{\infty}-T_{22}^{\infty})}{4\mu}\left[\,r + \frac{a^4}{r^3} + \frac{2(1-2\bar{\nu})a^2}{r}\,\right]\sin(2\theta)$$

$$+ \frac{T_{12}^{\infty}}{2\mu}\left[\,r + \frac{a^4}{r^3} + \frac{2(1-2\bar{\nu})a^2}{r}\,\right]\cos(2\theta)\ . \qquad (18.27)$$

19. The second law of thermodynamics

The thermodynamic procedure proposed by Green and Naghdi (1977,1978) introduces the balance of entropy (4.16) to determine the temperature field θ, and it suggests that the balance of angular momentum (4.21) and the balance of energy (4.31) place restrictions on constitutive equations which ensure that these balance laws are satisfied for all thermomechanical processes. In particular, the reduced forms of the balances of angular momentum and energy are given by (4.52a,b) and (4.52c,d), respectively. Moreover, it was shown in section 5 that for a thermoelastic material these equations require the entropy η and the stress $\mathbf{T}$ to be determined by derivatives of the Helmholtz free energy ψ, (5.7a,b). Also, for a thermoelastic material it was shown that the internal specific rate of entropy production ξ' associated with material dissipation vanishes. This procedure has the advantage that restrictions on constitutive equations can be obtained without making any statement of the second law of thermodynamics.

Various statements of the second law have been proposed which relate to purely thermal processes and coupled thermomechanical processes. All of these statements attempt to propose mathematical expressions for restrictions on constitutive equations that ensure that theoretical predictions are consistent with observations.

For example, one thermal statement of the second law requires heat to flow from hot to cold regions. Mathematically, this means that

$$-\mathbf{q} \bullet \mathbf{g} > 0 \quad \text{for } \mathbf{g} = \partial\theta/\partial\mathbf{x} \neq 0 \quad . \tag{19.1}$$

Alternatively, since temperature remains positive, (4.41) can be used to rewrite the restriction (19.1) in terms of the entropy flux $\mathbf{p}$ instead of the heat flux $\mathbf{q}$ to obtain

$$-\mathbf{p} \bullet \mathbf{g} > 0 \quad \text{for } \mathbf{g} = \partial\theta/\partial\mathbf{x} \neq 0 \quad . \tag{19.2}$$

For either case, the constitutive equations (5.10) and (5.9b) will satisfy the restrictions (19.1) and (19.2), respectively, provided that the heat conduction coefficient κ is positive

$$\kappa > 0 \quad . \tag{19.3}$$

Thus, (5.10) and (5.9b), associated with Fourier's law, require the entropy flux $\mathbf{p}$ and the heat flux $\mathbf{q}$ to be in the opposite direction to the temperature gradient $\mathbf{g}$.

Another example is the notion that friction causes heat generation. It has been shown in section 5 that ξ' vanishes for a thermoelastic material which is considered an ideal

material with no material dissipation [also see Rubin (1992)]. Therefore, within the context of continuum mechanics, a nonzero value of ξ' indicates that the material is non-ideal or dissipative. However, using (4.46) it can be shown that the rate of heat expelled from the body can be written in the form

$$-(\rho r - \operatorname{div} \mathbf{q}) = -\rho\theta\dot{\eta} + \rho\theta\xi' \quad . \tag{19.4}$$

Thus, if ξ' remains positive then material dissipation causes a tendency for heat to be expelled by the body (which is consistent with the notion of friction). Consequently, another statement of the second law is that the rate of material dissipation must be non-negative

$$\rho\theta\xi' \geq 0 \quad . \tag{19.5}$$

Although this restriction is trivially satisfied for a thermoelastic material, it places important restrictions on the constitutive equations of more complicated materials like viscous fluids and elastic-plastic or elastic-viscoplastic solids.

Finally, it is noted from (4.45) that the two restrictions (19.2) and (19.5) require the internal rate of entropy production ξ to be non-negative

$$\rho\theta\xi = -\mathbf{p} \bullet \mathbf{g} + \rho\theta\xi' \geq 0 \quad . \tag{19.6}$$

20. Material dissipation

A thermoelastic material is an ideal material in the sense that the internal rate of production of entropy ξ' due to material dissipation vanishes (5.7c). The simplest model which includes material dissipation can be developed by adding linear viscous damping to the thermoelastic response. Specifically, it is assumed that the stress tensor $\mathbf{T}$ separates additively into two parts

$$\mathbf{T} = \hat{\mathbf{T}} + \overset{v}{\mathbf{T}} \ . \tag{20.1}$$

The first part $\hat{\mathbf{T}}$ characterizes the thermoelastic response and takes the form (5.12)

$$\hat{\mathbf{T}} = -\hat{p}\,\mathbf{I} + \hat{\mathbf{T}}' \ , \ \hat{p} = -K\{\mathbf{e} \bullet \mathbf{I} - 3\alpha(\theta - \theta_0)\} \ , \ \hat{\mathbf{T}}' = 2\mu\,\mathbf{e}' \ , \tag{20.2a,b,c}$$

$$\hat{T}_{ij} = -\hat{p}\,\delta_{ij} + \hat{T}'_{ij} \ , \ \hat{p} = -K\{e_{mm} - 3\alpha(\theta - \theta_0)\} \ , \ \hat{T}'_{ij} = 2\mu\,e'_{ij} \ , \tag{20.2d,e,f}$$

and the second part $\overset{v}{\mathbf{T}}$ is due to viscous dissipation and takes the form

$$\overset{v}{\mathbf{T}} = -\overset{v}{p}\,\mathbf{I} + \overset{v}{\mathbf{T}}' \ , \ \overset{v}{p} = -d_1\,\dot{\mathbf{e}} \bullet \mathbf{I} \ , \ \overset{v}{\mathbf{T}}' = 2d_2\,\dot{\mathbf{e}}' \ , \tag{20.3a,b,c}$$

$$\overset{v}{T}_{ij} = -\overset{v}{p}\,\delta_{ij} + \overset{v}{T}'_{ij} \ , \ \overset{v}{p} = -d_1\,\dot{e}_{mm} \ , \ \overset{v}{T}'_{ij} = 2d_2\,\dot{e}'_{ij} \ , \tag{20.3d,e,f}$$

where d_1 controls the dissipation due to dilatational deformation and d_2 controls the dissipation due to distortional deformation. Also, the values of the Helmholtz free energy ψ, the entropy η, the internal energy ε, and the entropy flux vector $\mathbf{p}$, are the same as those associated with a thermoelastic material

$$\psi = \hat{\psi} \ , \ \ \eta = \hat{\eta} \ , \ \ \varepsilon = \hat{\varepsilon} = \hat{\psi} + \theta\hat{\eta} \ , \ \mathbf{p} = \hat{\mathbf{p}} \ , \tag{20.4}$$

where $\hat{\psi}$ is given by (5.9a), $\hat{\eta}$ is given by (5.7a), $\hat{\mathbf{T}}$ is given by (5.7b), and $\hat{\mathbf{p}}$ is given by (5.9b), such that

$$\hat{\eta} = -\frac{\partial\hat{\psi}}{\partial\theta} \ , \ \ \hat{\mathbf{T}} = \rho_0\frac{\partial\hat{\psi}}{\partial\mathbf{e}} \ . \tag{20.5}$$

Now, the reduced form of the energy equation (5.5) becomes

$$\rho_0\theta\xi' = -\rho_0\left[\,\hat{\eta} + \frac{\partial\hat{\psi}}{\partial\theta}\,\right]\dot{\theta} + \left[\,\mathbf{T} - \rho_0\frac{\partial\hat{\psi}}{\partial\mathbf{e}}\,\right] \bullet \dot{\mathbf{e}} \ . \tag{20.6}$$

Thus, with the help of (20.1) and (20.5), it follows that the rate of material dissipation is given by

$$\rho_0 \theta \xi' = \overset{v}{\mathbf{T}} \bullet \dot{\mathbf{e}} \geq 0 \quad , \tag{20.7}$$

which must be nonnegative by the second law of thermodynamics (19.5). Moreover, substitution of (20.3) into (20.7) yields

$$\rho_0 \theta \xi' = d_1 (\dot{\mathbf{e}} \bullet \mathbf{I})^2 + 2d_2 (\dot{\mathbf{e}}' \bullet \dot{\mathbf{e}}') \geq 0 \ , \tag{20.8}$$

which is satisfied provided that both d_1 and d_2 are nonnegative

$$d_1 \geq 0 \ , \ d_2 \geq 0 \quad . \tag{20.9}$$

In the remainder of this course attention will be confined to the case of a nondissipative thermoelastic material with

$$d_1 = d_2 = 0 \ . \tag{20.10}$$

In order to understand the connection of the internal rate of production of entropy and dissipation, it is convenient to consider an idealized problem where the outer boundary ∂P of a body is free of surface tractions, and is insulated from heat flux (or entropy flux)

$$\mathbf{t} = 0 \ \text{ and } \ \mathbf{q} \bullet \mathbf{n} = 0 \ \ (\mathbf{p} \bullet \mathbf{n} = 0) \ \text{ on } \partial P \ . \tag{20.11}$$

Moreover, the body force $\mathbf{b}$ and external rate of heat supply r (or entropy supply s) are taken to be zero in the entire region P occupied by the body

$$\mathbf{b} = 0 \ \text{ and } \ r = 0 \ \ (s = 0) \ \text{ on } P \ . \tag{20.12}$$

Also, for simplicity, it is assumed that in its initial configuration the body has no strain, the temperature is the reference temperature, but the velocity field is nonzero and is inhomogeneous

$$\mathbf{e} = 0 \ , \ \theta = \theta_0 \ , \ \text{grad}\, \mathbf{L} \neq 0 \ \text{ at } \ t = 0 \ . \tag{20.13}$$

It then follows that the global form of the balance of entropy (4.16) reduces to

$$\frac{d}{dt} \int_P \rho \, \eta \, dv = \int_P \rho \, \xi \, dv \ , \tag{20.14}$$

and the global form of the balance of energy can be integrated to yield

$$\int_P \rho \, \varepsilon \, dv + \int_P \frac{1}{2} \rho \, \mathbf{v} \bullet \mathbf{v} \, dv = \mathcal{K}_0 \ , \tag{20.15}$$

where $\mathcal{K}_0$ is the kinetic energy of the body in its initial configuration. This means that the total energy (internal plus kinetic) remains constant.

Next, it is recalled from (19.6) that the internal rate of production of entropy separates into a thermal part and a material part, such that

$$\rho\theta\xi = -\,\mathbf{p} \bullet \mathbf{g} + \rho\theta\xi' \geq 0 \quad , \tag{20.16}$$

each of which is nonnegative due to the second law of thermodynamics (19.2) and (19.5)

$$-\,\mathbf{p} \bullet \mathbf{g} > 0 \quad \text{for } \mathbf{g} = \partial\theta/\partial\mathbf{x} \neq 0 \quad , \quad \rho\theta\xi' \geq 0 \quad . \tag{20.17}$$

Furthermore, for κ, d_1 and d_2 positive

$$\kappa > 0 \ , \ d_1 > 0 \ , \ d_2 > 0 \ , \tag{20.18}$$

it can be seen that the internal production of entropy vanishes only when the temperature is uniform and the strain rate vanishes

$$\xi = 0 \implies \partial\theta/\partial\mathbf{x} = 0 \ \text{ and } \ \dot{\mathbf{e}} = 0 \quad . \tag{20.19}$$

Now, it is observed from (20.14) that since ξ is nonnegative, the entropy continues to increase as long as the temperature is not uniform and the strain rate does not vanish. However, the entropy η and the internal energy ε for a thermoelastic material are functions of the strain $\mathbf{e}$ and the temperature θ only

$$\eta = \eta(\mathbf{e},\theta) \ , \ \varepsilon = \varepsilon(\mathbf{e},\theta) \quad . \tag{20.20}$$

Moreover, in view of (20.15), the total internal energy is bounded from above

$$\int_P \rho\, \varepsilon \, dv \leq \mathcal{K}_0 \quad . \tag{20.21}$$

Consequently, it is impossible for the entropy to increase without bound and for the internal energy to remain bounded. This means that the thermomechanical process must evolve so that eventually the temperature becomes uniform and the strain rate vanishes. It is particularly interesting to note that even if the material dissipation vanishes (20.10), the thermomechanical process will still evolve to a uniform temperature with vanishing strain rate since thermal heat conduction causes entropy production.

21. Wave propagation: Wave speeds in an infinite media, uniaxial strain waves, and vibrations of a bar in uniaxial stress

Typical wave propagation speeds in metals are about 5 km/s. Therefore, the wave travels so fast that there is essentially no time for heat transfer by heat conduction. Consequently, in the absence of external heat supply r, it is reasonable to consider wave propagation to be an adiabatic process.

For adiabatic processes the stress is determined by the constitutive equations (9.28)

$$T_{ij} = (\bar{K} - \frac{2\mu}{3})\, e_{mm}\, \delta_{ij} + 2\mu\, e_{ij} \quad , \tag{21.1}$$

where $\bar{K}$ is specified by (9.29)

$$\bar{K} = K\left[1 + \frac{9K\alpha^2\theta_0}{\rho_0 C_v}\right] \quad . \tag{21.2}$$

Also, the temperature is given by (9.27)

$$\theta = \theta_0 - \frac{3K\alpha\theta_0}{\rho_0 C_v}\, e_{mm} \quad . \tag{21.3}$$

It then follows that in the absence of body force (**b**=0), the strain-displacement relations (6.1) can be used together with (21.1) to rewrite the balance of linear momentum (6.21d) in the form

$$\rho_0\, \ddot{u}_i = (\bar{K} + \frac{\mu}{3})\, u_{m,mi} + \mu\, u_{i,mm} \quad . \tag{21.4}$$

WAVE SPEEDS IN AN INFINITE MEDIA

In order to show that there are two types of waves in an infinite media it is convenient to express the displacement field in terms of two potential functions ϕ and $\boldsymbol{\psi}$, such that

$$\mathbf{u} = \nabla\phi + \nabla \times \boldsymbol{\psi} \quad , \quad \nabla \bullet \boldsymbol{\psi} = 0 \quad ,$$

$$u_i = \phi_{,i} + \varepsilon_{ijk}\, \psi_{k,j} \quad , \quad \psi_{j,j} = 0 \quad . \tag{21.5}$$

Physically, the potential $\boldsymbol{\psi}$ is associated with pure distortional deformations, whereas the potential ϕ includes both dilatational and distortional deformations since

$$e_{ij} = \phi_{,ij} + \frac{1}{2}\left[\varepsilon_{imn}\, \psi_{n,mj} + \varepsilon_{jmn}\, \psi_{n,mi}\right] \quad ,$$

$$\mathbf{e} \bullet \mathbf{I} = \nabla \bullet \mathbf{u} = e_{mm} = \phi_{,mm} = \nabla^2 \phi \ ,$$

$$e'_{ij} = \left[\phi_{,ij} - \frac{1}{3}\phi_{,mm}\,\delta_{ij}\right] + \frac{1}{2}\left[\varepsilon_{imn}\,\psi_{n,mj} + \varepsilon_{jmn}\,\psi_{n,mi}\right] \ . \tag{21.6}$$

Now, it follows from (21.5) that

$$u_{m,mi} = \phi_{,mmi} \ , \ u_{i,mm} = \phi_{,mmi} + \varepsilon_{ijk}\,\psi_{k,mmj} \ , \tag{21.7}$$

so that the equations of motion (21.4) can be rewritten in the forms

$$\left[\rho_0\,\ddot{\phi} - (\bar{K} + \frac{4\mu}{3})\,\phi_{,mm}\right]_{,i} + \varepsilon_{ijk}\left[\rho_0\,\ddot{\psi}_k - \mu\,\psi_{k,mm}\right]_{,j} = 0 \ . \tag{21.8}$$

It then follows that the equations of motion will be satisfied if

$$\rho_0\,\ddot{\phi} = (\bar{K} + \frac{4\mu}{3})\,\phi_{,mm} \ , \ \rho_0\,\ddot{\psi}_k = \mu\,\psi_{k,mm} \ , \tag{21.9a,b}$$

which are wave equations of standard form.

To derive the wave speeds from these equations consider plane waves traveling in the k_i direction ($k_i k_i = 1$) with speed C and let

$$\phi = \phi(x) \ , \ \psi_i = \psi_i(x) \ , \ x = k_i x_i - Ct \ . \tag{21.10a,b,c}$$

Using this representation it follows that the dilatational wave equation (21.9a) will be satisfied for an arbitrary functional form $\phi(x)$ if the wave propagates with the dilatational wave speed

$$C = C_P = \sqrt{\frac{\bar{K} + \frac{4\mu}{3}}{\rho_0}} \ . \tag{21.11}$$

Similarly, it follows that the distortional wave equation (21.9b) will be satisfied for an arbitrary functional form $\psi_i(x)$ if the wave propagates with the distortional wave speed

$$C = C_S = \sqrt{\frac{\mu}{\rho_0}} \ . \tag{21.12}$$

In particular, it can be observed that the dilatational wave speed is faster than the distortional wave speed. With regard to earthquakes, (21.11) is the speed of the P-wave (primary wave) and (21.12) is the speed of the S-wave (secondary wave), so the P-wave arrives before the S-wave.

Also, it is noted that if the minus sign in front of C in (21.10c) is replaced by a plus sign then the wave travels in the negative k_i direction with the same speed as the wave traveling in the positive k_i direction.

UNIAXIAL STRAIN WAVES

In order to study the properties of materials at high strain rates it is most common to use plate impact experiments where a cylindrical flyer plate is propelled to a high velocity by a gas gun and impacts a cylindrical target plate. Since waves travel with a finite speed, points at the center of the plate experience uniaxial strain until release waves arrive from the plate's free lateral surfaces. Moreover, the time window for which the strain remains uniaxial can be controlled by specifying the thickness and radius of the plate.

Taking the $\mathbf{e}_1$ direction to be the direction of motion, it follows that for uniaxial strain

$$u_1 = u_1(x_1 - Ct) \ , \ u_2 = u_3 = 0 \ . \tag{21.13}$$

Thus, the only nontrivial equation of motion associated with (21.4) becomes

$$\rho_0 \ddot{u}_1 = (\bar{K} + \frac{4\mu}{3})\, u_{1,11} \ , \tag{21.14}$$

which is satisfied provided that the uniaxial strain wave speed C is the same as the P-wave speed (21.11). Moreover, it is important to note from (21.1) that the nonzero stresses associated with uniaxial strain are given by

$$T_{11} = (\bar{K} + \frac{4\mu}{3})\, u_{1,1} \ , \ T_{22} = (\bar{K} - \frac{2\mu}{3})\, u_{1,1} \ , \ T_{33} = (\bar{K} - \frac{2\mu}{3})\, u_{1,1} \ . \tag{21.15}$$

VIBRATIONS OF A BAR IN UNIAXIAL STRESS

Waves traveling along the axis of a thin bar are affected by the free lateral surfaces of the bar because the time required for waves to travel through the thickness of the bar is short. Consequently, the majority of the energy transmitted by the wave travels at a wave velocity associated with uniaxial stress conditions. Specifically, with the help of (9.17), and (9.18) with ε vanishing, it can be shown that

$$\theta-\theta_0 = -\left[\frac{3K\alpha\theta_0}{\rho_0 C_v + 9K\alpha^2\theta_0}\right]\frac{(1-2\nu)T_{11}}{E} \quad ,$$

$$T_{11} = \bar{E}\, e_{11} \ , \quad \bar{E} = E\left[\frac{\rho_0 C_v + 9K\alpha^2\theta_0}{\rho_0 C_v + 6K\alpha^2\theta_0(1+\nu)}\right] \ . \tag{21.15}$$

It is interesting to note that $\bar{E} > E$ since the material cools when it is stretched adiabatically.

Now, in the absence of body force (**b**=0), the balance of linear momentum (6.21d) in the $\mathbf{e}_1$ direction becomes

$$\rho_0\, \ddot{u}_1 = \bar{E}\, u_{1,11} \ . \tag{21.16}$$

Thus, taking u_1 in the form

$$u_1 = u_1(x_1 - Ct) \ , \tag{21.17}$$

the equation of motion (21.16) is satisfied if the adiabatic wave speed is given by

$$C = C_B = \sqrt{\frac{\bar{E}}{\rho_0}} \ . \tag{21.18}$$

Then, equation (21.16) can be written in the alternative form

$$\ddot{u}_1 = C_B^2\, u_{1,11} \ . \tag{21.19}$$

Next, consider a bar of length L which in its reference configuration occupies the region

$$0 \le x_1 \le L \ . \tag{21.20}$$

<u>Free vibrations of the bar</u>

For free vibrations of the bar, both the ends x_1=0 and x_1=L remain stress free. Therefore, from (21.15) it follows that the free boundary conditions require

$$u_{1,1}(0) = 0 \ , \quad u_{1,1}(L) = 0 \ . \tag{21.21}$$

Using separation of variables, the solution of equation (21.19) can be written in the form

$$u_1 = A \sin\omega t \ f(x_1) \ , \tag{21.22}$$

where A is the amplitude of the mode and ω is its frequency. Next, substituting this solution into the equation (21.19) yields

$$f_{,11} + k^2 f = 0 \quad , \quad k = \frac{\omega}{C_B} \quad , \tag{21.23}$$

where k is called the wave number. Thus, the solution of (21.23) associated with symmetric modes about the center of the bar is given by

$$u_1 = A \sin\omega t \ \sin\left[k(x_1 - L/2)\right] \quad , \tag{21.24}$$

with the boundary conditions (21.21) reducing to

$$\cos kL/2 = 0 \quad . \tag{21.25}$$

The solution of (21.25) predicts an infinite number of modes with wave number k_n and frequency ω_n characterized by

$$k_n = \frac{(2n-1)\pi}{L} \quad , \ \omega_n = \frac{(2n-1)\pi C_B}{L} \quad \text{for } n = 1,2,3,... \quad . \tag{21.26}$$

In particular, notice that the higher modes (higher values of n) have shorter wavelengths ($2\pi/k_n$) and higher frequencies, and that the frequency increases with decreasing length L. Moreover, the lowest free vibrational frequency of the symmetric mode corresponds to n=1 and has the frequency ω_1 and the period T_1 of vibration given by

$$\omega_1 = \frac{\pi C_B}{L} \quad , \ T_1 = \frac{2\pi}{\omega_1} = \frac{2L}{C_B} \quad . \tag{21.27}$$

Thus, the period of vibration is equal to the time required for a wave to travel from one end of the bar to the other end and back again (i.e. twice the length of the bar).

Similarly, the solution of (21.23) associated with anti-symmetric modes about the center of the bar is given by

$$u_1 = A \sin\omega t \ \cos\left[k(x_1 - L/2)\right] \quad , \tag{21.28}$$

with the boundary conditions (21.21) reducing to

$$\sin kL/2 = 0 \quad . \tag{21.29}$$

The solution of (21.28) predicts an infinite number of modes with wave number k_n and frequency ω_n characterized by

$$k_n = \frac{2(n-1)\pi}{L} \quad , \ \omega_n = \frac{2(n-1)\pi C_B}{L} \quad \text{for } n = 1,2,3,... \tag{21.30}$$

Thus, the lowest anti-symmetrical vibrational frequency corresponds to n=1 and has the frequency ω_1 and the period T_1 of vibration given by

$$\omega_1 = \frac{2\pi C_B}{L} \ , \ T_1 = \frac{2\pi}{\omega_1} = \frac{L}{C_B} \ , \tag{21.31}$$

which equals twice the frequency of the lowest symmetric mode.

<u>Fixed-Free vibrations of the bar</u>

As another example, consider the case when the end (x_1=0) is fixed and the end (x_1=L) is free. Therefore, from (21.15) it follows that the boundary conditions require

$$u_1(0) = 0 \ , \ u_{1,1}(L) = 0 \ . \tag{21.32a,b}$$

Again, using separation of variables, the solution can be written in the form (21.22) with the function f satisfying the equation (21.23). Now, the solution of (21.23) which satisfies the boundary condition (21.32a) becomes

$$u_1 = A \sin\omega t \ \sin(kx_1) \ , \tag{21.33}$$

where k is determined by the boundary condition (21.32b)

$$\cos(kL) = 0 \ . \tag{21.34}$$

The solution of (21.34) predicts an infinite number of modes with wave number k_n and frequency ω_n characterized by

$$k_n = \frac{(2n-1)\pi}{2L} \ , \ \omega_n = \frac{(2n-1)\pi C_B}{2L} \quad \text{for } n = 1,2,3,... \ . \tag{21.35}$$

Therefore, it follows that the lowest vibrational frequency corresponds to n=1 and has the frequency ω_1 and the period T_1 of vibration given by

$$\omega_1 = \frac{\pi C_B}{2L} \ , \ T_1 = \frac{2\pi}{\omega_1} = \frac{4L}{C_B} \ . \tag{21.36}$$

This period of vibration corresponds to the wave traveling four lengths of the bar.

It is of interest to note that this result can be obtained from the symmetrical free vibrational mode, since the middle of the bar (x_1=L/2) remains stationary in the symmetric mode (21.24). This means that the effective fixed-free length of the free vibrational bar is L/2. In other words, if the free bar has length 2L then its effective

fixed-free length with L. Consequently, the results (21.35) can be obtained by replacing L in (21.26) by 2L.

22. Bending of a rectangular plate due to mechanical and thermal loads

Consider a rectangular plate with length L, height H, and width W, which in its reference configuration occupies the region characterized by

$$-\frac{L}{2} \le x_1 \le \frac{L}{2} \ , \ -\frac{H}{2} \le x_2 \le \frac{H}{2} \ , \ -\frac{W}{2} \le x_3 \le \frac{W}{2} \ . \tag{22.1}$$

The top (x_2=H/2) and bottom (x_2=–H/2) surfaces of the plate are taken to be traction free so that

$$\mathbf{t}(x_1,\pm H/2,x_3;\pm\mathbf{e}_2) = 0 \ , \ \ T_{i2}(x_1,\pm H/2,x_3) = 0 \ . \tag{22.2}$$

Also, the boundary conditions on the edges of the plate are specified in a Saint Venant sense (7.11) such that the resultant forces are zero, but the resultant moments are not. Specifically, the boundary conditions on the edges are specified by

$$\int_{-W/2}^{W/2}\int_{-H/2}^{H/2} \mathbf{t}(L/2,x_2,x_3;\mathbf{e}_1)\, dx_2\, dx_3 = 0 \ ,$$

$$M_3\, \mathbf{e}_3 = \int_{-W/2}^{W/2}\int_{-H/2}^{H/2} (\mathbf{x} - \frac{L}{2}\mathbf{e}_1) \times \mathbf{t}(L/2,x_2,x_3;\mathbf{e}_1)\, dx_2\, dx_3 \ ,$$

$$\int_{-W/2}^{W/2}\int_{-H/2}^{H/2} \mathbf{t}(-L/2,x_2,x_3;-\mathbf{e}_1)\, dx_2\, dx_3 = 0 \ ,$$

$$-M_3\, \mathbf{e}_3 = \int_{-W/2}^{W/2}\int_{-H/2}^{H/2} (\mathbf{x} + \frac{L}{2}\mathbf{e}_1) \times \mathbf{t}(-L/2,x_2,x_3;-\mathbf{e}_1)\, dx_2\, dx_3 \ ,$$

$$\int_{-H/2}^{H/2}\int_{-L/2}^{L/2} \mathbf{t}(x_1,x_2,W/2;\mathbf{e}_3)\, dx_1\, dx_2 = 0 \ ,$$

$$M_1\, \mathbf{e}_1 = \int_{-H/2}^{H/2}\int_{-L/2}^{L/2} (\mathbf{x} - \frac{W}{2}\mathbf{e}_3) \times \mathbf{t}(x_1,x_2,W/2;\mathbf{e}_3)\, dx_1\, dx_2 \ ,$$

$$\int_{-H/2}^{H/2}\int_{-L/2}^{L/2} \mathbf{t}(x_1,x_2,-W/2;\mathbf{e}_3)\, dx_1 dx_2 = 0 \ ,$$

$$-M_1\, \mathbf{e}_1 = \int_{-H/2}^{H/2}\int_{-L/2}^{L/2} (\mathbf{x} + \frac{W}{2}\mathbf{e}_3) \times \mathbf{t}(x_1,x_2,-W/2;-\mathbf{e}_3)\, dx_1\, dx_2 \ , \tag{22.3}$$

where M_1 and M_3 are the moments applied to the edges. It then follows from (22.2) and (22.3) that the total resultant force and total resultant moment (about any fixed point) applied to the entire plate both vanish. Consequently, the plate will be in equilibrium if the body force also vanishes (**b**=0). Moreover, the boundary conditions (22.3) can be expanded to yield

$$\int_{-W/2}^{W/2}\int_{-H/2}^{H/2} T_{i1}(\pm L/2,x_2,x_3)\, dx_2\, dx_3 = 0 \ ,$$

$$\int_{-W/2}^{W/2}\int_{-H/2}^{H/2} \left[x_2\, T_{31}(\pm L/2,x_2,x_3) - x_3\, T_{21}(\pm L/2,x_2,x_3)\right] dx_2\, dx_3 = 0 \ ,$$

$$\int_{-W/2}^{W/2}\int_{-H/2}^{H/2} x_3\, T_{11}(\pm L/2,x_2,x_3)\, dx_2\, dx_3 = 0 \ ,$$

$$M_3 = -\int_{-W/2}^{W/2}\int_{-H/2}^{H/2} x_2\, T_{11}(\pm L/2,x_2,x_3)\, dx_2\, dx_3 \ ,$$

$$\int_{-H/2}^{H/2}\int_{-L/2}^{L/2} T_{i3}(x_1,x_2,\pm W/2)\, dx_1\, dx_2 = 0 \ ,$$

$$M_1 = \int_{-H/2}^{H/2}\int_{-L/2}^{L/2} x_2\, T_{33}(x_1,x_2,\pm W/2)\, dx_1\, dx_2 \ ,$$

$$\int_{-H/2}^{H/2}\int_{-L/2}^{L/2} x_1\, T_{33}(x_1,x_2,\pm W/2)\, dx_1\, dx_2 = 0 \ ,$$

$$\int_{-H/2}^{H/2}\int_{-L/2}^{L/2} \left[x_1\, T_{23}(x_1,x_2,\pm W/2) - x_2\, T_{13}(x_1,x_2,\pm W/2)\right] dx_1\, dx_2 = 0 \ . \qquad (22.4)$$

Examination of the boundary conditions (22.3) and (22.4) suggests that the stresses be specified by

$$T_{11} = -\left[\frac{12M_3}{WH^3}\right] x_2 \ , \ T_{33} = \left[\frac{12M_1}{LH^3}\right] x_2 \ , \ \text{all other } T_{ij} = 0 \ . \qquad (22.5)$$

These stresses also satisfy the equilibrium equations

$$T_{ij,j} = 0 \ . \qquad (22.6)$$

In addition to the mechanical loads associated with the moments M_1 and M_3 it is of interest to specify a thermal load by taking the temperature gradient through the thickness H of the plate to be nonzero. Specifically, the temperature field is specified by

$$\theta = \theta_0 + G_2\, x_2 \ , \qquad (22.7)$$

where G_2 is a constant. It then follows from the constitutive equation (6.22g) that the heat flux $\mathbf{q}$ is given by

$$\mathbf{q} = -\,\kappa G_2\, \mathbf{e}_2 \ , \qquad (22.8)$$

so that heat flows in the negative $\mathbf{e}_2$ direction. This physically corresponds to a plate which is heated on its top surface (x_2=H/2) and cooled on its bottom surface (x_2=–H/2).

In this regard, it is recalled that the case of a uniform thermal expansion was solved in section 8.

Next, using the constitutive equations (9.9) it follows that the strains associated with the stresses (22.5) and the temperature field (22.7) become

$$e_{11} = A\, x_2 \;\; , \; e_{22} = B\, x_2 \;\; , \; e_{33} = C\, x_2 \;\; , \; e_{12} = e_{13} = e_{23} = 0 \;\; , \tag{22.9}$$

where the constants A, B, C are specified by

$$A = -\frac{12M_3}{EWH^3} - \frac{12\nu M_1}{ELH^3} + \alpha G_2 \;\; , \;\; B = \frac{12\nu M_3}{EWH^3} - \frac{12\nu M_1}{ELH^3} + \alpha G_2 \;\; ,$$

$$C = \frac{12\nu M_3}{EWH^3} + \frac{12M_1}{ELH^3} + \alpha G_2 \;\; . \tag{22.10}$$

Since the strains (22.9) are linear functions of the coordinate x_2 it follows that they satisfy the compatibility equations (6.3) so that a displacement field exists.

Moreover, the constitutive equation (6.22j) indicates that the internal energy

$$\rho_0 \varepsilon = \rho_0 C_v\, G_2\, x_2 + 3K\alpha\theta_0\, e_{mm} \; . \tag{22.11}$$

is independent of time so that in the absence of external heat supply (r=0) the balance of energy (6.21f)

$$\rho_0\, \dot{\varepsilon} = -\, q_{j,j} \;\; . \tag{22.12}$$

is satisfied.

In order to complete the solution it is necessary to integrate the strain-displacement relations (6.1) to determine expressions for the displacement field. Specifically, it follows from (22.9) that

$$u_{1,1} = A\, x_2 \;\; , \; u_{2,2} = B\, x_2 \;\; , \; u_{3,3} = C\, x_2 \;\; , \tag{22.13a,b,c}$$

$$u_{1,2} + u_{2,1} = 0 \;\; , \; u_{1,3} + u_{3,1} = 0 \;\; , \; u_{2,3} + u_{3,2} = 0 \;\; . \tag{22.13d,e,f}$$

Thus, integration of the first three of these equations yields

$$u_1 = A\, x_1 x_2 + f_1(x_2,x_3) \;\; , \;\; u_2 = \frac{B}{2}\, x_2^2 + f_2(x_1,x_3) \;\; ,$$

$$u_3 = C\, x_2 x_3 + f_3(x_1,x_2) \;\; , \tag{22.14}$$

where f_1, f_2 and f_3 are functions of integration. These functions are determined by substituting (22.14) into the remaining three equations of (22.13). Specifically, (22.13d) requires

$$A\, x_1 + f_{1,2} + f_{2,1} = 0 \ . \tag{22.15}$$

Now, from (22.15a) it can be seen that since f_2 does not depend on x_2, then $f_{1,2}$ also cannot depend on x_2 so it must be a function $g_1(x_3)$ of x_3 only

$$f_{1,2} = g_1(x_3) \ , \quad f_1(x_2,x_3) = g_1(x_3)\, x_2 + h_1(x_3) \ , \tag{22.16a,b}$$

where $h_1(x_3)$ is a function of x_3 only. Substituting this result back into (22.15) yields

$$A\, x_1 + g_1(x_3) + f_{2,1} = 0 \ , \quad f_2(x_1,x_3) = -\frac{A}{2}\, x_1^2 - g_1(x_3)\, x_1 + h_2(x_3) \ , \tag{22.17a,b}$$

where $h_2(x_3)$ is another function of x_3 only. Thus, with the help of (2.14), (2.16) and (2.17), the displacements u_1 and u_2 become

$$u_1 = A\, x_1 x_2 + g_1(x_3)\, x_2 + h_1(x_3) \ , \tag{22.18a}$$

$$u_2 = -\frac{A}{2}\, x_1^2 + \frac{B}{2}\, x_2^2 - g_1(x_3)\, x_1 + h_2(x_3) \ . \tag{22.18b}$$

Next, (2.14) and (22.18a) are substituted into (22.13e) to deduce that

$$g_{1,3}\, x_2 + h_{1,3} + f_{3,1} = 0 \ . \tag{22.19}$$

Now, since f_3 does not depend on x_3, and g_1 and h_1 depend only on x_3, it follows that $g_{1,3}$ and $h_{1,3}$ are constants, and $f_{3,1}$ is a function of x_2 only

$$g_{1,3} = D \ , \quad g_1 = D\, x_3 + H_{12} \ ,$$

$$h_{1,3} = H_{13} \ , \quad h_1 = H_{13}\, x_3 + c_1 \ ,$$

$$f_{3,1} = -D\, x_2 - H_{13} \ , \quad f_3 = -D\, x_1 x_2 - H_{13}\, x_1 + h_3(x_2) \ , \tag{22.20}$$

where D, H_{12}, H_{13}, c_1 are constants and h_3 is a function of x_2 only. Thus, with the help of (22.14), (22.18b) and (22.20) the displacements u_2 and u_3 become

$$u_2 = -\frac{A}{2}\, x_1^2 + \frac{B}{2}\, x_2^2 - D\, x_1 x_3 - H_{12}\, x_1 + h_2(x_3) \ , \tag{22.21a}$$

$$u_3 = C\, x_2 x_3 - D\, x_1 x_2 - H_{13}\, x_1 + h_3(x_2) \ . \tag{22.21b}$$

To determine the remaining functions, (22.21) are substituted into (22.13f) to obtain

$$-D\, x_1 + h_{2,3} + C\, x_3 - D\, x_1 + h_{3,2} = 0 \ . \tag{22.22}$$

Since h_2 depends on x_3 only, and h_3 depends only on x_2, it follows that D vanishes and $h_{3,2}$ is constant

$$D = 0 \ ,$$

$$h_{3,2} = -H_{23} \ , \ \ h_3 = -H_{23}\, x_2 + c_3 \ ,$$

$$h_{2,3} = -C\, x_3 + H_{23} \ , \ h_2 = -\frac{C}{2} x_3^2 + H_{23}\, x_3 + c_2 \ , \tag{22.23}$$

where H_{23}, c_2, c_3 are constants.

Thus, collecting the results (22.18a), (22.20), (22.21) and (22.23), the displacement field becomes

$$u_1 = A\, x_1 x_2 + H_{12}\, x_2 + H_{13}\, x_3 + c_1 \ ,$$

$$u_2 = -\frac{A}{2} x_1^2 + \frac{B}{2} x_2^2 - \frac{C}{2} x_3^2 - H_{12}\, x_1 + H_{23}\, x_3 + c_2 \ ,$$

$$u_3 = C\, x_2 x_3 - H_{13}\, x_1 - H_{23}\, x_2 + c_3 \ , \tag{22.24}$$

where A, B and C are given by (22.10). Moreover, it can be observed that c_i represent rigid body translations and H_{12}, H_{13}, H_{23} represent rigid body rotations.

As special cases consider:

PURE BENDING IN THE $\mathbf{e}_1$–$\mathbf{e}_2$ PLANE (M_1=G_2=0)

$$u_1 = -\left[\frac{12M_3}{EWH^3}\right] x_1 x_2 + H_{12}\, x_2 + H_{13}\, x_3 + c_1 \ ,$$

$$u_2 = \left[\frac{6M_3}{EWH^3}\right] \left[\, x_1^2 + \nu\, x_2^2 - \nu\, x_3^2 \,\right] - H_{12}\, x_1 + H_{23}\, x_3 + c_2 \ ,$$

$$u_3 = \left[\frac{12\nu M_3}{EWH^3}\right] x_2 x_3 - H_{13}\, x_1 - H_{23}\, x_2 + c_3 \ . \tag{22.25}$$

PURE BENDING IN THE $\mathbf{e}_2$–$\mathbf{e}_3$ PLANE (M_3=G_2=0)

$$u_1 = -\left[\frac{12\nu M_1}{ELH^3}\right] x_1 x_2 + H_{12}\, x_2 + H_{13}\, x_3 + c_1 \ ,$$

$$u_2 = -\left[\frac{6M_1}{ELH^3}\right]\left[-\nu\, x_1^2 + \nu\, x_2^2 + x_3^2\right] - H_{12}\, x_1 + H_{23}\, x_3 + c_2\ ,$$

$$u_3 = \left[\frac{12M_1}{ELH^3}\right] x_2 x_3 - H_{13}\, x_1 - H_{23}\, x_2 + c_3\ . \qquad (22.26)$$

FREE THERMAL BENDING (M_1=M_3=0)

$$u_1 = \left[\alpha G_2\right] x_1 x_2 + H_{12}\, x_2 + H_{13}\, x_3 + c_1\ ,$$

$$u_2 = \left[\frac{\alpha G_2}{2}\right]\left[-x_1^2 + x_2^2 - x_3^2\right] - H_{12}\, x_1 + H_{23}\, x_3 + c_2\ ,$$

$$u_3 = \left[\alpha G_2\right] x_2 x_3 - H_{13}\, x_1 - H_{23}\, x_2 + c_3\ . \qquad (22.27)$$

23. Composite plates

The objective of this section is to consider simple deformations of composite plates which are loaded by both mechanical and thermal loads. Specifically, consider a composite rectangular plate which is made of two different materials (see Fig. 23.1). The plate has length L, and width W. The heights of the bottom and top portions of the plate are H_1 and H_2, respectively. The bottom portion of the plate occupies the region

$$-\frac{L}{2} \le x_1 \le \frac{L}{2} \ , \ -H_1 \le x_2 \le 0 \ , \ -\frac{W}{2} \le x_3 \le \frac{W}{2} \ , \tag{23.1}$$

and has the material properties

$$\{ \rho_{01} , E_1 , \nu_1 , \alpha_1 , C_{v1} \} \ , \tag{23.2}$$

whereas the top portion of the plate occupies the region

$$-\frac{L}{2} \le x_1 \le \frac{L}{2} \ , \ 0 \le x_2 \le H_2 \ , \ -\frac{W}{2} \le x_3 \le \frac{W}{2} \ , \tag{23.3}$$

and has the material properties

$$\{ \rho_{02} , E_2 , \nu_2 , \alpha_2 , C_{v1} \} \ . \tag{23.4}$$

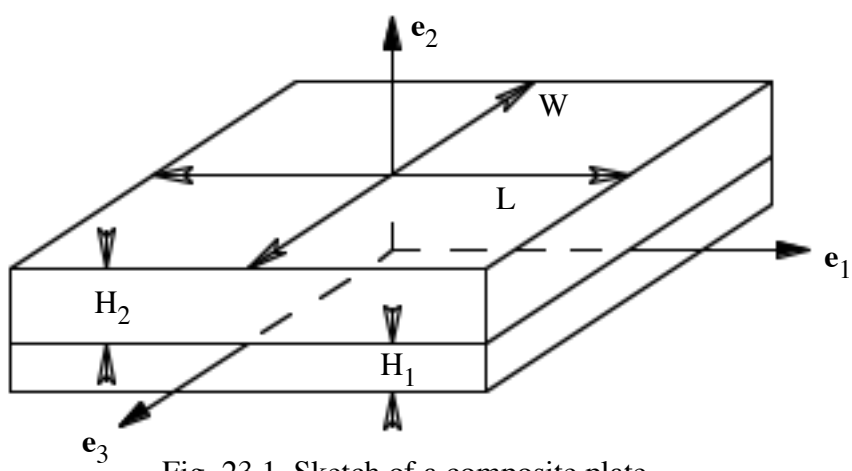

Fig. 23.1 Sketch of a composite plate.

For simplicity, attention will be focused only on simple problems of equilibrium with no body force

$$\mathbf{b} = 0 \ . \tag{23.5}$$

Also, it will be assumed that heat is supplied ($r \neq 0$) only during the process of heating the body uniformly to a uniform temperature

$$\theta = \text{constant} \ , \tag{23.6}$$

so that the heat flux vector **q** vanishes. In general, it is necessary to specify boundary conditions which are valid at each point of the exterior boundary of the composite plate. Also, it is necessary to specify conditions at the interface (x_2=0) which characterize the bonding between the top and bottom portions of the plate.

For definiteness, let the displacement, strain and stress fields be $\{u_i^{(1)}, e_{ij}^{(1)}, T_{ij}^{(1)}\}$ in the bottom portion of the plate and be $\{u_i^{(2)}, e_{ij}^{(2)}, T_{ij}^{(2)}\}$ in its top portion. Assuming that the two portions of the plate are bonded perfectly at the interface (x_2=0), it follows that material points that were in contact before loading remain in contact. This means that the displacement field must be continuous at the contact surface

$$u_i^{(1)}(x_1,0,x_3) = u_i^{(2)}(x_1,0,x_3) \quad . \tag{23.7}$$

In addition, it is necessary to specify kinetic conditions at this surface. Specifically, it is necessary to require the traction vector $\bar{\mathbf{t}}$ applied by the top portion of the plate on its bottom portion to be equal and opposite to the traction vector applied by the bottom portion of the plate on its top portion

$$\mathbf{t}(x_1,0,x_3;\mathbf{e}_2) = \bar{\mathbf{t}} \text{ applied to the bottom portion of the plate } ,$$

$$\mathbf{t}(x_1,0,x_3;-\mathbf{e}_2) = -\bar{\mathbf{t}} \text{ applied to the top portion of the plate } . \tag{23.8}$$

Now, using the relationship between the stress vector and the stress tensor it follows that

$$\mathbf{T}^{(1)}(x_1,0,x_3)\,\mathbf{e}_2 = \bar{\mathbf{t}} \;,\quad \mathbf{T}^{(2)}(x_1,0,x_3)\,(-\mathbf{e}_2) = -\bar{\mathbf{t}} \;,$$

$$\mathbf{T}^{(1)}(x_1,0,x_3)\,\mathbf{e}_2 = \mathbf{T}^{(2)}(x_1,0,x_3)\,\mathbf{e}_2 \;,$$

$$T_{i2}^{(1)}(x_1,0,x_3) = T_{i2}^{(2)}(x_1,0,x_3) \;, \tag{23.9}$$

which require continuity of three components of stress.

Since the two materials are different, they respond differently to thermal and mechanical loads so the solution of boundary value problems of contact problems such as the one under consideration usually are quite complicated and lead to inhomogeneous deformation fields in each portion of the plate. However, it is possible to develop intuition about the potential deformation field associated with this incompatibility of the materials by considering a very special case where surface tractions are applied in such a way that the deformation fields in each of the portions of the plate remain homogeneous.

To this end, consider the following boundary value problem. Let the top $(x_2=H_2)$ and bottom $(x_2=-H_1)$ surfaces of the plate be traction free so that

$$\mathbf{t}(x_1,H_2,x_3;\mathbf{e}_2) = 0 \ , \quad T^{(2)}_{i2}(x_1,H_2,x_3) = 0 \ ,$$

$$\mathbf{t}(x_1,-H_1,x_3;-\mathbf{e}_2) = 0 \ , \quad T^{(1)}_{i2}(x_1,-H_1,x_3) = 0 \ . \tag{23.10}$$

Also, specify the boundary conditions on the edges of the plate in a Saint Venant sense (7.11), such that the resultant forces and moments on the edges are specified by

$$P\,\mathbf{e}_1 = \int_{-W/2}^{W/2}\int_{-H_1}^{H_2} \mathbf{t}(L/2,x_2,x_3;\mathbf{e}_1)\,dx_2\,dx_3 \ ,$$

$$M_3\,\mathbf{e}_3 = \int_{-W/2}^{W/2}\int_{-H_1}^{H_2} \left[\mathbf{x} - \frac{L}{2}\mathbf{e}_1 - \frac{(H_2-H_1)}{2}\mathbf{e}_2\right] \times \mathbf{t}(L/2,x_2,x_3;\mathbf{e}_1)\,dx_2\,dx_3 \ ,$$

$$-P\,\mathbf{e}_1 = \int_{-W/2}^{W/2}\int_{-H_1}^{H_2} \mathbf{t}(-L/2,x_2,x_3;-\mathbf{e}_1)\,dx_2\,dx_3 \ ,$$

$$-M_3\,\mathbf{e}_3 = \int_{-W/2}^{W/2}\int_{-H_1}^{H_2} \left[\mathbf{x} + \frac{L}{2}\mathbf{e}_1 - \frac{(H_2-H_1)}{2}\mathbf{e}_2\right] \times \mathbf{t}(-L/2,x_2,x_3;-\mathbf{e}_1)\,dx_2\,dx_3 \ ,$$

$$\int_{-H_1}^{H_2}\int_{-L/2}^{L/2} \mathbf{t}(x_1,x_2,W/2;\mathbf{e}_3)\,dx_1\,dx_2 = 0 \ ,$$

$$M_1\,\mathbf{e}_1 = \int_{-H_1}^{H_2}\int_{-L/2}^{L/2} \left[\mathbf{x} - \frac{(H_2-H_1)}{2}\mathbf{e}_2 - \frac{W}{2}\mathbf{e}_3\right] \times \mathbf{t}(x_1,x_2,W/2;\mathbf{e}_3)\,dx_1\,dx_2 \ ,$$

$$\int_{-H_1}^{H_2}\int_{-L/2}^{L/2} \mathbf{t}(x_1,x_2,-W/2;\mathbf{e}_3)\,dx_1 dx_2 = 0 \ ,$$

$$-M_1\,\mathbf{e}_1 = \int_{-H_1}^{H_2}\int_{-L/2}^{L/2} \left[\mathbf{x} - \frac{(H_2-H_1)}{2}\mathbf{e}_2 + \frac{W}{2}\mathbf{e}_3\right] \times \mathbf{t}(x_1,x_2,-W/2;-\mathbf{e}_3)\,dx_1\,dx_2 \ . \tag{23.11}$$

Here, P is a specified resultant force, and M_1 and M_3 are the resultant moments applied to the edges that are determined by the solution of the problem. It then follows from (23.10) and (23.11) that the total resultant force and total resultant moment (about any fixed point) applied to the entire plate both vanish. Moreover, the boundary conditions (23.11) can be expanded to yield

$$P = \int_{-W/2}^{W/2}\int_{-H_1}^{H_2} T_{11}(\pm L/2,x_2,x_3)\,dx_2\,dx_3 \ ,$$

$$0 = \int_{-W/2}^{W/2} \int_{-H_1}^{H_2} T_{21}(\pm L/2, x_2, x_3)\, dx_2\, dx_3 = 0 \ ,$$

$$0 = \int_{-W/2}^{W/2} \int_{-H_1}^{H_2} T_{31}(\pm L/2, x_2, x_3)\, dx_2\, dx_3 = 0 \ ,$$

$$\int_{-W/2}^{W/2} \int_{-H_1}^{H_2} \left[\left\{ x_2 - \frac{(H_2 - H_1)}{2} \right\} T_{31}(\pm L/2, x_2, x_3) - x_3\, T_{21}(\pm L/2, x_2, x_3) \right] dx_2\, dx_3 = 0 \ ,$$

$$\int_{-W/2}^{W/2} \int_{-H_1}^{H_2} x_3\, T_{11}(\pm L/2, x_2, x_3)\, dx_2\, dx_3 = 0 \ ,$$

$$M_3 = -\int_{-W/2}^{W/2} \int_{-H_1}^{H_2} \left\{ x_2 - \frac{(H_2 - H_1)}{2} \right\} T_{11}(\pm L/2, x_2, x_3)\, dx_2\, dx_3 \ ,$$

$$\int_{-H_1}^{H_2} \int_{-L/2}^{L/2} T_{i3}(x_1, x_2, \pm W/2)\, dx_1\, dx_2 = 0 \ ,$$

$$M_1 = \int_{-H_1}^{H_2} \int_{-L/2}^{L/2} \left\{ x_2 - \frac{(H_2 - H_1)}{2} \right\} T_{33}(x_1, x_2, \pm W/2)\, dx_1\, dx_2 \ ,$$

$$\int_{-H_1}^{H_2} \int_{-L/2}^{L/2} x_1\, T_{33}(x_1, x_2, \pm W/2)\, dx_1\, dx_2 = 0 \ ,$$

$$\int_{-H_1}^{H_2} \int_{-L/2}^{L/2} \left[x_1\, T_{23}(x_1, x_2, \pm W/2) - x_2\, T_{13}(x_1, x_2, \pm W/2) \right] dx_1\, dx_2 = 0 \ . \qquad (23.12)$$

Here, simple solutions are considered of the forms

$$e_{11}^{(1)} = e_{11}^{(2)} = e_{11} = \text{constant} \ , \quad e_{22}^{(1)} = \text{constant} \ , \quad e_{22}^{(2)} = \text{constant} \ ,$$

$$e_{33}^{(1)} = e_{33}^{(2)} = e_{33} = \text{constant} \ , \quad e_{12}^{(1)} = e_{12}^{(2)} = 0 \ ,$$

$$e_{13}^{(1)} = e_{13}^{(2)} = 0 \ , \quad e_{23}^{(1)} = e_{23}^{(2)} = 0 \ , \qquad (23.13)$$

where the constants $\{e_{11}, e_{22}^{(1)}, e_{22}^{(2)}, e_{33}\}$ must be determined by the solution. In then follows from the constitutive equations (6.22) that the stress fields are constant in each portion of the plate so the balance laws (6.21) are satisfied pointwise. Also, it follows that

$$T_{12}^{(1)} = T_{12}^{(2)} = 0 \ , \quad T_{13}^{(1)} = T_{13}^{(2)} = 0 \ , \quad T_{23}^{(1)} = T_{23}^{(2)} = 0 \ . \qquad (23.14)$$

Moreover, the contact conditions (23.9) and the boundary conditions (23.10) can be used to deduce the additional result that

$$T_{22}^{(1)} = T_{22}^{(2)} = 0 \quad . \tag{23.15}$$

Then, the constitutive equations in the forms (9.9) can be used to obtain

$$e_{11} = \frac{T_{11}^{(1)}}{E_1} - \frac{\nu_1 T_{33}^{(1)}}{E_1} + \alpha_1(\theta-\theta_0) \ , \ e_{22}^{(1)} = -\frac{\nu_1 T_{11}^{(1)}}{E_1} - \frac{\nu_1 T_{33}^{(1)}}{E_1} + \alpha_1(\theta-\theta_0) \ ,$$

$$e_{33} = -\frac{\nu_1 T_{11}^{(1)}}{E_1} + \frac{T_{33}^{(1)}}{E_1} + \alpha_1(\theta-\theta_0) \ , \tag{23.16}$$

in the bottom portion of the plate, and

$$e_{11} = \frac{T_{11}^{(2)}}{E_2} - \frac{\nu_2 T_{33}^{(2)}}{E_2} + \alpha_2(\theta-\theta_0) \ , \ e_{22}^{(2)} = -\frac{\nu_2 T_{11}^{(2)}}{E_2} - \frac{\nu_2 T_{33}^{(2)}}{E_2} + \alpha_2(\theta-\theta_0) \ ,$$

$$e_{33} = -\frac{\nu_2 T_{11}^{(2)}}{E_2} + \frac{T_{33}^{(2)}}{E_2} + \alpha_2(\theta-\theta_0) \ , \tag{23.17}$$

in its top portion.

Next, equating the expressions for e_{11} and e_{33} in (23.16) and (23.17) yields

$$\frac{T_{11}^{(1)}}{E_1} - \frac{\nu_1 T_{33}^{(1)}}{E_1} - \frac{T_{11}^{(2)}}{E_2} + \frac{\nu_2 T_{33}^{(2)}}{E_2} = -(\alpha_1-\alpha_2)(\theta-\theta_0) \ ,$$

$$-\frac{\nu_1 T_{11}^{(1)}}{E_1} + \frac{T_{33}^{(1)}}{E_1} + \frac{\nu_2 T_{11}^{(2)}}{E_2} - \frac{T_{33}^{(2)}}{E_2} = -(\alpha_1-\alpha_2)(\theta-\theta_0) \ , \tag{23.18}$$

which are two equations for the four unknowns $\{T_{11}^{(1)}, T_{33}^{(1)}, T_{11}^{(2)}, T_{33}^{(2)}\}$. The remaining two equations are obtained from $(23.12)_1$ and $(23.12)_7$

$$W\left[H_1 T_{11}^{(1)} + H_2 T_{11}^{(2)}\right] = P \ , \ \ L\left[H_1 T_{33}^{(1)} + H_2 T_{33}^{(2)}\right] = 0 \quad . \tag{23.19}$$

Alternatively, these equations yield

$$T_{11}^{(2)} = -\left[\frac{H_1}{H_2}\right] T_{11}^{(1)} + \frac{P}{H_2 W} \ , \ T_{33}^{(2)} = -\left[\frac{H_1}{H_2}\right] T_{33}^{(1)} \ , \tag{23.20}$$

so that (23.18) can be written in the matrix form

$$\begin{Bmatrix} A_{11} & A_{12} \\ A_{12} & A_{11} \end{Bmatrix} \begin{Bmatrix} T_{11}^{(1)} \\ T_{33}^{(1)} \end{Bmatrix} = \begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} \ ,$$

$$A_{11} = \left[\frac{1}{E_1} + \frac{H_1}{H_2}\frac{1}{E_2}\right] \ , \ A_{12} = -\left[\frac{\nu_1}{E_1} + \frac{H_1}{H_2}\frac{\nu_2}{E_2}\right] \ ,$$

$$B_1 = \frac{P}{H_2 W E_2} - (\alpha_1 - \alpha_2)(\theta - \theta_0) \ , \ B_2 = -\frac{\nu_2 P}{H_2 W E_2} - (\alpha_1 - \alpha_2)(\theta - \theta_0) \ , \qquad (23.21)$$

which has the solution

$$T_{11}^{(1)} = \frac{A_{11}B_1 - A_{12}B_2}{A_{11}^2 - A_{12}^2} \ , \ T_{33}^{(1)} = \frac{A_{11}B_2 - A_{12}B_1}{A_{11}^2 - A_{12}^2} \ . \qquad (23.22)$$

Now, the equations (23.12) will be satisfied provided that M_1 and M_3 are specified by

$$M_1 = L\left[-\frac{H_1^2}{2}T_{33}^{(1)} + \frac{H_2^2}{2}T_{33}^{(2)}\right] = -H_1 L\left[\frac{H_1+H_2}{2}\right]T_{33}^{(1)} ,$$

$$M_3 = -W\left[-\frac{H_1^2}{2}T_{11}^{(1)} + \frac{H_2^2}{2}T_{11}^{(2)}\right] + \left[\frac{H_2-H_1}{2}\right]P = H_1 W\left[\frac{H_1+H_2}{2}\right]T_{11}^{(1)} - \left[\frac{H_1}{2}\right]P , \qquad (23.23)$$

where use has been made of the equations (23.20).

In summary, once the force P and the temperature θ have been specified, the stresses are given by (23.14, (23.15), (23.20) and (23.22), and the strains are given by (23.13), (23.16) and (23.17). Also, the moments M_1 and M_3 are given by (23.23). It is important to emphasize that since M_1 and M_3 are not zero, the plate would bend if these moments were not supplied.

In order to analyze the physical meaning of this solution, it is convenient to consider simpler problems. Specifically, consider the case when the top and bottom portions of the plate have the same geometry and the same Young's modulus

$$H_1 = H_2 = H \ , \ E_1 = E_2 = E \ . \qquad (23.24)$$

It then follows from (23.20), (23.21) and (23.22) that

$$A_{11} = \frac{2}{E} \ , \ A_{12} = -\frac{(\nu_1+\nu_2)}{E} \ ,$$

$$B_1 = \frac{P}{HWE} - (\alpha_1 - \alpha_2)(\theta - \theta_0) \ , \ B_2 = -\frac{\nu_2 P}{HWE} - (\alpha_1 - \alpha_2)(\theta - \theta_0) \ , \qquad (23.25)$$

and that

$$T_{11}^{(1)} = \frac{\{2 - \nu_2(\nu_1+\nu_2)\}\frac{P}{HW} - (2+\nu_1+\nu_2)(\alpha_1-\alpha_2)(\theta-\theta_0)}{4\left[1 - \left\{\frac{\nu_1+\nu_2}{2}\right\}^2\right]} \ ,$$

$$T_{11}^{(2)} = \frac{\{2-\nu_1(\nu_1+\nu_2)\}\frac{P}{HW} + (2+\nu_1+\nu_2)(\alpha_1-\alpha_2)(\theta-\theta_0)}{4\left[1-\{\frac{\nu_1+\nu_2}{2}\}^2\right]} \quad ,$$

$$T_{33}^{(1)} = \frac{(\nu_1-\nu_2)\frac{P}{HW} - (2+\nu_1+\nu_2)(\alpha_1-\alpha_2)(\theta-\theta_0)}{4\left[1-\{\frac{\nu_1+\nu_2}{2}\}^2\right]} \quad , \quad T_{33}^{(2)} = -T_{33}^{(1)} \quad ,$$

$$M_1 = -\frac{(\nu_1-\nu_2)\frac{HLP}{W} - H^2L(2+\nu_1+\nu_2)(\alpha_1-\alpha_2)(\theta-\theta_0)}{4\left[1-\{\frac{\nu_1+\nu_2}{2}\}^2\right]}$$

$$M_3 = \frac{\{\frac{\nu_1^2-\nu_2^2}{2}\}HP - H^2W(2+\nu_1+\nu_2)(\alpha_1-\alpha_2)(\theta-\theta_0)}{4\left[1-\{\frac{\nu_1+\nu_2}{2}\}^2\right]} \quad . \tag{23.26}$$

First of all, notice that if the materials are the same then

$$\nu_1 = \nu_2 = \nu \quad , \quad \alpha_1 = \alpha_2 = \alpha \quad ,$$

$$T_{11}^{(1)} = T_{11}^{(2)} = \frac{P}{2HW} \quad , \quad T_{33}^{(1)} = T_{33}^{(2)} = 0 \quad , \quad M_1 = M_3 = 0 \quad , \tag{23.27}$$

which is consistent with uniaxial stress in the $\mathbf{e}_1$ direction.

Next, consider the case of mechanical loading only, such that

$$\theta = \theta_0 \quad ,$$

$$T_{11}^{(1)} = \frac{\{2-\nu_2(\nu_1+\nu_2)\}\frac{P}{HW}}{4\left[1-\{\frac{\nu_1+\nu_2}{2}\}^2\right]} \quad , \quad T_{11}^{(2)} = \frac{\{2-\nu_1(\nu_1+\nu_2)\}\frac{P}{HW}}{4\left[1-\{\frac{\nu_1+\nu_2}{2}\}^2\right]} \quad ,$$

$$T_{33}^{(1)} = \frac{(\nu_1-\nu_2)\frac{P}{HW}}{4\left[1-\{\frac{\nu_1+\nu_2}{2}\}^2\right]} \quad , \quad T_{33}^{(2)} = -T_{33}^{(1)} \quad ,$$

$$M_1 = -\frac{(\nu_1-\nu_2)\,\frac{HLP}{W}}{4\left[1-\left\{\frac{\nu_1+\nu_2}{2}\right\}^2\right]} \quad , \quad M_3 = \frac{\left\{\frac{\nu_1^2-\nu_2^2}{2}\right\}HP}{4\left[1-\left\{\frac{\nu_1+\nu_2}{2}\right\}^2\right]} \quad . \tag{23.28}$$

Now, for ν_1 greater than ν_2, it is necessary to apply a tension $T_{33}^{(1)}$ to the bottom portion of the plate and a compression $T_{33}^{(2)}$ to its top portion in order to prevent the Poisson effect of contracting the bottom portion of the plate more than its top portion. Also, the tension $T_{11}^{(1)}$ in the top portion of the plate is greater than the tension $T_{11}^{(2)}$ in its bottom portion. Moreover, notice that even if the dimensions L and W are equal, the magnitudes of the bending moments M_1 and M_3 are not equal.

Finally, consider the case of thermal loading only, such that

$$P = 0 \ ,$$

$$T_{11}^{(1)} = T_{33}^{(1)} = -\frac{(2+\nu_1+\nu_2)(\alpha_1-\alpha_2)(\theta-\theta_0)}{4\left[1-\left\{\frac{\nu_1+\nu_2}{2}\right\}^2\right]} \quad , \quad T_{11}^{(2)} = T_{33}^{(2)} = -T_{11}^{(1)} \ ,$$

$$M_1 = \frac{H^2L(2+\nu_1+\nu_2)(\alpha_1-\alpha_2)(\theta-\theta_0)}{4\left[1-\left\{\frac{\nu_1+\nu_2}{2}\right\}^2\right]} \ ,$$

$$M_3 = -\frac{H^2W(2+\nu_1+\nu_2)(\alpha_1-\alpha_2)(\theta-\theta_0)}{4\left[1-\left\{\frac{\nu_1+\nu_2}{2}\right\}^2\right]} = -\frac{W}{L}M_1 \ . \tag{23.29}$$

Now, for α_1 greater than α_2 and θ greater than θ_0, it is necessary to apply a compression $T_{11}^{(1)} = T_{33}^{(1)}$ to the bottom portion of the plate and a tension $T_{11}^{(2)} = T_{33}^{(2)}$ to its top portion in order to prevent the temperature expanding the bottom portion of the plate more than its top portion.

24. Flamant's problem: A concentrated line force on a two-dimensional half space

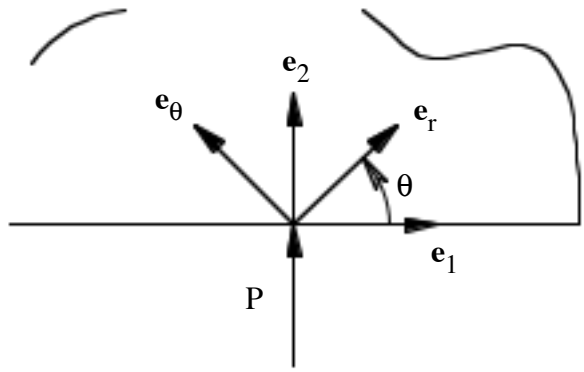

Fig. 24.1 Sketch of a concentrated line force P applied to a two-dimensional half space.

Flamant's problem considers a concentrated line force applied to a two-dimensional half space [see Fig. 24.1]. The body force vanishes

$$b_r = b_\theta = 0 \quad , \tag{24.1}$$

and the temperature θ^* is taken to be uniform but not necessarily equal to θ_0^*. Also, the surface (x_2=0) of the half space is free of surface tractions except at the point (x_1=x_2=0) where the concentrated force is applied

$$\mathbf{t} = \mathbf{T}(r,0)\,[-\mathbf{e}_\theta(0)] = 0 \;\text{ for } r > 0$$

$$T_{r\theta}(r,0) = 0 \;\text{ and }\; T_{\theta\theta}(r,0) = 0 \;\text{ for }\; r > 0 \quad , \tag{24.2a,b}$$

$$\mathbf{t} = \mathbf{T}(r,\pi)\,[\mathbf{e}_\theta(\pi)] = 0 \;\text{ for } r > 0 \quad ,$$

$$T_{r\theta}(r,\pi) = 0 \;\text{ and }\; T_{\theta\theta}(r,\pi) = 0 \;\text{ for }\; r > 0 \quad . \tag{24.2c,d}$$

Figure 24.2 shows a semi-circle of radius a centered at the concentrated line force P (per unit length in the $\mathbf{e}_3$ direction). Since this semi-circle must be in equilibrium, the surface traction **t** applied to the curved part of the semi-circle must balance the applied force

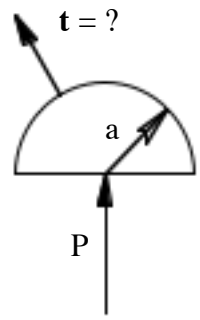

Fig. 24.2 Sketch of a semi-circle of radius a centered at the concentrated line force.

$$\int_0^{\pi} \mathbf{t}(a,\theta;\mathbf{e}_r)\, a\, d\theta + P\, \mathbf{e}_2 = 0 \ ,$$

$$\int_0^{\pi} \left[T_{rr}(a,\theta)\, \mathbf{e}_r(\theta) + T_{r\theta}(a,\theta)\, \mathbf{e}_\theta(\theta)\right] a\, d\theta + P\, \mathbf{e}_2 = 0 \ ,$$

$$\int_0^{\pi} \left[\left\{T_{rr}(a,\theta)\cos\theta - T_{r\theta}(a,\theta)\sin\theta\right\} \mathbf{e}_1 + \left\{ T_{rr}(a,\theta)\sin\theta + T_{r\theta}(a,\theta)\cos\theta\right] a\, d\theta$$

$$+ P\, \mathbf{e}_2 = 0 \ ,$$

$$\int_0^{\pi} \left[T_{rr}(a,\theta)\cos\theta - T_{r\theta}(a,\theta)\sin\theta \right] a\, d\theta = 0 \ , \tag{24.3a}$$

$$\int_0^{\pi} \left[T_{rr}(a,\theta)\sin\theta + T_{r\theta}(a,\theta)\cos\theta \right] a\, d\theta + P = 0 \ . \tag{24.3b}$$

Moreover, since these equations must be valid for any value of the radius a, they suggest that the stress field should depend on (1/r). Thus, examination of the Michell solution indicates that the simplest stress field which satisfies this condition as well as the boundary conditions (24.2) is given by

$$T_{rr} = -\left[\frac{2A_1}{r}\right] \sin\theta \ , \ T_{\theta\theta} = 0 \ , \ T_{r\theta} = 0 \ , \tag{24.4}$$

where A_1 is a constant. It can easily be seen that this stress field satisfies the equations of equilibrium (18.6) and the compatibility equations (18.7). Moreover, it can be seen that this stress field satisfies the boundary conditions (24.3) provided that A_1 is given by

$$A_1 = \frac{P}{\pi} \ , \tag{24.5}$$

so the stress field becomes

$$T_{rr} = -\left[\frac{2P}{\pi r}\right]\sin\theta \ , \ T_{\theta\theta} = 0 \ , \ T_{r\theta} = 0 \ . \tag{24.6}$$

In order to determine the displacement field associated with the solution (24.6), use is made of the constitutive equations (16.14) with the temperature set to the constant value θ_1^*

$$e_{rr} = -\left[\frac{(1-\bar{\nu})P}{\mu\pi r}\right]\sin\theta + (1+\bar{\nu})\bar{\alpha}(\theta_1^*-\theta_0^*) \ ,$$

$$e_{\theta\theta} = \left[\frac{\bar{\nu}P}{\mu\pi r}\right]\sin\theta + (1+\bar{\nu})\bar{\alpha}(\theta_1^*-\theta_0^*) \ , \ e_{r\theta} = 0 \ . \tag{24.7}$$

Next, integration of the strain-displacement relations (16.9) yields

$$u_r = -\left[\frac{(1-\bar{\nu})P}{\mu\pi}\right]\ln(r)\sin\theta + (1+\bar{\nu})\bar{\alpha}(\theta_1^*-\theta_0^*)\, r + \frac{df(\theta)}{d\theta} \ ,$$

$$u_\theta = -\frac{P}{\mu\pi}\left[\bar{\nu} + (1-\bar{\nu})\ln(r)\right]\cos\theta - f(\theta) + g(r) \ , \tag{24.8}$$

where $f(\theta)$ and $g(r)$ are functions of integration. Moreover, substituting these results into the expression (16.9) for the strain $e_{r\theta}$ and use of (24.7) yields the equation

$$\frac{1}{r}\left[\frac{d^2f(\theta)}{d\theta^2} + f(\theta) - \frac{(1-2\bar{\nu})P}{\mu\pi}\cos\theta\right] + \left[\frac{dg(r)}{dr} - \frac{1}{r}\, g(r)\right] = 0 \ . \tag{24.9}$$

Thus, in view of the expressions (16.8) for rigid body displacements it follows that the solution of (24.9) can be expressed in the form

$$f(\theta) = c_1\sin\theta - c_2\cos\theta + \frac{(1-2\bar{\nu})P}{2\mu\pi}\left[\cos\theta + \theta\sin\theta\right] \ , \ g(r) = -H_{12}\, r \ , \tag{24.10}$$

where c_α and H_{12} are constants. Then, the displacements become

$$u_r = -\left[\frac{(1-\bar{\nu})P}{\mu\pi}\right]\ln(r)\sin\theta + (1+\bar{\nu})\bar{\alpha}(\theta_1^*-\theta_0^*)\, r + c_1\cos\theta + c_2\sin\theta$$

$$+ \frac{(1-2\bar{\nu})P}{2\mu\pi}\,\theta\cos\theta \ ,$$

$$u_\theta = -\frac{P}{\mu\pi}\left[\,\bar{\nu} + (1-\bar{\nu})\ln(r)\,\right]\cos\theta - c_1\sin\theta + c_2\cos\theta - \frac{(1-2\bar{\nu})P}{2\mu\pi}\left[\cos\theta + \theta\sin\theta\,\right]$$

$$- H_{12}\, r\ . \tag{24.11}$$

Now, using the relations

$$r = \sqrt{x_1^2+x_2^2}\ ,\ \cos\theta = \frac{x_1}{\sqrt{x_1^2+x_2^2}}\ ,\ \sin\theta = \frac{x_2}{\sqrt{x_1^2+x_2^2}}\ ,\ \theta = \tan^{-1}\left[\frac{x_2}{x_1}\right], \tag{24.12}$$

the rectangular Cartesian components of the stress tensor associated with the results (24.6) and (24.11) become

$$\mathbf{T} = T_{rr}\,(\mathbf{e}_r\otimes\mathbf{e}_r) = T_{rr}\left[\cos^2\theta\,(\mathbf{e}_1\otimes\mathbf{e}_1) + \sin^2\theta\,(\mathbf{e}_2\otimes\mathbf{e}_2)\right.$$

$$\left. + \sin\theta\cos\theta\,(\mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1)\right],$$

$$T_{11} = -\frac{2P}{\pi}\left[\frac{x_1^2x_2}{(x_1^2+x_2^2)^2}\right]\ ,\ T_{22} = -\frac{2P}{\pi}\left[\frac{x_2^3}{(x_1^2+x_2^2)^2}\right]\ , \tag{24.13a,b}$$

$$T_{12} = -\frac{2P}{\pi}\left[\frac{x_1x_2^2}{(x_1^2+x_2^2)^2}\right]\ , \tag{24.13c}$$

and those of the displacement vector become

$$\mathbf{u} = u_r\,\mathbf{e}_r + u_\theta\,\mathbf{e}_\theta = \left[u_r\cos\theta - u_\theta\sin\theta\right]\mathbf{e}_1 + \left[u_r\sin\theta + u_\theta\cos\theta\right]\mathbf{e}_2\ ,$$

$$u_1 = \frac{P}{2\mu\pi}\left[\frac{x_1x_2}{x_1^2+x_2^2} + (1-2\bar{\nu})\tan^{-1}\left\{\frac{x_2}{x_1}\right\}\right] + (1+\bar{\nu})\bar{\alpha}(\theta_1^*-\theta_0^*)\,x_1 + c_1 + H_{12}\,x_2\ , \tag{24.14a}$$

$$u_2 = -\frac{P}{2\mu\pi}\left[\frac{x_1^2}{x_1^2+x_2^2} + (1-\bar{\nu})\ln\left\{x_1^2+x_2^2\right\}\right] + (1+\bar{\nu})\bar{\alpha}(\theta_1^*-\theta_0^*)\,x_2 + c_2 - H_{12}\,x_1\ . \tag{24.14b}$$

25. Hertz contact: Contact of an elastic cylinder with a smooth rigid half space

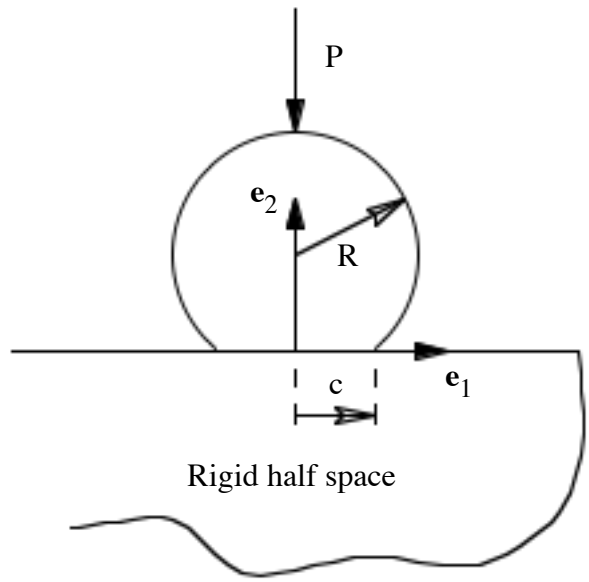

Fig. 25.1 Sketch of an elastic cylinder in contact with a smooth rigid half space .

The objective of this section is to develop an approximate solution for contact of an elastic cylinder with a smooth rigid half space [see section 12.2 in Barber, 1996]. The undeformed radius of the cylinder is R and the half length of the deformed contact region is c. Also, P is the magnitude of the line force (per unit length in the $\mathbf{e}_3$ direction) applied by the half space on the cylinder over the contact region.

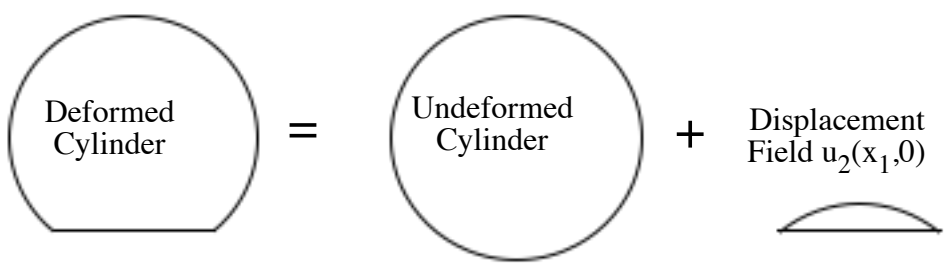

Fig. 25.2 Sketch of the displacement field $u_2(x_1,0)$ that must be applied to the cylinder.

Within the context of the linear theory of elasticity, the exact formulation of this problem is as follows. Since the rigid half space is smooth, it can only apply a contact stress in the positive $\mathbf{e}_2$ direction. The distribution $p(x_1)$ of this stress must be determined so that the displacement u_2 (in the $\mathbf{e}_2$ direction) of cylinder causes the cylinder to remain in contact

with the half space. Figure 25.2 shows a sketch of the displacement field u_2 that must be applied to the cylinder to cause the contact region to become flat.

Moreover, even within the context of the linear theory of elasticity, the solution becomes nonlinear because the extent of the contact region also needs to be determined. For more general contact problems it is possible that the solution will predict that the contact stress becomes negative over portions of the presumed contact region. For this case, the actual solution must be reformulated to allow portions of the presumed contact region to separate from the half space with no contact stress being applied there.

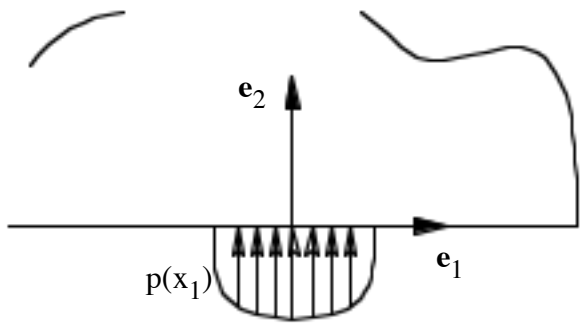

Fig. 25.3 Sketch of the line pressure distribution $p(x_1)$ applied to the contact region.

One of the major assumptions in the Hertz solution of this problem is that the contact region is very small relative to the radius of the cylinder ($c/R<<1$). This means that the curvature of the surface of the cylinder is negligible. Consequently, it is possible to generalize Flamant's solution of section 24 to model the line pressure distribution $p(x_1)$ (per unit length in the $\mathbf{e}_3$ direction) and the displacement field $u_2(x_1,0)$ that must be applied to the cylinder [see Fig. 25.3]. To this end, it is first recalled that the delta function $\delta(x_1-x)$ has the property that

$$p(x_1) = \int_{-c}^{c} p(x)\, \delta(x_1-x)\, dx \quad , \tag{25.1}$$

for any continuous function $p(x_1)$ defined to be nonzero over the domain ($-c \le x_1 \le c$). Therefore, it is possible to use the solutions (24.13) and (24.14) as the Green's function for the solution of a distributed line load $p(x_1)$ applied to the surface $x_2=0$ to obtain the stress field

$$T_{11}(x_1,x_2) = -\frac{2}{\pi}\int_{-c}^{c}\left[\, p(x)\left\{\frac{(x_1-x)^2 x_2}{\left\{(x_1-x)^2+x_2^2\right\}^2}\right\}\right] dx \ ,$$

$$T_{22}(x_1,x_2) = -\frac{2}{\pi}\int_{-c}^{c} p(x)\left[\frac{x_2^3}{\left\{(x_1-x)^2+x_2^2\right\}^2}\right] dx \ ,$$

$$T_{12}(x_1,x_2) = -\frac{2}{\pi}\int_{-c}^{c} p(x)\left[\frac{(x_1-x)x_2^2}{\left\{(x_1-x)^2+x_2^2\right\}^2}\right] dx \ , \tag{25.2}$$

and the displacement field

$$u_1(x_1,x_2) = \frac{1}{2\mu\pi}\int_{-c}^{c} p(x)\left[\frac{(x_1-x)x_2}{\left\{(x_1-x)^2+x_2^2\right\}} + (1-2\bar{\nu})\tan^{-1}\left\{\frac{x_2}{(x_1-x)}\right\}\right] dx$$

$$+ (1+\bar{\nu})\bar{\alpha}(\theta_1-\theta_0)\, x_1 + c_1 + H_{12}\, x_2 \ ,$$

$$u_2(x_1,x_2) = -\frac{1}{2\mu\pi}\int_{-c}^{c} p(x)\left[\frac{(x_1-x)^2}{\left\{(x_1-x)^2+x_2^2\right\}} + (1-\bar{\nu})\ln\left\{(x_1-x)^2+x_2^2\right\}\right] dx$$

$$+ (1+\bar{\nu})\bar{\alpha}(\theta_1^*-\theta_0^*)\, x_2 + c_2 - H_{12}\, x_1 \ . \tag{25.3}$$

Now, in the limit that x_2 approaches zero, it can be shown that

$$u_1(x_1,0) = \frac{(1-2\bar{\nu})}{2\mu}\int_{x_1}^{c} p(x)\, dx + (1+\bar{\nu})\bar{\alpha}(\theta_1^*-\theta_0^*)\, x_1 + c_1 \ ,$$

$$u_2(x_1,0) = -\frac{1}{2\mu\pi}\int_{-c}^{c} p(x)\, dx - \frac{(1-\bar{\nu})}{2\mu\pi}\int_{-c}^{c} p(x)\ln\left\{(x_1-x)^2\right\} dx + c_2 - H_{12}\, x_1 \ . \tag{25.4}$$

If the contact pressure $p(x_1)$ is a symmetric function

$$p(x_1) = p(-x_1) \ , \tag{25.5}$$

then it can be shown that

$$\int_{-c}^{c} p(x)\, f(x)\, dx = \int_{0}^{c} p(x)\, f(x)\, dx + \int_{-c}^{0} p(x)\, f(x)\, dx \ ,$$

$$\int_{-c}^{c} p(x)\, f(x)\, dx = \int_{0}^{c} p(x)\left\{f(x) + f(-x)\right\} dx \ . \tag{25.6}$$

Thus,

$$\int_{-c}^{c} p(x)\, \ln\{(x_1-x)^2\}\, dx = \int_{0}^{c} p(x)\, \ln\{ (x_1^2-x^2)^2\}\, dx \quad . \tag{25.7}$$

Next, it can be seen that these displacements will correspond to symmetric deformation provided that the constants c_1, c_2 and H_{12} associated with rigid body displacements are given by

$$c_1 = -\frac{(1-2\bar{\nu})}{4\mu} \int_{-c}^{c} p(x)\, dx \quad ,$$

$$c_2 = \frac{1}{2\mu\pi} \int_{-c}^{c} p(x)\, dx + \frac{(1-\bar{\nu})}{2\mu\pi} \int_{0}^{c} p(x)\, \ln\{(c^2-x^2)^2\}\, dx + C_2 \quad , \; H_{12} = 0 \quad , \tag{25.8}$$

where C_2 is an alternative constant associated with rigid body displacement in the $\mathbf{e}_2$ direction. Then, the displacements reduce to

$$u_1(x_1,0) = -\frac{(1-2\bar{\nu})}{4\mu} \int_{-c}^{x_1} p(x)\, dx + \frac{(1-2\bar{\nu})}{4\mu} \int_{x_1}^{c} p(x)\, dx + (1+\bar{\nu})\bar{\alpha}(\theta_1-\theta_0)\, x_1 \quad , \tag{25.9a}$$

$$u_2(x_1,0) = -\frac{(1-\bar{\nu})}{2\mu\pi} \int_{0}^{c} p(x)\, \ln\left\{ \frac{(x_1^2-x^2)^2}{(c^2-x^2)^2} \right\} dx + C_2 \quad . \tag{25.9b}$$

This choice of c_2 simplifies the expression for the displacement u_2 at $x_1=\pm c$

$$u_2(\pm c,0) = C_2 \quad . \tag{25.10}$$

Given the displacement field $u_2(x_1,0)$, equation (25.9b) represents an integral equation for determining the contact pressure $p(x_1)$. In order to solve this integral equation it is convenient to differentiate it once with respect to x_1 to obtain

$$\frac{du_2(x_1,0)}{dx_1} = -\frac{(1-\bar{\nu})}{\mu\pi} \int_{0}^{c} p(x) \left[\frac{2x_1}{x_1^2-x^2}\right] dx \quad . \tag{25.11}$$

However,

$$\frac{2x_1}{x_1^2-x^2} = \frac{1}{x_1-x} + \frac{1}{x_1+x} \quad ,$$

$$\int_{0}^{c} \frac{p(x)}{x_1+x}\, dx = \int_{-c}^{0} \frac{p(x)}{x_1-x}\, dx \quad , \tag{25.12}$$

so that (25.11) can be written in the alternative form

$$\frac{du_2(x_1,0)}{dx_1} = -\frac{(1-\bar{\nu})}{\mu\pi} \int_{-c}^{c} \frac{p(x)}{x_1-x}\, dx \quad . \tag{25.13}$$

Taking C_2 to be zero in (25.9b), the displacement $u_2(x_1,0)$ which is required to flatten the portion of the cylinder is given by

$$u_2(x_1,0) = \sqrt{R^2-x_1^2} - \sqrt{R^2-c^2} \quad \text{for} - c \le x_1 \le c \quad ,$$

$$\Delta = u_2(0,0) = R - \sqrt{R^2-c^2} \quad , \tag{25.14}$$

where Δ is the displacement of the center of the cylinder if the cylinder is considered to be rigid. Now, since the contact region is assumed to be small ($c/R << 1$), the expressions (25.14) can be approximated by

$$u_2(x_1,0) = \frac{1}{2R}(c^2 - x_1^2) \quad , \quad \Delta = u_2(0,0) = \frac{c^2}{2R} \quad , \tag{25.15}$$

so that the integral equation (25.13) simplifies to

$$\frac{x_1}{R} = \frac{(1-\bar{\nu})}{\mu\pi} \int_{-c}^{c} \frac{p(x)}{x_1-x}\, dx \quad . \tag{25.16}$$

It will now be shown that the solution of this equation takes the form

$$p(x_1) = \frac{2P}{\pi c^2}\sqrt{c^2-x_1^2} \quad , \quad \text{for } |x_1| < c \quad , \tag{25.17}$$

where the coefficient was determined by satisfying the expression

$$P = \int_{-c}^{c} p(x)\, dx \quad . \tag{25.18}$$

Now, with the help of (25.17) and the change of variables

$$x = c\cos\alpha \ , \quad x_1 = c\cos\beta \quad , \tag{25.19}$$

the integral equation (25.16) can be rewritten in the form

$$\frac{c}{R}\cos\beta = \frac{2P(1-\bar{\nu})}{\mu\pi^2 c} \int_0^{\pi} \frac{\sin^2\alpha}{\cos\beta-\cos\alpha}\, d\alpha \ . \tag{25.20}$$

Moreover, it can be shown that

$$\frac{\sin^2\alpha}{\cos\beta-\cos\alpha} = \frac{d}{d\alpha}\,\mathrm{Re}\,\Big[\alpha\cos\beta + \ln\Big\{\sin\Big(\frac{\beta}{2}-\frac{\alpha}{2}\Big)\Big\}\sin\beta$$

$$-\ln\Big\{\sin\Big(\frac{\beta}{2}+\frac{\alpha}{2}\Big)\Big\}\Big]\sin\beta + \sin\alpha\Big]\ , \tag{25.21}$$

where Re[x] denotes the real part of x. Thus, using this result it follows that

$$\int_0^{\pi} \frac{\sin^2\alpha}{\cos\beta-\cos\alpha}\,d\alpha = \pi\cos\beta\ ,$$

$$\int_{-c}^{c} \frac{p(x)}{x_1-x}\,dx = \frac{2P}{\pi c^2}\int_{-c}^{c}\frac{\sqrt{c^2-x_1^2}}{x_1-x}\,dx = \frac{2P}{c^2}\,x_1 \tag{25.22}$$

and the equation (25.16) is satisfied provided that

$$P = \frac{\mu\pi c^2}{2R(1-\bar{\nu})}\ ,\quad c = \sqrt{\frac{2R(1-\bar{\nu})P}{\mu\pi}}\ . \tag{25.23a,b}$$

In particular, note that the extent of the contact region c is a nonlinear function of the force.

From a practical point of view, it is of interest to determine the relationship between the total line force P applied in the contact region to the displacement Δ of the center of the cylinder. To this end, (25.15) is used to rewrite (25.23a) in the form

$$P = \frac{\mu\pi}{(1-\bar{\nu})}\,\Delta\ , \tag{25.24}$$

which indicates that the line force P and the displacement Δ are linearly related. Furthermore, it is of interest to note that in contrast with the result of this two-dimensional problem, the three-dimensional problem of indentation of a sphere into a half-space yields a nonlinear relationship between the applied force and the displacement.

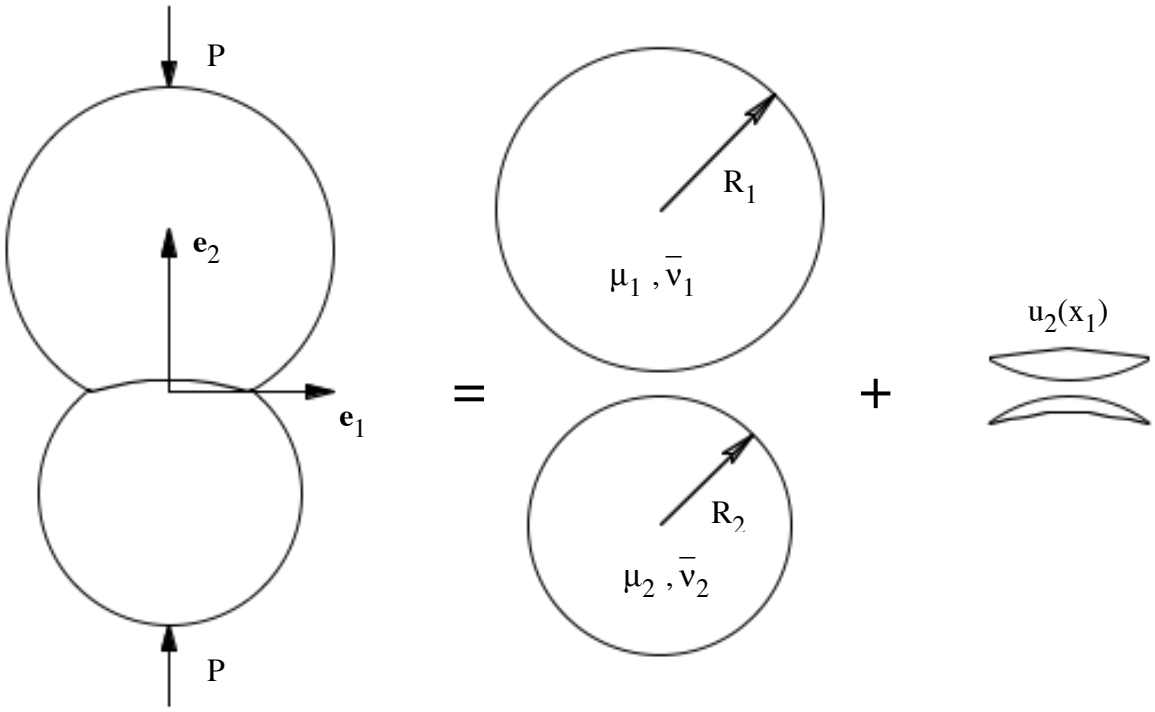

Fig. 25.4 Sketch of the two-dimensional Hertz problem.

Next, consider the two-dimensional Hertz problem shown in Fig. 25.4 where two elastic cylinders are pressed into contact by a line force P. One cylinder has material constants μ_1 and $\bar{\nu}_1$ and undeformed radius R_1, whereas the other cylinder has material constants μ_2 and $\bar{\nu}_2$ and undeformed radius R_2. In general, the line of contact between the two cylinders in the x_3=0 plane is curved. Using the approximations of the previous analysis, the cylindrical surfaces are approximated as parabolas and the pressure distribution applied to the contact region is determined by the solution for a pressure distribution on a flat half-space.

Specifically, the pressure distribution is taken in the form (25.17) which satisfies the equation (25.18) and ensures that each of the cylinders is in equilibrium. Next, using the approximation that the contact line is described by a parabola $f(x_1)$

$$f(x_1) = \frac{b}{2}(c^2 - x_1^2) \quad , \tag{25.25}$$

it follows that the magnitude of the displacement u_2 compressing the cylinder of radius R_1 is given by

$$u = \sqrt{R_1^2 - x_1^2} - \sqrt{R_1^2 - c^2} + f(x_1) \approx \frac{1}{2}\left[\frac{1}{R_1} + b\right](c^2 - x_1^2) \ . \tag{25.26}$$

Thus, replacing u_2 in (25.13) by the expression (25.26), replacing μ, $\bar{\nu}$ by μ_1, $\bar{\nu}_1$, respectively, and using (25.22), the kinematic contact condition reduces to

$$\left[\frac{1}{R_1} + b\right] x_1 = \frac{2P(1-\bar{\nu}_1)}{\mu_1 \pi^2 c^2} \int_{-c}^{c} \frac{\sqrt{c^2 - x_1^2}}{x_1 - x}\, dx = \frac{2P(1-\bar{\nu}_1)}{\mu_1 \pi c^2}\, x_1 \ , \tag{25.27}$$

which is satisfied provided that

$$\frac{\mu_1}{(1-\bar{\nu}_1)}\left[\frac{1}{R_1} + b\right] = \frac{2P}{\pi c^2} \ . \tag{25.28}$$

Similarly, the magnitude of the displacement u_2 compressing the cylinder of radius R_2 is given by

$$u = \sqrt{R_2^2 - x_1^2} - \sqrt{R_2^2 - c^2} - f(x_1) \approx \frac{1}{2}\left[\frac{1}{R_2} - b\right](c^2 - x_1^2) \ . \tag{25.29}$$

Thus, replacing u_2 in (25.13) by the expression (25.29), replacing μ, $\bar{\nu}$ by μ_2, $\bar{\nu}_2$, respectively, and using (25.22), the kinematic contact condition reduces to

$$\left[\frac{1}{R_2} - b\right] x_1 = \frac{2P(1-\bar{\nu}_2)}{\mu_2 \pi^2 c^2} \int_{-c}^{c} \frac{\sqrt{c^2 - x_1^2}}{x_1 - x}\, dx = \frac{2P(1-\bar{\nu}_2)}{\mu_2 \pi c^2}\, x_1 \ , \tag{25.30}$$

which is satisfied provided that

$$\frac{\mu_2}{(1-\bar{\nu}_2)}\left[\frac{1}{R_2} - b\right] = \frac{2P}{\pi c^2} \ . \tag{25.31}$$

The equations (25.28) and (25.31) represent two equations to determine the extent 2c and the shape b of the contact region. In particular, the solutions of these equations yields the results that

$$b = \frac{\dfrac{\mu_2}{(1-\bar{\nu}_2)R_2} - \dfrac{\mu_1}{(1-\bar{\nu}_1)R_1}}{\dfrac{\mu_1}{(1-\bar{\nu}_1)} + \dfrac{\mu_2}{(1-\bar{\nu}_2)}} ,$$

$$c = \sqrt{\frac{\dfrac{2P}{\pi}\left[\dfrac{\mu_1}{(1-\bar{\nu}_1)} + \dfrac{\mu_2}{(1-\bar{\nu}_2)}\right]}{\dfrac{\mu_1}{(1-\bar{\nu}_1)}\dfrac{\mu_2}{(1-\bar{\nu}_2)}\left[\dfrac{1}{R_1} + \dfrac{1}{R_2}\right]}} . \tag{25.32}$$

Also, letting Δ_1 be the displacement (towards the contact surface) of the center of the cylinder of radius R_1, and Δ_2 be the displacement (towards the contact surface) of the center of the cylinder of radius R_2, it can be shown that

$$\Delta_1 = R_1 - \sqrt{R_1^2 - c^2} \approx \frac{c^2}{2R_1} ,$$

$$\Delta_2 = R_2 - \sqrt{R_2^2 - c^2} \approx \frac{c^2}{2R_2} . \tag{25.33}$$

Thus, the relationship between the line force P and the total displacement Δ of the centers of the cylinders

$$\Delta = \Delta_1 + \Delta_2 = \frac{c^2}{2}\left[\frac{1}{R_1} + \frac{1}{R_2}\right] , \tag{25.34}$$

can be written in the form

$$P = \left[\frac{\pi \dfrac{\mu_1}{(1-\bar{\nu}_1)}\dfrac{\mu_2}{(1-\bar{\nu}_2)}}{\dfrac{\mu_1}{(1-\bar{\nu}_1)} + \dfrac{\mu_2}{(1-\bar{\nu}_2)}}\right] \Delta , \tag{25.35}$$

which again is a linear relationship.

It is interesting to note that if the two materials are identical then the expression b in (25.32) reduces to

$$b = \frac{1}{R_2} - \frac{1}{R_1} \quad . \tag{25.36}$$

This means that the contact surface is such that the smaller cylinder penetrates the larger cylinder. Moreover, if the cylinder of radius R_2 is rigid then $\mu_2=\infty$ and the expression (25.32) reduces to

$$b = \frac{1}{R_2} \quad , \tag{25.37}$$

as expected. Also, the contact surface will be flat (b=0) if

$$\frac{\mu_1}{(1-\bar{\nu}_1)R_1} = \frac{\mu_2}{(1-\bar{\nu}_2)R_2} \quad . \tag{25.38}$$

26. Two-dimensional climb dislocation solution

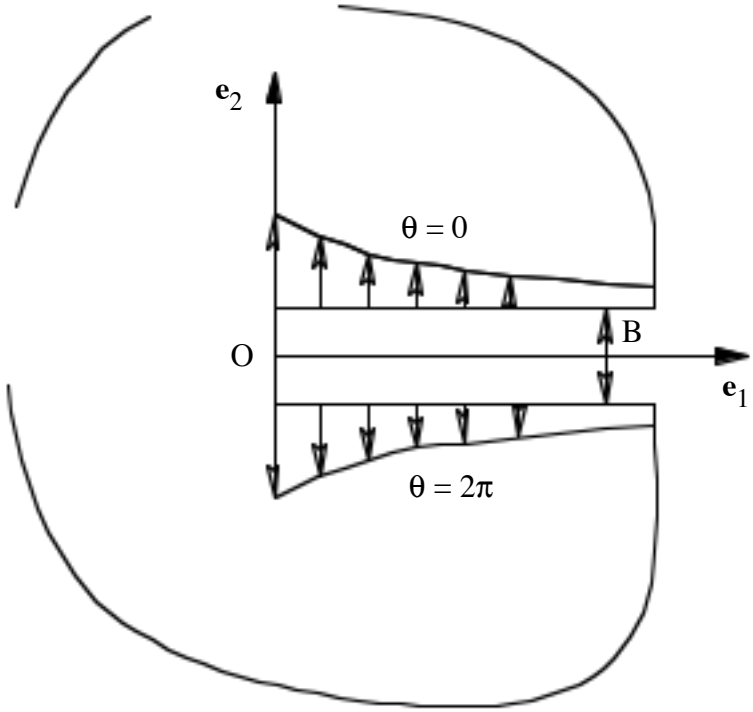

Fig. 26.1 Sketch of the compressive stresses on the surfaces of a two-dimensional climb dislocation.

Following the development in section 13.2 of Barber (1992), it is convenient to consider the solution for a two-dimensional climb dislocation to develop the Green's function for the solution of a two-dimensional crack considered in the next section. Fig. 26.1 shows a sketch of the compressive stresses on the surfaces of a two-dimensional climb dislocation which is modeled as a slit of constant thickness B. For this problem the body force vanishes

$$b_r = b_\theta = 0 \quad , \tag{26.1}$$

and the temperature θ^* is taken to be uniform but not necessarily equal to θ_0^*.

The solution of this problem is not only singular at the origin but it is not single valued, since the stresses and displacements of the surface θ=0 are different from those of the surface θ=2π. Specifically, consider the Michell solution of the form

$$T_{rr} = \frac{b_1}{r}\cos\theta \quad , \quad T_{\theta\theta} = \frac{b_1}{r}\cos\theta \quad , \quad T_{r\theta} = \frac{b_1}{r}\sin\theta \quad , \tag{26.2}$$

where b_1 is a constant to be determined. It can easily be seen that this stress field satisfies the equations of equilibrium (18.6) and the compatibility equations (18.7).

In order to determine the displacement field associated with the solution (26.2), use is made of the constitutive equations (16.14) with the temperature set to the constant value θ_1

$$e_{rr} = \left[\frac{(1-2\bar{\nu})b_1}{2\mu r} \right] \cos\theta + (1+\bar{\nu})\bar{\alpha}(\theta_1^* - \theta_0^*) ,$$

$$e_{\theta\theta} = \left[\frac{(1-2\bar{\nu})b_1}{2\mu r} \right] \cos\theta + (1+\bar{\nu})\bar{\alpha}(\theta_1^* - \theta_0^*) \ , \ e_{r\theta} = \frac{b_1}{2\mu r} \sin\theta \ . \tag{26.3}$$

Next, integration of the strain-displacement relations (16.9) yields

$$u_r = \left[\frac{(1-2\bar{\nu})b_1}{2\mu} \right] \ln(r) \cos\theta + (1+\bar{\nu})\bar{\alpha}(\theta_1^* - \theta_0^*)\, r + \frac{df(\theta)}{d\theta} ,$$

$$u_\theta = \left[\frac{(1-2\bar{\nu})b_1}{2\mu} \right] \left[1 - \ln(r) \right] \sin\theta - f(\theta) + g(r) \ , \tag{26.4}$$

where $f(\theta)$ and $g(r)$ are functions of integration. Moreover, substituting these results into the expression (16.9) for the strain $e_{r\theta}$ and using of (26.3) yields the equation

$$\frac{1}{r} \left[\frac{d^2 f(\theta)}{d\theta^2} + f(\theta) - \frac{2(1-\bar{\nu})b_1}{\mu} \sin\theta \right] + \left[\frac{dg(r)}{dr} - \frac{1}{r} g(r) \right] = 0 \ . \tag{26.5}$$

Thus, in view of the expressions (16.8) for rigid body displacements, it follows that the solution of (26.4) can be expressed in the form

$$f(\theta) = c_1 \sin\theta - c_2 \cos\theta + \frac{(1-\bar{\nu})b_1}{\mu} \left[\sin\theta - \theta \cos\theta \right] , \ g(r) = - H_{12}\, r \ , \tag{26.6}$$

where c_α and H_{12} are constants. Then, the displacements become

$$u_r = \left[\frac{(1-2\bar{\nu})b_1}{2\mu} \right] \ln(r) \cos\theta + (1+\bar{\nu})\bar{\alpha}(\theta_1^* - \theta_0^*)\, r + c_1 \cos\theta + c_2 \sin\theta$$

$$+ \left[\frac{(1-\bar{\nu})b_1}{\mu} \right] \theta \sin\theta ,$$

$$u_\theta = \left[\frac{(1-2\bar{\nu})b_1}{2\mu} \right] \left[1 - \ln(r) \right] \sin\theta - c_1 \sin\theta + c_2 \cos\theta - \frac{(1-\bar{\nu})b_1}{\mu} \left[\sin\theta - \theta \cos\theta \right]$$

$$- H_{12}\, r \ . \tag{26.7}$$

Next, the value of b_1 is determined by using this expression for u_θ to determine the value B of the gap caused by the dislocation

$$\delta = \mathbf{e}_2 \bullet \left[\mathbf{u}(r,0) - \mathbf{u}(r,2\pi) \right] = u_\theta(0) + u_\theta(2\pi) = - \frac{2\pi(1-\bar{\nu})b_1}{\mu} \ . \tag{26.8}$$

Thus, a dislocation of strength B can be defined by taking

$$b_1 = - \frac{\mu B}{2\pi(1-\bar{\nu})} \ , \tag{26.9}$$

so that the displacement field becomes

$$u_r = - \left[\frac{(1-2\bar{\nu})B}{4\pi(1-\bar{\nu})} \right] \ln(r) \cos\theta + (1+\bar{\nu})\bar{\alpha}(\theta_1^* - \theta_0^*)\, r + c_1 \cos\theta + c_2 \sin\theta$$

$$- \left[\frac{B}{2\pi} \right] \theta \sin\theta \ ,$$

$$u_\theta = - \left[\frac{(1-2\bar{\nu})B}{4\pi(1-\bar{\nu})} \right] \left[1 - \ln(r) \right] \sin\theta - c_1 \sin\theta + c_2 \cos\theta + \left[\frac{B}{2\pi} \right] \left[\sin\theta - \theta \cos\theta \right]$$

$$- H_{12}\, r \ , \tag{26.10}$$

and the stress field becomes

$$T_{rr} = - \left[\frac{\mu B}{2\pi(1-\bar{\nu})r} \right] \cos\theta \ , \ T_{\theta\theta} = - \left[\frac{\mu B}{2\pi(1-\bar{\nu})r} \right] \cos\theta \ ,$$

$$T_{r\theta} = - \left[\frac{\mu B}{2\pi(1-\bar{\nu})r} \right] \sin\theta \ , \tag{26.11}$$

Now, using the relations

$$r = \sqrt{x_1^2 + x_2^2} \ , \ \cos\theta = \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \ , \ \sin\theta = \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \ , \ \theta = \tan^{-1}\left[\frac{x_2}{x_1} \right] , \tag{26.12}$$

the rectangular Cartesian components of the displacement vector associated with the results (24.11) and (24.12) become

$$\mathbf{u} = u_r\,\mathbf{e}_r + u_\theta\,\mathbf{e}_\theta = \left[u_r\cos\theta - u_\theta\sin\theta\right]\mathbf{e}_1 + \left[u_r\sin\theta + u_\theta\cos\theta\right]\mathbf{e}_2 \ ,$$

$$u_1 = -\left[\frac{(1-2\bar{\nu})B}{4\pi(1-\bar{\nu})}\right]\ln\left\{\sqrt{x_1^2+x_2^2}\right\} - \left[\frac{B}{4\pi(1-\bar{\nu})}\right]\left[\frac{x_2^2}{x_1^2+x_2^2}\right]$$

$$+ (1+\bar{\nu})\bar{\alpha}(\theta_1-\theta_0)\,x_1 + c_1 + H_{12}\,x_2 \ , \tag{26.13a}$$

$$u_2 = \left[\frac{B}{4\pi(1-\bar{\nu})}\right]\left[\frac{x_1x_2}{x_1^2+x_2^2}\right] - \frac{B}{2\pi}\tan^{-1}\left\{\frac{x_2}{x_1}\right\}$$

$$+ (1+\bar{\nu})\bar{\alpha}(\theta_1^*-\theta_0^*)\,x_2 + c_2 - H_{12}\,x_1 \ . \tag{26.13b}$$

and those of the stress tensor become

$$\mathbf{T} = T_{rr}\,(\mathbf{e}_r\otimes\mathbf{e}_r) + T_{\theta\theta}\,(\mathbf{e}_\theta\otimes\mathbf{e}_\theta) + T_{r\theta}\,(\mathbf{e}_r\otimes\mathbf{e}_\theta+\mathbf{e}_\theta\otimes\mathbf{e}_r)$$

$$\mathbf{T} = T_{rr}\left[\cos^2\theta\,(\mathbf{e}_1\otimes\mathbf{e}_1) + \sin^2\theta\,(\mathbf{e}_2\otimes\mathbf{e}_2) + \sin\theta\cos\theta\,(\mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1)\right]$$

$$+ T_{\theta\theta}\left[\sin^2\theta\,(\mathbf{e}_1\otimes\mathbf{e}_1) + \cos^2\theta\,(\mathbf{e}_2\otimes\mathbf{e}_2) - \sin\theta\cos\theta\,(\mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1)\right]$$

$$+ T_{r\theta}\left[2\sin\theta\cos\theta(-\,\mathbf{e}_1\otimes\mathbf{e}_1+\mathbf{e}_2\otimes\mathbf{e}_2) + (\cos^2\theta - \sin^2\theta)(\mathbf{e}_1\otimes\mathbf{e}_2+\mathbf{e}_2\otimes\mathbf{e}_1)\right] \ ,$$

$$\mathbf{T} = -\left[\frac{\mu B}{2\pi(1-\bar{\nu})r}\right]\cos\theta\left[(1-2\sin^2\theta)(\mathbf{e}_1\otimes\mathbf{e}_1) + (1+2\sin^2\theta)(\mathbf{e}_2\otimes\mathbf{e}_2)\right.$$

$$\left. + (\cos^2\theta-\sin^2\theta)\sin\theta\,(\mathbf{e}_1\otimes\mathbf{e}_2+\mathbf{e}_2\otimes\mathbf{e}_1)\right] \ ,$$

$$T_{11} = -\left[\frac{\mu B}{2\pi(1-\bar{\nu})}\right]\left[\frac{x_1(x_1^2-x_2^2)}{(x_1^2+x_2^2)^2}\right] \ , \quad T_{22} = -\left[\frac{\mu B}{2\pi(1-\bar{\nu})}\right]\left[\frac{x_1(x_1^2+3x_2^2)}{(x_1^2+x_2^2)^2}\right] \ , \tag{26.14a}$$

$$T_{12} = -\left[\frac{\mu B}{2\pi(1-\bar{\nu})}\right]\left[\frac{x_2(x_1^2-x_2^2)}{(x_1^2+x_2^2)^2}\right] \ . \tag{26.14b}$$

In particular, notice that the stress field is self equilibrating since the stresses vanish far way from the dislocation.

27. Two-dimensional crack in a tensile field

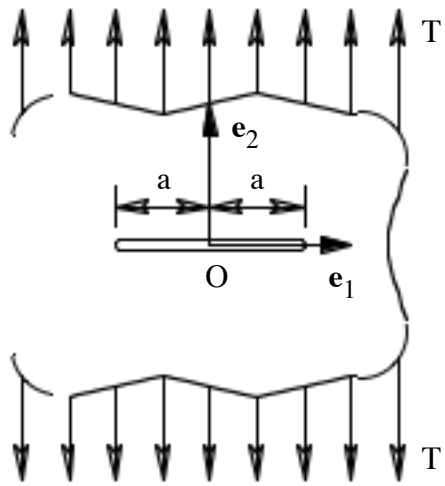

Fig. 27.1 Sketch of a two-dimensional crack subjected to far field tension T.

Following the development in section 13.3 of Barber (1992), it is convenient to use the solution of section 26 as the Green's function for a two-dimensional crack subjected to far field tension (Fig. 27.1). For this problem the body force vanishes

$$b_r = b_\theta = 0 \ , \tag{27.1}$$

and the temperature θ^* is taken to be uniform but not necessarily equal to θ_0^*.

The main idea is to consider the simple solution of uniaxial stress T in the $\mathbf{e}_2$ direction

$$T_{11} = T_{12} = 0 \ , \ T_{22} = T \ ,$$

$$e_{11} = -\frac{\nu T}{E} \ , \ e_{22} = \frac{T}{E} \ , \ e_{12} = 0 \ ,$$

$$u_1 = -\frac{\nu T}{E} x_1 \ , \ u_2 = \frac{T}{E} x_2 \ , \tag{27.2}$$

and to use superposition of the solution of section 26 to eliminate the tensions on the crack surfaces.

Specifically, consider a distribution of dislocations $B(x_1)$ which model the crack opening in the region $-a \le x_1 \le a$. Then, using (26.14) and superposition of (27.2), the stress field associated with this distribution can be written in the form

$$T_{11}(x_1,x_2) = -\left[\frac{\mu}{2\pi(1-\bar{\nu})}\right]\int_{-a}^{a} B(x)\left\{\frac{(x_1-x)[(x_1-x)^2-x_2^2]}{[(x_1-x)^2+x_2^2]^2}\right\}dx \ , \qquad (27.3a)$$

$$T_{22}(x_1,x_2) = T - \left[\frac{\mu}{2\pi(1-\bar{\nu})}\right]\int_{-a}^{a} B(x)\left\{\frac{(x_1-x)[(x_1-x)^2+3x_2^2]}{[(x_1-x)^2+x_2^2]^2}\right\}dx \ , \qquad (27.3b)$$

$$T_{12}(x_1,x_2) = -\left[\frac{\mu}{2\pi(1-\bar{\nu})}\right]\int_{-a}^{a} B(x)\left\{\frac{x_2[(x_1-x)^2-x_2^2]}{[(x_1-x)^2+x_2^2]^2}\right\}dx \ , \qquad (27.3c)$$

where x denotes the location of the root of the dislocation of strength B(x). Similarly, the displacement field associated with this distribution of dislocations can be obtained using (26.13) and superposition of (27.2) to deduce that

$$u_1(x_1,x_2) = -\left[\frac{(1-2\bar{\nu})}{4\pi(1-\bar{\nu})}\right]\int_{-a}^{a} B(x)\left[(1-2\bar{\nu})\ \ln\left\{\sqrt{(x_1-x)^2+x_2^2}\right\} + \left\{\frac{x_2^2}{(x_1-x)^2+x_2^2}\right\}\right]dx$$

$$-\frac{\bar{\nu}T}{E}x_1 + (1+\bar{\nu})\bar{\alpha}(\theta_1^*-\theta_0^*)\,x_1 + c_1 + H_{12}\,x_2 \ , \qquad (27.4a)$$

$$u_2(x_1,x_2) = \left[\frac{1}{4\pi(1-\bar{\nu})}\right]\int_{-a}^{a} B(x)\left[\left\{\frac{(x_1-x)x_2}{(x_1-x)^2+x_2^2}\right\} - 2(1-\bar{\nu})\ \tan^{-1}\left\{\frac{x_2}{(x_1-x)}\right\}\right]dx$$

$$+\frac{T}{E}x_2 + (1+\bar{\nu})\bar{\alpha}(\theta_1^*-\theta_0^*)\,x_2 + c_2 - H_{12}\,x_1 \ , \qquad (27.4b)$$

where the effects of temperature and rigid body displacements have been included. Moreover, the solution (27.3) will correspond to a crack with stress-free surfaces if

$$T_{22}(x_1,0) = T - \left[\frac{\mu}{2\pi(1-\bar{\nu})}\right]\int_{-a}^{a}\frac{B(x)}{x_1-x}\,dx = 0 \quad \text{for } -a < x_1 < a \ . \qquad (27.5)$$

It will now be shown that the solution of this integral equation becomes

$$B(x_1) = \left[\frac{2\pi(1-\bar{\nu})TC}{\mu}\right]\frac{x_1}{\sqrt{a^2-x_1^2}} \quad \text{for } -a < x_1 < a \ , \qquad (27.6)$$

where C is a constant to be determined. To this end, (27.6) is substituted into (27.5) to obtain the equation

$$C\int_{-a}^{a}\frac{x}{(x_1-x)\sqrt{a^2-x^2}}\,dx = 1 \qquad \text{for } -a < x_1 < a \ . \qquad (27.7)$$

Next, it can be shown that

$$\frac{x}{(x_1-x)\sqrt{a^2-x^2}} = \mathrm{Re}\Big[\frac{\partial f(x,x_1)}{\partial x}\Big] ,$$

$$f(x,x_1) = -\tan^{-1}\Big\{\frac{x\sqrt{a^2-x^2}}{a^2-x^2}\Big\} + \frac{x_1}{\sqrt{a^2-x_1^2}}\ \ln\Big\{\frac{2\sqrt{a^2-x^2}\sqrt{a^2-x_1^2} + 2(a^2-x_1x)}{x_1(x_1-x)\sqrt{a^2-x_1^2}}\Big\} \quad , \quad (27.8)$$

where Re[x] denote the real part of x and $f(x,x_1)$ is an auxiliary function. Then, the integral in (27.7) can be written in the form

$$\int_{-a}^{a} \frac{x}{(x_1-x)\sqrt{a^2-x^2}}\ dx = \mathrm{Re}\ \big[f(a,x_1) - f(-a,x_1)\big] \ . \qquad (27.9)$$

To evaluate this expression, it is convenient to define an additional auxiliary function

$$g(x,x_1) = f(x,x_1) - f(-x,x_1) ,$$

$$g(x,x_1) = -2\tan^{-1}\Big\{\frac{x\sqrt{a^2-x^2}}{a^2-x^2}\Big\} + \frac{x_1}{\sqrt{a^2-x_1^2}}\ \ln\Big\{\frac{2\sqrt{a^2-x^2}\sqrt{a^2-x_1^2} + 2(a^2-x_1x)}{x_1(x_1-x)\sqrt{a^2-x_1^2}}\Big\}$$

$$-\frac{x_1}{\sqrt{a^2-x_1^2}}\ \ln\Big\{\frac{2\sqrt{a^2-x^2}\sqrt{a^2-x_1^2} + 2(a^2+x_1x)}{x_1(x_1+x)\sqrt{a^2-x_1^2}}\Big\} \ . \qquad (27.10)$$

Now, it can easily be seen that in the limit as x approaches a from below

$$\underset{x\varnothing a^-}{\mathrm{limit}}\ \ g(x,x_1) = -\pi + h(x_1) \ ,$$

$$h(x_1) = \frac{x_1}{\sqrt{a^2-x_1^2}}\Big[\ln\Big\{-\frac{a}{x_1\sqrt{a^2-x_1^2}}\Big\} - \ln\Big\{\frac{a}{x_1\sqrt{a^2-x_1^2}}\Big\}\Big] \ . \qquad (27.11)$$

Thus, the integral (27.9) can be expressed in the form

$$\int_{-a}^{a} \frac{x}{(x_1-x)\sqrt{a^2-x^2}}\ dx = -\pi + \mathrm{Re}\ \big[h(x_1)\big] . \qquad (27.12)$$

Moreover, recalling that

$$\ln(x) = \ln(|x|)\ \text{ for } x > 0 \ , \quad \ln(x) = i\,\pi + \ln(|x|) \text{ for } x < 0 \ ,$$

$$i = \sqrt{-1}\ \ , \ \frac{1}{i} = -i\ \ , \ \ln(i) = \frac{i\pi}{2}\ , \ln(-i) = -\frac{i\pi}{2}\ , \qquad (27.13)$$

it can be shown that

For $0 < x_1 < a$

$$h(x_1) = \frac{x_1}{\sqrt{a^2-x_1^2}}\left[i\,\pi + \ln\left\{\frac{a}{x_1\sqrt{a^2-x_1^2}}\right\} - \ln\left\{\frac{a}{x_1\sqrt{a^2-x_1^2}}\right\}\right] = \frac{i\pi x_1}{\sqrt{a^2-x_1^2}} \quad , \quad (27.14a)$$

For $-a < x_1 < 0$

$$h(x_1) = \frac{x_1}{\sqrt{a^2-x_1^2}}\left[\ln\left\{\frac{a}{|x_1|\sqrt{a^2-x_1^2}}\right\} - \ln\left\{\frac{a}{|x_1|\sqrt{a^2-x_1^2}}\right\} - i\,\pi\right] = -\frac{i\pi x_1}{\sqrt{a^2-x_1^2}} \quad , (27.14b)$$

For $a < x_1$

$$h(x_1) = -\frac{ix_1}{\sqrt{x_1^2-a^2}}\left[\frac{i\pi}{2} + \ln\left\{\frac{a}{x_1\sqrt{x_1^2-a^2}}\right\} + \frac{i\pi}{2} - \ln\left\{\frac{a}{x_1\sqrt{x_1^2-a^2}}\right\}\right] = \frac{\pi x_1}{\sqrt{x_1^2-a^2}} \quad , \quad (27.14c)$$

For $x_1 < -a$

$$h(x_1) = -\frac{ix_1}{\sqrt{x_1^2-a^2}}\left[-\frac{i\pi}{2} + \ln\left\{\frac{a}{|x_1|\sqrt{x_1^2-a^2}}\right\} - \frac{i\pi}{2} - \ln\left\{\frac{a}{|x_1|\sqrt{x_1^2-a^2}}\right\}\right] = -\frac{\pi x_1}{\sqrt{x_1^2-a^2}} \; . \; (27.14d)$$

It then follows that the integral (27.12) becomes

$$\int_{-a}^{a} \frac{x}{(x_1-x)\sqrt{a^2-x^2}}\, dx = -\pi \quad \text{for } |x_1| < a \quad , \qquad (27.15a)$$

$$\int_{-a}^{a} \frac{x}{(x_1-x)\sqrt{a^2-x^2}}\, dx = \pi\left[-1 + \frac{|x_1|}{\sqrt{x_1^2-a^2}}\right] \quad \text{for } |x_1| > a \quad . \qquad (27.15b)$$

Consequently, the constant C in (27.7) can be determined using the result (27.15a) to obtain

$$C = -\frac{1}{\pi} \quad . \qquad (27.16)$$

Next, with the help of (27.3b), (27.6) and (27.16), the stress T_{22} ahead of the crack tip can be expressed in the form

$$T_{22}(x_1,0) = T\left[\,1 + \frac{1}{\pi}\int_{-a}^{a} \frac{x}{(x_1-x)\sqrt{a^2-x^2}}\, dx\,\right] \quad \text{for } |x_1| > a \quad . \qquad (27.17)$$

Consequently, using (27.15b) this expression becomes

$$T_{22}(x_1,0) = \frac{T\,|x_1|}{\sqrt{x_1^2 - a^2}} \quad \text{for } |x_1| > a \ . \tag{27.18}$$

This solution is the basis for linear elastic fracture mechanics. In particular, it indicates that the stress field is square root singular at the crack tip. Moreover, it is convenient to define the stress intensity factor K_I for this mode I crack by the expression

$$K_I = \lim_{x_1 \to a^+} \left[T_{22}(x_1,0)\sqrt{x-a} \right] = \lim_{x_1 \to a^+} \left[\frac{T\,|x_1|\sqrt{x-a}}{\sqrt{x_1^2 - a^2}} \right] \ ,$$

$$K_I = T\sqrt{\frac{a}{2}} \ . \tag{27.19}$$

Linear elastic fracture mechanics assumes that brittle materials have a material constant K_{Ic}, called the fracture toughness, which limits the value of K_I for which the crack remains stationary. In other words, when K_I attains the critical value K_{Ic}, then crack propagation is initiated. Thus, when the far field tension T attains the critical value T_c

$$T_c = K_{Ic}\sqrt{\frac{2}{a}} \ , \tag{27.20}$$

crack propagation begins. It is important to emphasize that larger cracks (larger values of a) require less tension to propagate than smaller cracks.

Physically, it is also of interest to determine the crack opening displacement δ defined by (26.8). Specifically, using (27.4b) it follows that

$$\delta(x_1) = u_2(x_1,0^+) - u_2(x_1,0^-) \ ,$$

$$\delta(x_1) = -\left[\frac{1}{2\pi}\right] \int_{-a}^{a} B(x) \left[\lim_{x_2 \to 0^+} \tan^{-1}\left\{ \frac{x_2}{(x_1 - x)} \right\} - \lim_{x_2 \to 0^-} \tan^{-1}\left\{ \frac{x_2}{(x_1 - x)} \right\} \right] dx$$

$$\text{for } |x_1| < a \ . \tag{27.21}$$

In order to determine the correct values of the inverse tangent function it is necessary to separate the integral into two parts

$$\delta(x_1) = -\left[\frac{1}{2\pi}\right] \int_{-a}^{x_1} B(x) \left[\lim_{x_2 \to 0^+} \tan^{-1}\left\{ \frac{x_2}{(x_1 - x)} \right\} - \lim_{x_2 \to 0^-} \tan^{-1}\left\{ \frac{x_2}{(x_1 - x)} \right\} \right] dx$$

$$- \left[\frac{1}{2\pi}\right] \int_{x_1}^{a} B(x) \left[\lim_{x_2 \to 0^+} \tan^{-1}\left\{ \frac{x_2}{(x_1 - x)} \right\} - \lim_{x_2 \to 0^-} \tan^{-1}\left\{ \frac{x_2}{(x_1 - x)} \right\} \right] dx \ ,$$

$$\delta(x_1) = - \left[\frac{1}{2\pi}\right] \int_{-a}^{x_1} B(x) \left[0 - 2\pi \right] dx - \left[\frac{1}{2\pi}\right] \int_{x_1}^{a} B(x) \left[\pi - \pi\right] dx \ ,$$

$$\delta(x_1) = \int_{-a}^{x_1} B(x)\, dx = - \left[\frac{(1-\bar{\nu})T}{\mu}\right] \int_{-a}^{x_1} \frac{x}{\sqrt{a^2 - x^2}}\, dx \ . \tag{27.22}$$

Thus, evaluation of this integral yields the crack opening displacement

$$\delta(x_1) = \frac{2(1-\bar{\nu})T}{\mu} \sqrt{a^2 - x_1^2} \ . \tag{27.23}$$

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Appendix A: Eigenvalues, Eigenvectors, and Principal Invariants of a Tensor

In this appendix we briefly review some basic properties of eigenvalues and eigenvectors. The vector **v** is said to be an eigenvector of a real second order symmetric tensor **T** with the associated eigenvalue σ if

$$\mathbf{T}\,\mathbf{v} = \sigma\,\mathbf{v} \;\;,\;\; T_{ij}\,v_j = \sigma\,v_i \;\;. \tag{A1a,b}$$

It follows that the characteristic equation for determining the three values of the eigenvalue σ is given by

$$\det(\mathbf{T} - \sigma\mathbf{I}) = -\sigma^3 + \sigma^2 I_1 - \sigma I_2 + I_3 = 0 \;\;, \tag{A2}$$

where I_1, I_2, I_3 are the principal invariants of an arbitrary real tensor **T**

$$I_1(\mathbf{T}) = \mathbf{T} \bullet \mathbf{I} = \mathrm{tr}\,\mathbf{T} = T_{mm} \;\;, \tag{A3a}$$

$$I_2(\mathbf{T}) = \frac{1}{2}\;[(\mathbf{T} \bullet \mathbf{I})^2 - (\mathbf{T} \bullet \mathbf{T}^T)] = \frac{1}{2}\;[(T_{mm})^2 - T_{mn}T_{nm}] \;\;, \tag{A3b}$$

$$I_3(\mathbf{T}) = \det \mathbf{T} = \frac{1}{6}\;\varepsilon_{ijk}\varepsilon_{lmn}T_{il}T_{jm}T_{kn} \;\;. \tag{A3c}$$

It can be shown that since **T** is a real symmetric tensor the three roots of the cubic equation (A2) are real. Also, it can be shown that the three independent eigenvectors **v** obtained by solving (A1) can be chosen to form an orthonormal set of vectors.

Recalling that **T** can be separated into its spherical part T **I** and its deviatoric part **T**' such that

$$\mathbf{T} = T\,\mathbf{I} + \mathbf{T}' \;\;,\;\; T_{ij} = T\,\delta_{ij} + T_{ij}' \;\;, \tag{A4a,b}$$

$$T = \frac{1}{3}(\mathbf{T} \bullet \mathbf{I}) = \frac{1}{3}(T_{mm}) \;\;,\;\; \mathbf{T}' \bullet \mathbf{I} = T_{mm}' = 0 \;\;, \tag{A4c,d}$$

it follows that when **v** is an eigenvector of **T** it is also an eigenvector of **T**'

$$\mathbf{T}'\,\mathbf{v} = (\mathbf{T} - T\,\mathbf{I})\,\mathbf{v} = (\sigma - T)\,\mathbf{v} = \sigma'\,\mathbf{v} \;\;, \tag{A5}$$

with the associated eigenvalue σ' related to σ by

$$\sigma = \sigma' + T \;\;. \tag{A6}$$

However since the first principal invariant of **T**' vanishes we may write the characteristic equation for σ' in the form

$$\det(\mathbf{T}' - \sigma'\mathbf{I}) = -(\sigma')^3 + \sigma'\left(\frac{\sigma_e^2}{3}\right) + J_3 = 0 \;\;, \tag{A7}$$

where we have defined the alternative invariants σ_e and J_3 by

$$\sigma_e^2 = \frac{3}{2}\,\mathbf{T}' \bullet \mathbf{T}' = -\,3\,I_2(\mathbf{T}') \;\;,\;\; J_3 = \det \mathbf{T}' = I_3(\mathbf{T}') \;\;. \tag{A8a,b}$$

Note that if σ_e vanishes then $\mathbf{T}'$ vanishes so that from (A7) σ' vanishes and from (A6) it follows that there is only one distinct eigenvalue

$$\sigma = T \;\;. \tag{A9}$$

On the other hand, if σ_e does not vanish we may divide (A7) by $(\sigma_e/3)^3$ to obtain

$$\left(\frac{3\sigma'}{\sigma_e}\right)^3 - 3\left(\frac{3\sigma'}{\sigma_e}\right) - 2\,\hat{J}_3 = 0 \;\;, \tag{A10}$$

where the invariant $\hat{J}_3$ is defined by

$$\hat{J}_3 = \frac{27\,J_3}{2\,\sigma_e^3} \;\;. \tag{A11}$$

Since (A10) is in the standard form for a cubic, the solution can be obtained easily using the trigonometric form

$$\sin 3\beta = -\,\hat{J}_3 \;\;,\; -\frac{\pi}{6} \le \beta \le \frac{\pi}{6} \;\;, \tag{A12a}$$

$$\sigma_1' = \frac{2\sigma_e}{3}\cos\left(\frac{\pi}{6} + \beta\right) \;\;, \tag{A12b}$$

$$\sigma_2' = \frac{2\sigma_e}{3}\sin(\beta) \;\;, \tag{A12c}$$

$$\sigma_3' = -\frac{2\sigma_e}{3}\cos\left(\frac{\pi}{6} - \beta\right) \;\;, \tag{A12d}$$

where the eigenvalues $\sigma_1', \sigma_2', \sigma_3'$ are ordered so that

$$\sigma_1' \ge \sigma_2' \ge \sigma_3' \;\;. \tag{A13}$$

Once these values have been determined the three values of σ may be calculated using (A6).

Furthermore, we note that the value of β or $\hat{J}_3$ may be used to identify three states of deviatoric stress denoted by: triaxial compression (TXC); torsion (TOR); and triaxial extension (TXE); and defined by

$$\beta = \frac{\pi}{6} \ , \ \hat{J}_3 = -1 \ , \ \text{(TXC)} \ , \tag{A14a}$$

$$\beta = 0 \ , \ \hat{J}_3 = 0 \ , \ \text{(TOR)} \ , \tag{A14b}$$

$$\beta = -\frac{\pi}{6} \ , \ \hat{J}_3 = 1 \ , \ \text{(TXE)} \ . \tag{A14c}$$

HOMEWORK PROBLEM SETS

PROBLEM SET 1

Problem 1.1 Expand the following equations for an index range of three:

(a) $I = C_{ij}\, x_i\, x_j$, (b) $\phi = A_{jj}\, B_{kk}$, (c) $C = A_{ij}\, B_{ij}$. (P1.1)

Problem 1.2 Verify the identities

(a) $\delta_{ii} = 3$, (b) $\delta_{ij}\, \delta_{ij} = \delta_{ii}$, (c) $\delta_{ij}\, a_{jk} = a_{ik}$. (P1.2)

Problem 1.3 Expand the relationship

$$t_i = T_{ij}\, n_j \ . \tag{P1.3}$$

where t_i are the components of the stress vector, T_{ij} are the components of the stress tensor and n_i are the components of the unit outward normal.

Problem 1.4 Expand the equations of the balance of linear momentum

$$\rho \ddot{u}_i = \rho b_i + T_{ij,j} \ , \tag{P1.4}$$

where u_i are the components of the displacement vector, ρ is the mass density, b_i are the components of the body force, a superposed dot denotes partial differentiation with respect to time t, and a comma denotes partial differentiation with respect to the position x_i.

Problem 1.5

(a) Verify that

$$x_{i,j} = \delta_{ij} \ . \tag{P1.5}$$

(b) Using the result of part (b), write a simplified indicial expression for $(x_i\, x_i)_{,j}$.

(c) Using the result of part (a), write a simplified indicial expression for $(x_i\, x_i)_{,jj}$.

Problem 1.6: Consider the equations

$$R_{ij} = e_{ij,mm} + e_{mm,ij} - e_{im,mj} - e_{jm,im} = 0 \ . \tag{P1.6a}$$

For the special case when

$$e_{ij} = e_{ji} \quad , e_{i3} = 0 \quad , \; e_{ij,3} = 0 \quad , \tag{P1.6b}$$

expand and simplify the expressions for R_{11} and R_{22} and show that the equation for R_{12} is automatically satisfied.

Problem 1.7 Let $a_i = (1,2,3)$ and $b_i = (3,2,1)$ be the components of the vectors **a** and **b**, respectively. Also, let **T** be a second order tensor defined by

$$\mathbf{T} = \mathbf{a}\otimes\mathbf{b} \quad . \tag{P1.7}$$

Determine the components T_{ij} of **T**.

Problem 1.8 Starting with the representations $\mathbf{A} = A_{ij}\, \mathbf{e}_i\otimes\mathbf{e}_j$ and $\mathbf{B} = B_{ij}\, \mathbf{e}_i\otimes\mathbf{e}_j$, prove that

$$\mathbf{A} \bullet \mathbf{B} = A_{ij}\, B_{ij} \quad . \tag{P1.8}$$

Problem 1.9 The magnitude of $|\mathbf{T}|$ of the second order tensor **T** is defined by

$$|\mathbf{T}|^2 = \mathbf{T} \bullet \mathbf{T} = T_{ij}\, T_{ij} \quad . \tag{P1.9}$$

Using the results of (P1.1c) and (P1.8), show that this expression is positive definite (i.e. it is positive whenever **T** is nonzero).

Problem 1.10 Let **A** be a second order tensor with components A_{ij} which is represented by

$$\mathbf{A} = A_{ij}\, \mathbf{e}_i\otimes\mathbf{e}_j \quad . \tag{P1.10a}$$

Show that the components A^T_{ij} of $\mathbf{A}^T$ are given by

$$A^T_{ij} = \mathbf{A}^T \bullet (\mathbf{e}_i\otimes\mathbf{e}_j) = A_{ji} \quad . \tag{P1.10b}$$

Problem 1.11 Let **T** be the second order stress tensor with components T_{ij}. Use indicial notation and the result (2.18) to show that restriction associated with the balance of angular momentum

$$\mathbf{e}_j \times \mathbf{T}\, \mathbf{e}_j = 0 \quad , \tag{P1.11a}$$

requires the stress tensor **T** to be symmetric.

$$\mathbf{T}^T = \mathbf{T} \quad , \quad T_{ji} = T_{ij} \quad . \tag{P1.11b}$$

Problem 1.12 Let **A** and **B** be second order tensors with components A_{ij} and B_{ij}, respectively. Using the representation

$$\mathbf{AB} = A_{im}B_{mj}\, \mathbf{e}_i \otimes \mathbf{e}_j \quad , \tag{P1.12a}$$

Prove that

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T \quad . \tag{P1.12b}$$

Problem 1.13 Let **A** and **B** be second order tensors with components A_{ij} and B_{ij}, respectively, and let **a** and **b** be vectors with components a_i and b_i, respectively. Prove that

$$\mathbf{A\,a} \bullet \mathbf{Bb} = \mathbf{a} \bullet \mathbf{A}^T\mathbf{B\,b} \quad . \tag{P1.13}$$

Problem 1.14 Let **T** be a second order tensor with components T_{ij}

$$T_{ij} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad . \tag{P1.14}$$

(a) Calculate the symmetric part $T_{(ij)}$ of T_{ij}.

(b) Calculate the skew-symmetric part $T_{[ij]}$ of T_{ij}.

(c) Calculate the spherical part $T\,\delta_{ij}$ of T_{ij}.

(d) Calculate the deviatoric part T'_{ij} of T_{ij}.

Problem 1.15 Let **A** be a symmetric second order tensor and **B** be a skew-symmetric second order tensor

$$\mathbf{A}^T = \mathbf{A} \quad , \quad \mathbf{B}^T = -\,\mathbf{B} \quad . \tag{P1.15a,b}$$

Prove that a symmetric tensor is orthogonal to a skew-symmetric tensor

$$\mathbf{A} \bullet \mathbf{B} = 0 \quad . \tag{P.1.15c}$$

PROBLEM SET 2

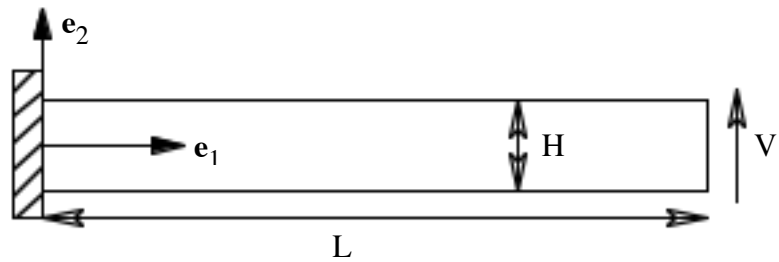


Fig. P.2.1 Sketch of a cantilever beam

Problem 2.1 Consider a rectangular cantilever beam of length L, height H, and depth W which occupies the region of space such that

$$0 \le x_1 \le L \ , \ -\frac{H}{2} \le x_2 \le \frac{H}{2} \ , \ -\frac{W}{2} \le x_3 \le \frac{W}{2} \ . \tag{P2.1a}$$

Next, consider the displacement field u_i associated with two-dimensional plane strain deformation of the beam which is given by

$$u_1 = -\left[\left\{\frac{V}{\mu HW}\right\} + \bar{\nu}\left\{\frac{V}{4\mu HW}\right\}\right] x_2 - (1-\bar{\nu})\left\{\frac{3V}{\mu H^3 W}\right\}\left\{2Lx_1 - x_1^2\right\} x_2$$

$$+ \left\{\frac{2V}{\mu H^3 W}\right\}\left\{\frac{3H^2}{4} x_2 - x_2^3\right\} + \bar{\nu}\left\{\frac{V}{\mu H^3 W}\right\} x_2^3 \ , \tag{P.2.1b}$$

$$u_2 = -\bar{\nu}\left\{\frac{V}{4\mu HW}\right\} L + \left[\left\{\frac{V}{\mu HW}\right\} + \bar{\nu}\left\{\frac{V}{4\mu HW}\right\}\right] x_1$$

$$+ \bar{\nu}\left\{\frac{3V}{\mu H^3 W}\right\}(L-x_1)\, x_2^2 \ + (1-\bar{\nu})\left\{\frac{V}{\mu H^3 W}\right\}\left\{3Lx_1^2 - x_1^3\right\} \ , \tag{P2.1c}$$

$$u_3 = 0 \ , \tag{P2.1d}$$

where V is the shear force applied to the end x_1=L, and μ and $\bar{\nu}$ are constants. Show that the strains e_{ij} associated with this displacement field are given by

$$e_{11} = -(1-\bar{\nu})\left\{\frac{6V}{\mu H^3 W}\right\}\left\{L - x_1\right\} x_2 \ , \ e_{22} = \bar{\nu}\left\{\frac{6V}{\mu H^3 W}\right\}(L-x_1)\, x_2 \ , \tag{P2.1e,f}$$

$$e_{12} = \left\{\frac{3V}{\mu H^3 W}\right\}\left\{\frac{H^2}{4} - x_2^2\right\} \ , \ e_{13} = e_{23} = e_{33} = 0 \ . \tag{P2.1g,h}$$

Problem 2.2 The displacement field given in problem 2.1 approximates built-in end conditions at x_1=0 but it is not exact. In particular, the displacements u_1 and u_2 do not vanish at x_1=0. However, specific averages of these displacements do vanish at x_1=0. To see this, define the average displacement field **w** by the formula

$$\mathbf{w}(x_1) = \frac{1}{H}\int_{-H/2}^{H/2} \mathbf{u}\, dx_2 \quad . \tag{P2.2a}$$

Calculate the value of **w** associated with the displacement field in Problem 2.1 and show that it vanishes at the end x_1=0

$$\mathbf{w}(0) = 0 \quad . \tag{P2.2b}$$

Problem 2.3 Similarly, define the director displacement $\boldsymbol{\delta}$ as the average gradient through the thickness by

$$\boldsymbol{\delta}(x_1) = \frac{1}{H}\int_{-H/2}^{H/2} \frac{\partial \mathbf{u}}{\partial x_2}\, dx_2 = \frac{1}{H}\left[\mathbf{u}(x_1,H/2) - \mathbf{u}(x_1,-H/2)\right] \quad . \tag{P2.3a}$$

Calculate the value of $\boldsymbol{\delta}$ associated with the displacement field in Problem 2.1 and show that it vanishes at the end x_1=0

$$\boldsymbol{\delta}(0) = 0 \quad . \tag{P2.3b}$$

Problem 2.4 Determine the terms in the displacement field in Problem 2.1 which are associated with homogeneous deformation and explain the physical meaning of these terms.

Problem 2.5: Recall from (3.16) that the strain E of a material fiber that was in the direction **S** in the reference configuration is given by the formula

$$E = \mathbf{e} \bullet (\mathbf{S}\otimes\mathbf{S}) = e_{ij}\, S_i\, S_j \quad . \tag{P2.5}$$

Consider material fibers which are in the direction $\mathbf{S} = \mathbf{e}_1$ along the axis of the beam, and take V to be positive and $\bar{\nu}$ to be less than 1/2.

(a) Show that these fibers are contracted at the top surface of the beam (x_2=H/2).

(b) Show that these fibers are extended at the bottom surface of the beam (x_2=–H/2).

(c) Show that these fibers are unstretched along the middle line of the beam (x_2=0).

Problem 2.6 Recall from (3.21) that the reduction γ in the angle between the two line elements which were in the directions $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$ in the reference configuration is given by the formula

$$\gamma = 2\mathbf{e} \bullet \{\mathbf{S}^{(1)} \otimes \mathbf{S}^{(2)}\} \quad \text{for } \mathbf{S}^{(1)} \bullet \mathbf{S}^{(2)} = 0 \ . \tag{P2.6}$$

As a special case take

$$\mathbf{S}^{(1)} = \mathbf{e}_1 \ , \ \mathbf{S}^{(2)} = \mathbf{e}_2 \ . \tag{P2.6b}$$

(a) Show that the fibers located at the middle line of the beam (x_2=0) are sheared relative to each other since γ is nonzero.

(b) Show that the value of γ vanishes for fibers located at the top (x_2=H/2) and bottom (x_2=–H/2) of the beam.

Problem 2.7 Consider material fibers located at the center line of the beam (x_2=0).

(a) Show that the fiber in the

$$\mathbf{S} = \frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2) \ , \tag{P2.7a}$$

is extended.

(b) Also, show that the fiber

$$\mathbf{S} = \frac{1}{\sqrt{2}}(-\mathbf{e}_1 + \mathbf{e}_2) \ , \tag{P2.7a}$$

is contracted.

Problem 2.8: Calculate the volume change at points along the top of the beam (x_2=H/2).

PROBLEM SET 3

Problem 3.1 For two dimensional deformations the displacements are given by

$$u_1 = u_1(x_1,x_2) \ , \ u_2 = u_2(x_1,x_2) \ , \ u_3 = 0 \ . \tag{P3.1a}$$

Specifically, consider the strain field

$$e_{11} = A\, x_2^2 \ , \ e_{22} = A\, x_1^2 \ , \ e_{12} = 2B\, x_1\, x_2 \ , \ e_{i3} = 0 \ , \tag{P3.1b}$$

where A and B are constants.

(a) Use the strain-displacement relations and integrate the strain e_{11} to derive an expression for u_1.

(b) Use the strain-displacement relations and integrate the strain e_{22} to derive an expression for u_2.

(c) Show that the expression for the strain e_{12} is incompatible with the strain-displacement relations unless B=A.

(d) Show that the compatibility relations (3.58) will not be satisfied unless B=A.

(e) Set B=A and derive expressions for the functions of integration associated with parts (a) and (b).

Problem 3.2 Starting with the constitutive equation (5.9a) for the Helmholtz free energy rederive the expression (5.11) for its derivative.

Problem 3.3 Use the constitutive equations (6.22) for the stress T_{ij} and the strain-displacement relations (6.1), and show that balance of linear momentum (6.21d) can be rewritten in terms of the displacements in the form

$$\rho_0\, \ddot{u}_i = \rho_0\, b_i + \left[K + \frac{\mu}{3}\right] u_{m,mi} + \mu\, u_{i,mm} - 3K\alpha\theta_{,i} \ . \tag{P3.3}$$

Problem 3.4 Consider a rectangular cantilever beam of length L, height H, and depth W which occupies the region of space such that

$$0 \le x_1 \le L \ , \ -\frac{H}{2} \le x_2 \le \frac{H}{2} \ , \ -\frac{W}{2} \le x_3 \le \frac{W}{2} \ . \tag{P3.4a}$$

Let the beam be subjected to a body force g per unit mass in the negative $\mathbf{e}_2$ direction and consider the stress field

$$T_{11} = -\{\frac{6Q}{H^3}\}x_1^2\, x_2 + \{\frac{4Q}{H^3}\}x_2^3\ ,\ T_{12} = -\{\frac{6Q}{H^3}\}\{\frac{H^2}{4} - x_2^2\}x_1\ , \qquad \text{(P3.4b,c)}$$

$$T_{22} = \frac{Q}{4}\,[2 + \{\frac{6}{H}\}x_2 - \{\frac{8}{H^3}\}x_2^3\,] + \rho_0 g[x_2 + \frac{H}{2}\,] - \bar{Q}\ , \qquad \text{(P3.4d)}$$

$$T_{13} = T_{23} = T_{33} = 0\ \ ,\ \ Q = \bar{Q} + \hat{Q} - \rho g H\ , \qquad \text{(P3.4e,f)}$$

where ρ_0 is the mass density and $\hat{Q}$ and $\bar{Q}$ are constants.

(a) Show that this stress field satisfies the equilibrium equations.

(b) Show that the traction vector applied to the top surface (x_2=H/2) of the beam is given by

$$\mathbf{t} = \hat{\mathbf{t}} = \hat{Q}\,\mathbf{e}_2\ . \qquad \text{(P3.4g)}$$

(c) Show that the traction vector applied to the bottom surface (x_2=–H/2) of the beam is given by

$$\mathbf{t} = \bar{\mathbf{t}} = \bar{Q}\,\mathbf{e}_2\ . \qquad \text{(P3.4h)}$$

(d) The resultant force $\mathbf{F}_L$ applied to the end (x_1=L) of the beam has a normal component N_L and a shear component V_L, such that

$$\mathbf{F}_L = N_L\,\mathbf{e}_1 + V_L\,\mathbf{e}_2\ , \qquad \text{(P3.4i)}$$

Derive expressions for N_L and V_L.

(e) The resultant moment $\mathbf{M}_L$ applied to the end (x_1=L) of the beam about the centroid of the cross-section takes the form

$$\mathbf{M}_L = M_L\,\mathbf{e}_3\ . \qquad \text{(P3.4j)}$$

Derive an expression for M_L.

Problem 3.5

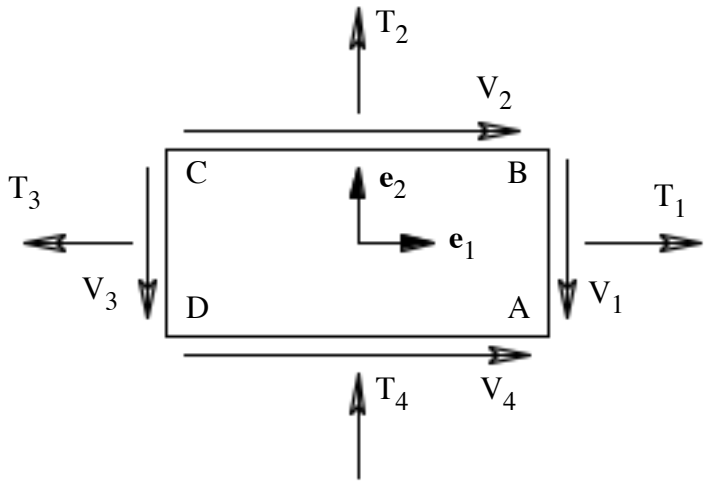


Fig. P3.5 Uniform loads on a rectangular parallelepiped.

The rectangular parallelepiped shown in Fig. P3.5 is subjected to uniform tractions in the $\mathbf{e}_1$–$\mathbf{e}_2$ plane on each of its faces AB, BC, CD, DA, which are characterized by the constants

$$\{ T_1, T_2, T_3, T_4, V_1, V_2, V_3, V_4 \} \ . \tag{P3.5}$$

(a) Determine expressions for the traction vectors **t** applied to each of these faces.

(b) Determine the values of the stress tensor $T_{\alpha\beta}$ associated with these boundary conditions on each of the faces.

(c) Determine restrictions on the constants (P3.5) which ensure that the stress field is uniform in the body.

Problem 3.6

For isothermal conditions ($\theta=\theta_0$), the constitutive equations for isotropic elastic materials can be written in terms of Lame's constants λ and μ, such that

$$T_{ij} = \lambda\, e_{mm}\, \delta_{ij} + 2\mu\, e_{ij} \ . \tag{P3.6}$$

(a) Invert the constitutive equations (10.4) [with $\theta=\theta_0$] to derive an expression for stress as a function of strain.

(b) Compare your result in (a) with the expression (P3.6) to determine an expression for Lame's constants λ and μ in terms of E and ν.

PROBLEM SET 4

Problem 4.1

(a) Use the results in Table 9.1 and show that the constitutive equations (9.9) can be written in the tensorial forms

$$e_{ij} = \frac{1}{2\mu}\left[\, T_{ij} - \left\{\frac{\nu}{1+\nu}\right\} T_{mm}\delta_{ij} \,\right] + \alpha(\theta-\theta_0)\,\delta_{ij} \ , \qquad \text{(P4.1a)}$$

(b) Invert these constitutive equations and use the results in Table 9.1 to show that

$$T_{ij} = 2\mu\left[\, e_{ij} + \left\{\frac{\nu}{1-2\nu}\right\} e_{mm}\,\delta_{ij} \,\right] - 2\mu\left[\frac{1+\nu}{1-2\nu}\right]\alpha(\theta-\theta_0)\,\delta_{ij} \ , \qquad \text{(P4.1b)}$$

$$T_{ij} = \left[\frac{E}{1+\nu}\right]\left[\, e_{ij} + \left\{\frac{\nu}{1-2\nu}\right\} e_{mm}\,\delta_{ij} \,\right] - \left[\frac{E}{1-2\nu}\right]\alpha(\theta-\theta_0)\,\delta_{ij} \ . \qquad \text{(P4.1c)}$$

Problem 4.2

Use superposition of the solutions (I) and (II) in sec. 14, and consider a cantilever beam which is fixed at x_1=0 and is loaded by a shear force V_L only at its end x_1=L. Also, neglect body forces. Determine expressions for:

(a) the constants $\{ N_{II}, V_{II}, M_{II}, c_1, c_2, \alpha\}$

(b) the stresses $T_{\alpha\beta}$ and check that they satisfy the equilibrium equations

(c) the average displacement w_2

Problem 4.3

For the Bernouilli-Euler beam theory, the equilibrium equations are given by (14.9), (14.10), (14.12) and (14.13). In these equations v is a constraint-response (an arbitrary function of x_1 which is determined by the equilibrium equations and the boundary conditions and is the response to the constraint that shear deformation vanishes) and the moment m is determined by the constitutive equation

$$m = EI\,\frac{d^2 w_2}{dx_1^2} \ , \quad I = \frac{H^3 W}{12} \ , \qquad \text{(P4.3a,b)}$$

where I is the second moment of area for a rectangular cross-section, and w_2 is the transverse displacement.

(a) Use the Bernouilli-Euler theory to calculate the displacement $w_2(x_1)$ for the cantilever beam described in Prob. 4.2.

(b) The solution in Prob. 4.2 does not restrict shear deformation and therefore is more accurate than the solution obtained in part (a). Denote the solution in Prob. 4.2 by $w_2^*(x_1)$ and show that the relative error in the displacement associated with the solution in part (a) at x_1=L and for plane stress is given by

$$E_w = \frac{w_2(L) - w_2^*(L)}{w_2^*(L)} = -\frac{\dfrac{(1+\nu)H^2}{2L^2}}{1 + \dfrac{(1+\nu)H^2}{2L^2}} \quad . \tag{P4.3c}$$

It follows that the actual beam is more flexible than the constrained Bernouilli-Euler beam. However, the error diminishes very rapidly as the beam becomes thin (H/L<<1).

Problem 4.4

Consider the problem of a cantilever rectangular beam in plane stress which is subjected to a uniform normal load Q on its top surface, no load on its bottom surface, and a shear force V_L and moment M_L at its end x_1=L. Neglect body forces. Using superposition of the solutions in sec. 14 determine expressions for:

(a) values of the constants $\{N_{II}, V_{II}, M_{II}, c_1, c_2, \alpha\}$

(b) the stresses $T_{\alpha\beta}$ and check that they satisfy the equilibrium equations

(c) the average displacement w_2

Problem 4.5

Consider a microbeam made of silicon which has length L=200μm, height H=5μm, and depth W=20μm. Also, using Table 6.1 the material properties become

$$E = 70.1 \times 10^3\ \mu N/\mu m^2 \ , \ \nu = 0.251 \ ,$$

$$\rho = 2.5 \times 10^{-3}\ ng/\mu m^3 \ , \ \sigma_T = 72.0 \times 10^3\ \mu N/\mu m^2 \quad . \tag{P4.5a}$$

where σ_T is the tensile strength.

The default dimensions and material properties in the Matlab program beam are those associated with this microbeam and the default conditions are those of plane stress. Also,

the default loading is associated with traction free top and bottom surfaces and a shear force

$$V(L) = 20.0\ \mu N\ , \qquad \text{(P4.5b)}$$

applied to the end of the beam, and the units are taken to be [μm, μN, ng, μs].

(a) Determine the values of the components of the traction vectors applied to the top and bottom surfaces of the beam which will cause the beam to be in a state of simple homogeneous shear.

(b) Apply these loads to the beam and check your results by plotting various stress components at various locations.

(c) Explain why the centerline of the beam is not horizontal.

Problem 4.6

Consider the same microbeam as in Prob. 4.5 with the same shear force V_L (P4.5b).

(a) Determine the value of the uniform normal surface traction $t_2(X_1,H/2)$ which must be applied to the top surface of the beam in order for the moment diagram to be that associated with pined-pined boundary conditions.

(b) Apply this surface traction and check your results by plotting the moment and shear diagrams of the beam.

(c) Plot the value of the normalized maximum tensile stress Smax/SigT ($=\sigma_{max}/\sigma_T$) versus X_1 and X_2 to determine approximately where the most critical point (X_1,X_2) is for tensile failure for this loading and its value there. Notice that this load is very far from the failure load.

(d) Use the analytical solution of Prob. 4.4 and determine the value of the moment M(L) which must be applied to the beam in order for the end X_1=L to be clamped. Ignore displacement of this end in the $\mathbf{e}_1$ direction.

(e) Apply this moment to the beam and test your result by plotting the moment diagram. Also, increase the scale factor Scale_u to 50 to exaggerate the displacements of the beam.

PROBLEM SET 5

Problem 5.1

Using the solution of section 17 for Lame's problem, consider a semi-infinite region with a cylindrical tunnel of radius a. The surface of this tunnel is taken to be stress-free and the stress field at infinity is taken to be a hydrostatic pressure of magnitude P.

(a) Derive an expression for the magnitude of P (in terms of the yield strength Y) which first causes yielding at the surface of the tunnel. Assume plane strain conditions and reference temperature $\theta^*=\theta_0$.

(b) Determine the amount of collapse of radius of this tunnel for this value of pressure P.

Problem 5.2

For two dimensional problems in cylindrical polar coordinates the stress tensor **T** is expressed in the form

$$\mathbf{T} = T_{rr}\,(\mathbf{e}_r\otimes\mathbf{e}_r) + T_{r\theta}\,(\mathbf{e}_r\otimes\mathbf{e}_\theta+\mathbf{e}_\theta\otimes\mathbf{e}_r) + T_{\theta\theta}\,(\mathbf{e}_\theta\otimes\mathbf{e}_\theta) + T_{33}\,(\mathbf{e}_3\otimes\mathbf{e}_3) \quad . \qquad \text{(P5.2a)}$$

Given the components of stress $\{T_{rr}, T_{r\theta}, T_{\theta\theta}, T_{33}\}$ and using the fact that

$$T_{ij} = \mathbf{T} \bullet (\mathbf{e}_i\otimes\mathbf{e}_j) \ , \qquad \text{(P5.2b)}$$

show that

$$T_{11} = \frac{T_{rr}+T_{\theta\theta}}{2} + \frac{T_{rr}-T_{\theta\theta}}{2}\cos(2\theta) - T_{r\theta}\sin(2\theta) \ ,$$

$$T_{12} = \frac{T_{rr}-T_{\theta\theta}}{2}\sin(2\theta) + T_{r\theta}\cos(2\theta) \ ,$$

$$T_{22} = \frac{T_{rr}+T_{\theta\theta}}{2} - \frac{T_{rr}-T_{\theta\theta}}{2}\cos(2\theta) + T_{r\theta}\sin(2\theta) \ . \qquad \text{(P5.2c)}$$

Problem 5.3

The solution in section 18 is valid for stresses applied at infinity. In order to estimate the stress concentration associated with a circular hole of radius a in a plate of finite thickness 2H (with $H\geq a\sqrt{2}$), consider the case when the stresses T_{11}^∞ and T_{22}^∞ are applied (with $T_{11}^\infty>0$ and $T_{12}^\infty=0$).

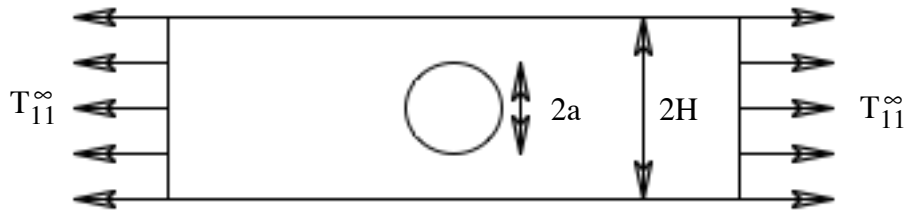


Fig. P.5.3 Uniaxial stress applied to a plate of finite thickness.

(a) Determine the value of T_{22}^{∞} which will cause the traction vector to vanish on the surfaces $x_2=\pm H$ at the point $x_1=0$.

(b) Determine an expression for the maximum stress concentration factor (relative to the magnitude of T_{11}^{∞}) at the surface of the hole associated with this loading.

(c) Determine the numerical value of this maximum stress concentration factor for $H=a\sqrt{2}$.

It is important to emphasize that this solution is only approximate because the traction vectors on the surfaces $x_2=\pm H$ do not vanish for all values of x_1.

Problem 5.4

Use the Matlab program "kirsh" for the following problems:

(a) Determine the form for the stress tensor at infinity (T_{11}^{∞}, T_{22}^{∞}, T_{12}^{∞}) which cause circular line elements to remain circular. Note that you can use Scale_u to amplify the effect of the displacements.

(b) Determine two sets of values of the far field stresses ($T_{11}^{\infty}\neq 0$, $T_{22}^{\infty}\neq 0$, $T_{12}^{\infty}=0$) which cause the same stress intensity factor as far field pure shear ($T_{11}^{\infty}=0$, $T_{22}^{\infty}=0$, $T_{12}^{\infty}=S>0$).

(c) Check the results of your calculations in Problem 5.1.

(d) Check the results of your calculations in Problem 5.3c.

PROBLEM SET 6

Problem 6.1

(a) Use the constitutive equations (6.22) for the heat flux vector and the internal energy, and show that the balance of energy (6.21f) can be rewritten in the form

$$\rho_0 C_v \dot{\theta} + 3K\alpha\theta_0 \dot{\mathbf{e}} \bullet \mathbf{I} = \rho_0 r + \kappa \nabla^2\theta \quad . \qquad \text{(P6.1a)}$$

In order to estimate the magnitude of each of the terms in this equation it is convenient to consider the following special cases.

(b) For Hopkinson bar experiments the strain rate $\dot{e}_{11}$ is about 1.0×10^3 s^{-1}. Assuming uniaxial strain (all other e_{ij}=0), calculate the magnitude of the temperature rate $\dot{\theta}$ associated with a pointwise adiabatic process (r=0, **q**=0) with constant internal energy for aluminum (see Table 6.1 for the material constants).

(c) If the time Δt of application of the strain rate in (b) is limited to 1.0×10^{-6} s then the strain will remain less that 0.1% which would ensure that the material remains elastic. Show that the temperature change $\Delta\theta$ associated with the solution in (b) is quite small.

(d) Calculate the value of $\nabla^2\theta$ assuming that the term associated with heat conduction in (P3.3a) has the same magnitude as that associated with the strain rate.

(e) An estimate of the characteristic length λ for wave propagation is the wave speed C times the duration Δt. Consequently, the temperature change $\Delta\theta$ associated with the result in (d) is about $\Delta\theta = \lambda^2 \nabla^2\theta$. Using the formula

$$C = \sqrt{\frac{K + \frac{4\mu}{3}}{\rho_0}} \qquad \text{(P6.1b)}$$

show that the value of $\Delta\theta$ associated with heat conduction is unrealistically large. This means that during wave propagation there is not enough time for heat conduction to be important so that the term $\kappa \nabla^2\theta$ is negligible in (P6.1a)

Problem 6.2

An experimentalist is using uniaxial stress waves to measure Young's modulus in steel. He measures the bar wave speed C_B to within 0.1% error. He asks you if it is accurate enough to use the standard formula

$$E = \rho_0 C_B^2 , \tag{P6.2}$$

for isothermal response to determine E. You know that it is more accurate to assume that wave propagation is an adiabatic process instead of an isothermal process. To answer his question, use the material properties given in Table 6.1 to calculate the percentage error in the isothermal value of E relative to the adiabatic value $\bar{E}$ given by (21.5).

Problem 6.3

Consider a rectangular parallelepiped which occupies the region of space defined by

$$-\frac{L}{2} \le x_2 \le \frac{L}{2} \ , \ -\frac{L}{2} \le x_2 \le \frac{L}{2} \ , \ -\frac{W}{2} \le x_3 \le \frac{W}{2} \ . \tag{P6.3a}$$

In the absence of body force (b_i=0), free shearing vibration of the parallelepiped is characterized by the displacement field

$$u_1 = A \sin(\omega t) \sin(kx_1) \cos(kx_2) ,$$

$$u_2 = - A \sin(\omega t) \cos(kx_1) \sin(kx_2) ,$$

$$u_3 = 0 \ , \tag{P6.3b}$$

where A, ω and k are constants.

(a) Calculate the strain field e_{ij} associated with this displacement field.

(b) Calculate the stress field T_{ij} associated with this displacement field.

(c) Assuming an adiabatic process, determine the temperature field.

(d) Show that the stress field will satisfy the stress-free boundary conditions provided that the wave number k is determined by the equation

$$k = k_n = \frac{(2n-1)\pi}{L} \quad \text{for} \quad n=1,2,3,... \tag{P6.3c}$$

(e) Show that the balance of linear momentum

$$\rho_0 \ddot{u}_i = T_{ij,j} \ , \tag{P6.3d}$$

is satisfied provided that the frequency ω of vibration is given by

$$\omega = \omega_n = \sqrt{2}\, C\, k_n \,, \quad C^2 = C_S^2 = \frac{\mu}{\rho_0} \,, \tag{P6.3e}$$

where C is the secondary (shear) wave speed C_S.

Problem 6.4

A cylindrical disk of radius R = 1 cm and thickness H = 1 cm is loaded in a dynamic plate impact experiment. The particle velocity at the center (r=0) of one face of the disk is measured using a velocity interferometer system. The data is analyzed assuming uniaxial strain conditions exist along the symmetry axis of the cylindrical sample. This assumption is valid only until stress relief waves propagate from the lateral surface of the disk to its center. The time associated with this wave propagation process determines the time window Δt for analyzing the data. Assuming that the disk is made of aluminum (see Table 6.1), determine the value of Δt.

PROBLEM SET 7

Problem 7.1

(a) Using the solution in section 22 for the bending moments M_1 and M_3 and taking $\theta=\theta_0$, determine the value of the bending moment M_1 which must be applied to the edges $x_3=\pm W/2$ in order for deformation of the plate due to the bending moment M_3 to be independent of the x_3 coordinate. This means that in order to bend the plate into a right cylindrical surface (like a beam) it is necessary to apply both the moments M_3 and M_1. This result is contrary to that associated with beam theory and it is due to the Poisson effect.

(b) Show that for this case, the strain e_{11} is related to the stress T_{11} by the formula

$$e_{11} = \frac{(1-\nu^2)T_{11}}{E} \quad , \tag{P7.1a}$$

which is different from the expression for uniaxial stress

$$e_{11} = \frac{T_{11}}{E} \quad , \tag{P7.1b}$$

used in standard beam theory.

Problem 7.2

(a) Calculate the average displacements w_i and rotations δ_i

$$w_i(x_1,x_3) = \frac{1}{H}\int_{-H/2}^{H/2} u_i(x_1,x_2,x_3)\, dx_2 \quad ,$$

$$\delta_i(x_1,x_3) = \frac{1}{H}\left[u_i(x_1,H/2,x_3) - u_i(x_1,-H/2,x_3)\right] \tag{P7.2a}$$

associated with the pure bending solution (22.25).

(b) Within the context of plate theory, an approximate three-dimensional displacement field $\bar{u}_i$ can be defined using w_i and δ_i, such that

$$\bar{u}_i(x_1,x_2,x_3) = w_i(x_1,x_3) + x_2\, \delta_i(x_1,x_3) \quad . \tag{P7.2b}$$

Derive expressions for the associated approximate strain field $\bar{e}_{ij}$

$$\bar{e}_{ij} = \frac{1}{2}(\bar{u}_{i,j} + \bar{u}_{j,i}) \quad . \tag{P7.2c}$$

In particular, note that the shear strains $\bar{e}_{12}, \bar{e}_{13}, \bar{e}_{23}$ vanish.

(c) Substitute these strains into the constitutive equation (P4.1c) (with $\theta=\theta_0$) to derive explicit expressions for the approximate stresses $\bar{T}_{ij}$

$$\bar{T}_{ij} = \left[\frac{E}{1+\nu}\right] \left[\bar{e}_{ij} + \left\{\frac{\nu}{1-2\nu}\right\} \bar{e}_{mm}\, \delta_{ij} \right] , \tag{P7.2d}$$

in terms of the moment M_3. Specifically, show that

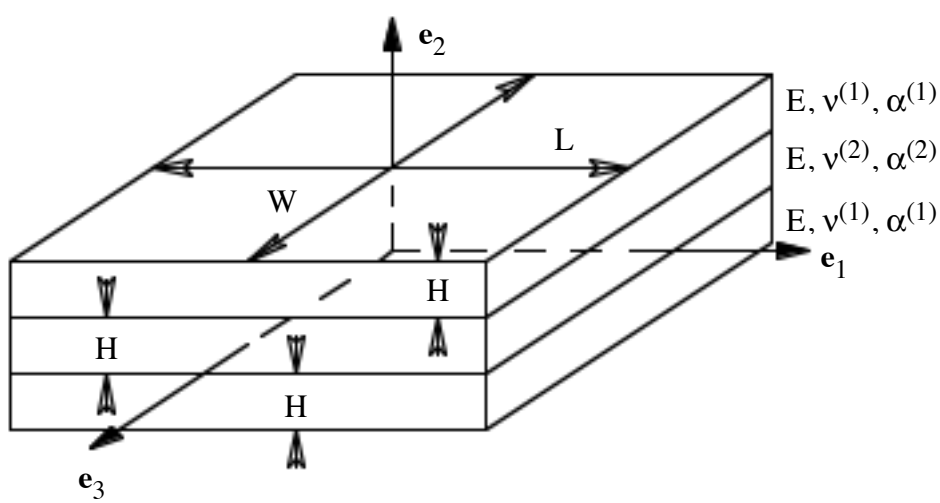
$$\bar{T}_{11} = -\left[\frac{(1-\nu-\nu^2)}{(1+\nu)(1-2\nu)}\right] \left[\frac{12M_3}{WH^3}\right] x_2 \quad , \tag{P7.2e}$$

which is different from the exact result (22.5)

$$T_{11} = -\left[\frac{12M_3}{WH^3}\right] x_2 \quad . \tag{P7.2f}$$

This means that the kinematic approximation (P7.2b) is not consistent with the exact solution. In particular, it is observed that e_{22} is nonzero in the exact solution but not in the approximate solution.

Problem 7.3



Consider a composite plate made of three layers with a symmetrical arrangement. The top and bottom layers are made of the same material with the material constants

$$\{ E , \nu^{(1)} , \alpha^{(1)} \} \quad , \tag{P7.3a}$$

and the middle layer is made of a different material with material constants

$$\{ E , \nu^{(2)} , \alpha^{(2)} \} \quad , \tag{P7.3b}$$

Each layer has length L in the $\mathbf{e}_1$ direction, thickness H in the $\mathbf{e}_2$ direction, and depth W in the $\mathbf{e}_3$ direction. Also, the origin of the axes is taken in the center of the middle plate. Assume that the plates are bonded perfectly, that all three plates are heated to the same uniform temperature $\theta > \theta_0$, and that the top and bottom surfaces ($x_2=\pm 3H/2$) are stress free and the edges ($x_1=\pm L/2$, and $x_3=\pm W/2$) are free from resultant forces and moments.

(a) Write expressions for the pointwise boundary conditions on the top and bottom surfaces of the composite plate.

(b) Write expressions for the pointwise boundary conditions at the interfaces of the middle plate.

For the following two questions assume that the deformation in each plate is homogeneous and that the top and bottom plates have the same state of stress.

(c) Write expressions for the integral boundary conditions on the edges of the composite plate.

(d) Consider a simple solution where all shear stresses and strains vanish, and determine the stress $\mathbf{T}^{(1)}$ and strain $\mathbf{e}^{(1)}$ fields in the top and bottom plates and the stress $\mathbf{T}^{(2)}$ and strain $\mathbf{e}^{(2)}$ fields in the middle plate.

PROBLEM SET 8

Problem 8.1

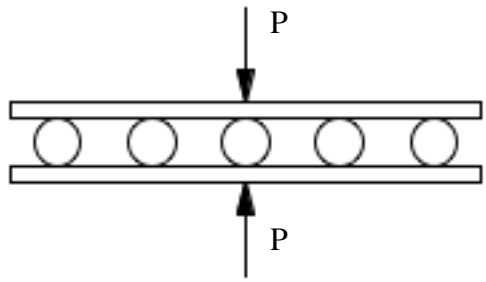


Fig. 8.1 Sketch of two plates separated by cylindrical roller bearings.

Figure 8.1 shows a sketch of two rigid plates that are separated by five elastic cylindrical roller bearings. Each of the cylinders has undeformed radius R. Assume plane strain conditions and reference temperature $\theta^*=\theta_0$.

(a) Assuming that a line force P (force per unit depth of the plate) is applied, determine the total contact region (area per unit depth) of all five cylinders associated with the top plate.

(b) Also, determine the gap between the plates in their loaded state.

Problem 8.2

One technique used to stimulate the production of oil from an oil well is called hydrofracture. This technique pumps fluid into the borehole at high pressure to cause a fracture to propagate from the borehole into the oil saturated rock. The simplest model for this process considers the in-situ stresses at a particular depth of the borehole to be equivalent to a hydrostatic pressure p_h. Assume plane strain conditions and reference temperature $\theta^*=\theta_0$.

(a) Using superposition of a uniform stress field ($T_{11}\neq 0$, $T_{22}\neq 0$, $T_{12}=0$) and the solution in section 27, determine the values of these uniform stresses which cause the combined solution to correspond to a hydrofracture with internal fluid pressure p_f at the depth with far field in-situ hydrostatic stress p_h.

(b) Determine the value of the stress intensity factor for a fracture of half length a in this situation.

(c) Assuming that the rock has a fracture toughness K_{Ic}, determine the minimum value of the borehole pressure p_f which will cause the fracture to begin to propagate.

(d) Sometimes during the hydrofracture process, small particles (called proppants) are mixed with the fluid to attempt to keep the fracture propped open when the pressure is released. In order to estimate the maximum allowable size of particles, determine the maximum value of the fracture opening during this hydrofracture process.