Fourier methods

• Continous signals

FS – Fourier Series : Periodic

FT – (Integral) Fourier Transform: Transients (aperiodic)

•Discrete (sampled) signals

DFS –Discrete Fourier Series

DFT – Discrete Fourier Transform

Theory

Every deterministic periodic signal having period T_0 can be decomposed using sinusoids and cosinusoids with determined frequencies and amplitude.

This decomposition is unique but needs an infinite number of terms for an exact reconstruction of the signal.

The decomposition allows the spectral representation of the signal in amplitude with respect to the frequency.

Different formulations of this decomposition are given and the differnet possible representation (amplitude, power, phase).

The Parseval theorem indicating the equality between powers in the time and frequency domains is also presented.

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Decomposition of a periodic function 1/3

Basic formulation :

Is a periodic signal of period T_0 , $x(t) = x(t + T_0)$, can be decomposed using the *Fourier series* :

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos\left(\frac{2\pi nt}{T_0}\right) + \sum_{n=1}^{+\infty} b_n \sin\left(\frac{2\pi nt}{T_0}\right)$$

a_n and b_n are unique coefficients of the series and given by :

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\cos\left(\frac{2\pi nt}{T_0}\right)dt$$

$$b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\sin\left(\frac{2\pi nt}{T_0}\right)dt$$

One sees that this is formed by components whose frequencies are the integer multiple of the fundamental frequency $1/T_0$ Hz. These frequencies are called harmonics and are $k\,.\,(1/T_0)$ Hz ; k = 1, 2, 3 .....
$a_0/2$ represents the average value of the signal.

Remark : The Fourier series of an even function contains only cosinusoids, while the Fourier series of an odd function contains only sinusoids.

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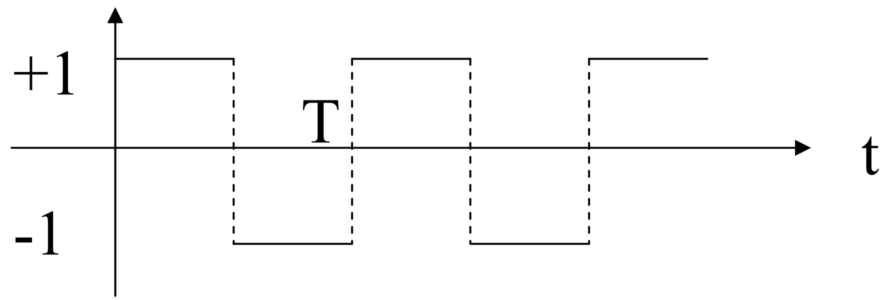
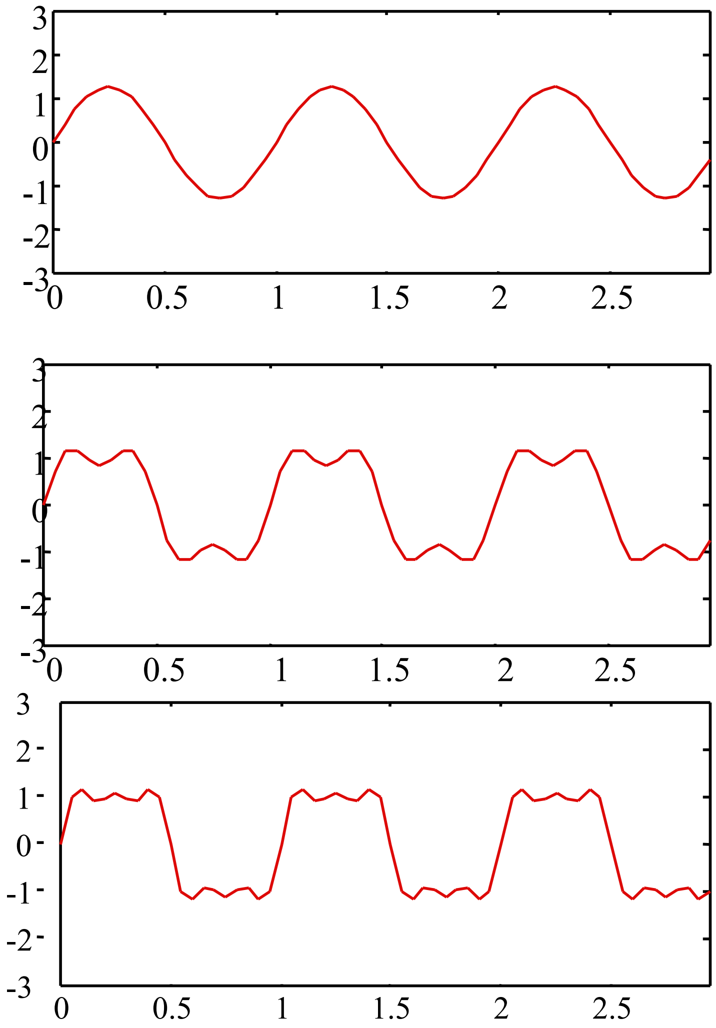


$$a_n = \frac{2}{T}\int_o^r x(t)\cos(2\pi\frac{nt}{T}) = 0$$

$$b_n = \frac{1}{T}\int_o^T x(t)\sin(2\pi\frac{nt}{T})dt = \begin{cases} 0 & n = 2,4,6... \\ \dfrac{2}{n\pi} & n = 1,3,5... \end{cases}$$

$$x(t) = \frac{4}{\pi}\left[Sin(\omega_o t) + \frac{1}{3}Sin(3\omega_o t) + \frac{1}{5}Sin(5\omega_o t)..... \right], \omega_o = \frac{2\pi}{T}$$

The convergence can be seen in the following figure.



Complex formulation :

One can also represent the Fourier series in the complex domain by using the relationship :
$e^{j\vartheta} = \cos\vartheta + j\sin\vartheta$

On obtient la représentation complexe de la série de Fourier :

$$x(t) = \sum_{n=-\infty}^{+\infty} C_n e^{j\frac{2\pi nt}{T_0}}$$

with $$C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-j\frac{2\pi nt}{T_0}}dt$$

C_n characterises both the amplitude and the phase of the component at the frequency n / T_0.

Remark : The passage in the complex area introduces the mathematical concept of negative frequencies, it is to tell that n is now taking values from $-\infty$ to $+\infty$.

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Decomposition of a periodic function 3/3

Other formulation :

By defining the following variables :

$$M_n = \sqrt{a_n^{\,2} + b_n^{\,2}}$$

$$tg\phi_n = \frac{b_n}{a_n}$$

One can rewrite the Fourier series decomposition under the form :

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} M_n \cos\left(\frac{2\pi nt}{T_0} + \phi_n\right)$$

M_n and ϕ_n are respectively the amplitude and the phase of the series terms and are related to the coefficients of the complex representations by :

$$|C_n| = \frac{M_n}{2}$$

and $arg(C_n) = -\phi_n$

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Unicity of the decomposition

Sinusoid and cosinusoid used for the decomposition in Fourier series are *orthogonal functions*, this means that they verify the property :

$$\int_{-T_0/2}^{T_0/2} \cos\left(\frac{2\pi nt}{T_0}\right)\sin\left(\frac{2\pi mt}{T_0}\right)dt = 0 \qquad \text{for all } m, n$$

This property allows to simplify considerably the derivation of the coefficients a_n and b_n.

We said that for all the formulations of the Fourier series the infinite sum of terms on the right side of the equation converges to $x(t)$ for every t. This is true excepted for points of discontinuity of $x(t)$. In these points, the infinite sum converges to the average of the right and the left limits of the function value in the point of discontinuity.

A Fourier series is theoretically infinite. In practise, because such representation is impossible, one uses a finite number of terms. One obtains thus a partial sum that represents the signal as better as the number of terms is large.

For a signal having a discontinuity, the utilisation of a finite sum leads to the Gibb phenomenon (existence of oscillations).

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Spectral representation

Several types of spectra represent a Fourier series, according to the value on the ordinate axes :

An **amplitude spectrum** is a layout of the amplitude M_n or $|C_n|$ with respect to the frequency.
A **phase spectrum** is a layout of the phase ϕ_n with respect to the frequency.
A **power spectrum** is a layout of M_n^2 and $|C_n|^2$ with respect to the frequency.

Remarks :

When the complex form of the Fourier series is used, the spectra traced for frequencies going from $-\infty$ to $+\infty$ are symmetric ($C_n = C_{-n}$ for all n going from 0 to $+\infty$).
Consequently, it is frequent that one prefers to trace spectra with frequencies going from 0 to $+\infty$ and to use the value $2C_n$ in ordinates.
The Amplitude spectrum $|C_n|$ and the power spectrum $|C_n|^2$ are even functions of the frequency.

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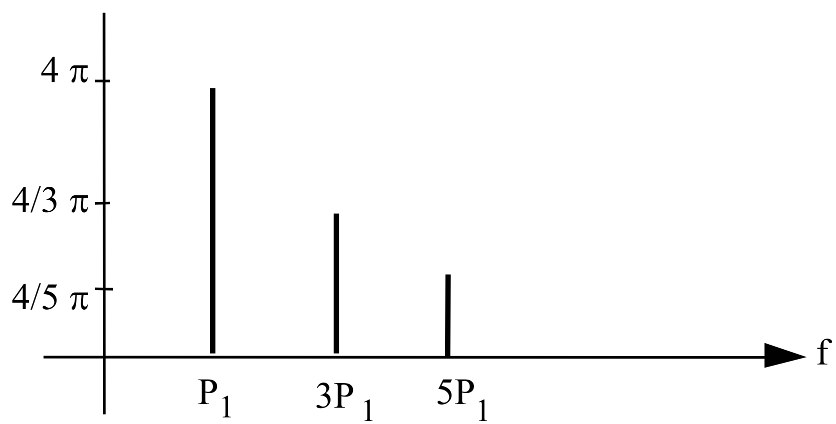
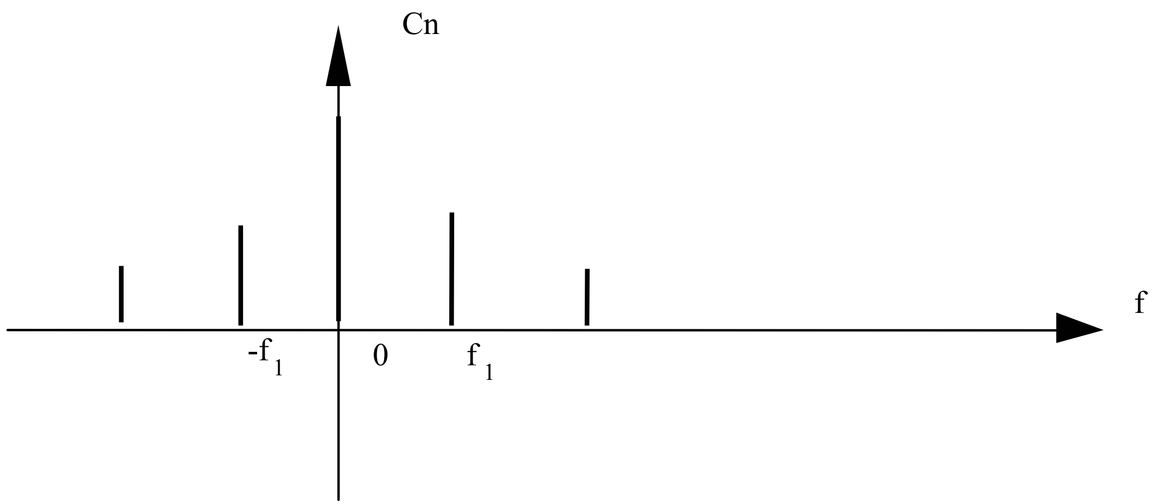
We may represent the decomposition by the spectrum. The 2-D representation is of magnitude at frequency location. The spectrum of the periodic signal is discrete: only elements with discrete frequencies nwo (n = 1,3,.....) exist.



The spectrum corresponding to the complex Fourier Series in 2 sided shows positive and negative frequencies.



The Parseval law

The periodic signal $x(t)$ has an average power given by :

$$P_{moy} = \frac{1}{T_0}\int_0^{T_0} x(t)^2\,dt$$

This can be expressed in function of the Fourier coefficients as :

$$P_{moy} = \left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty}\frac{M_n^{\,2}}{2} = \sum_{n=-\infty}^{\infty}|C_n|^2$$

This result is called Parseval theorem and it exhibits a fundamental result. The power in the temporal domain is equal to that in the spectral (frequency) domain.

Note that if partial sums are used then $x_P(t) = \sum_{n=-N}^{+N} C_n e^{j\frac{2\pi nt}{T_0}}$

and the average power of the partial sum becomes :

$$\frac{1}{T_0}\int_0^{T_0} x_P^{\,2}(t)\,dt = \sum_{n=-N}^{N}|C_n|^2$$

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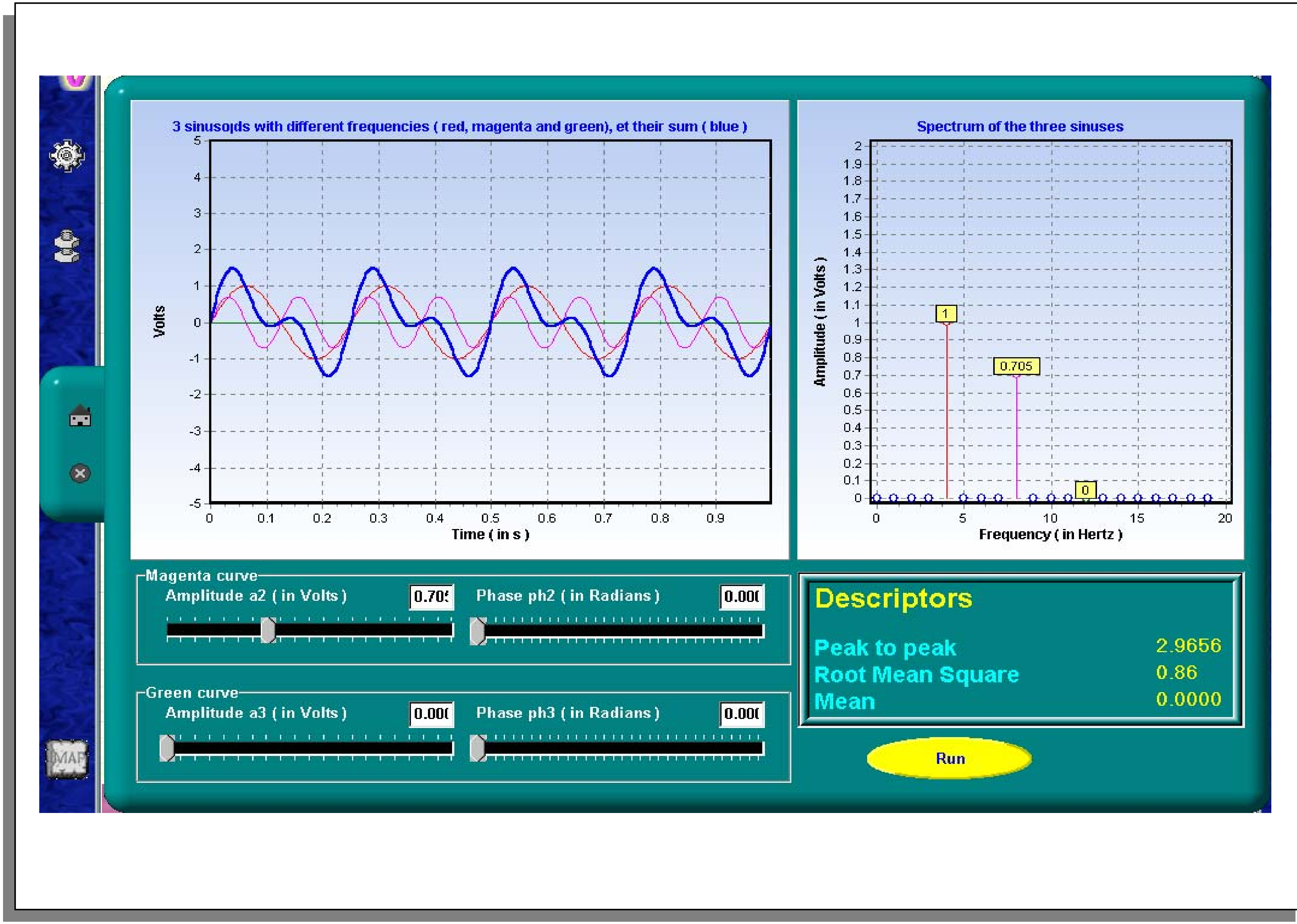
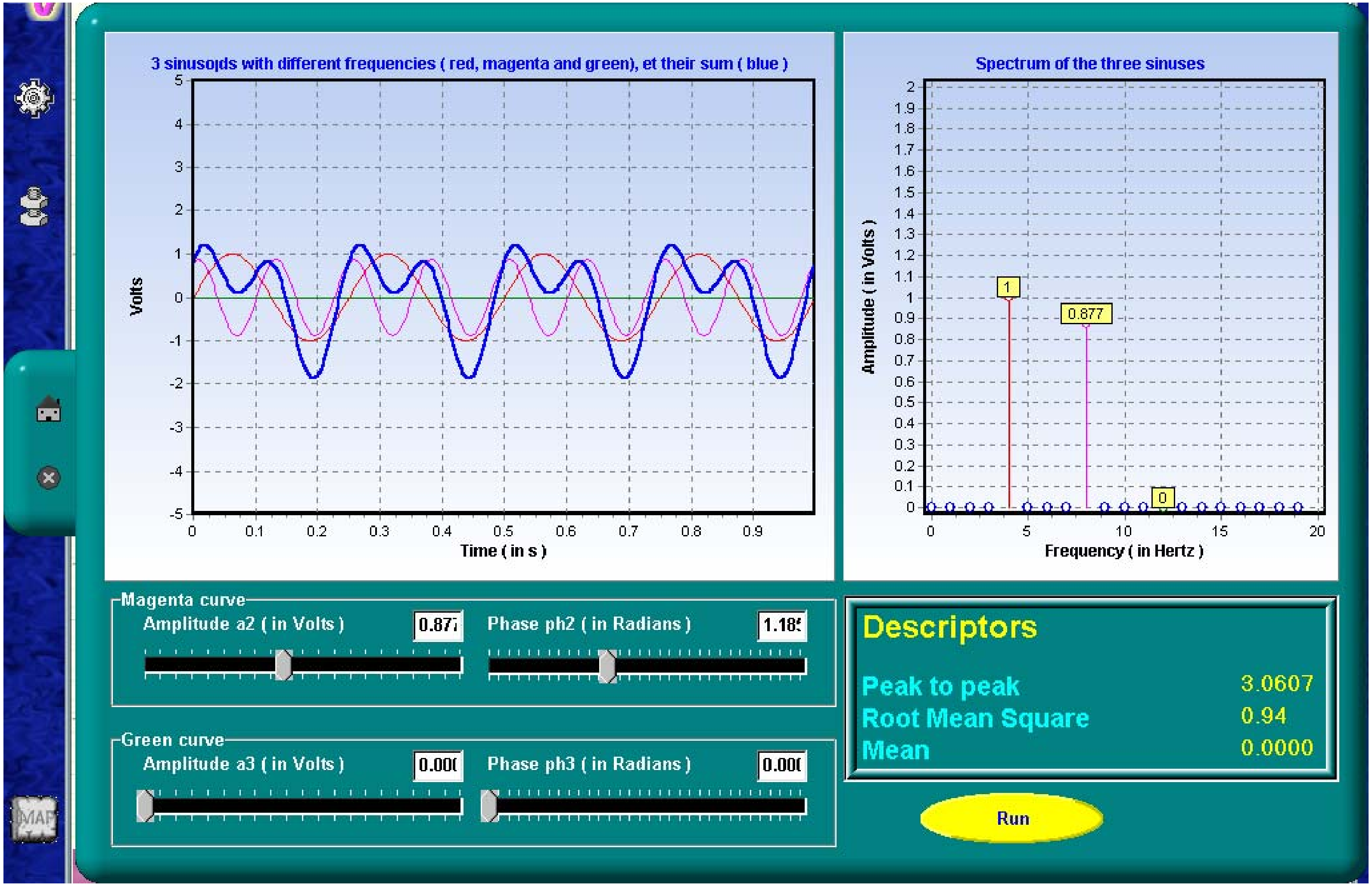


Signal Processing:Fourier



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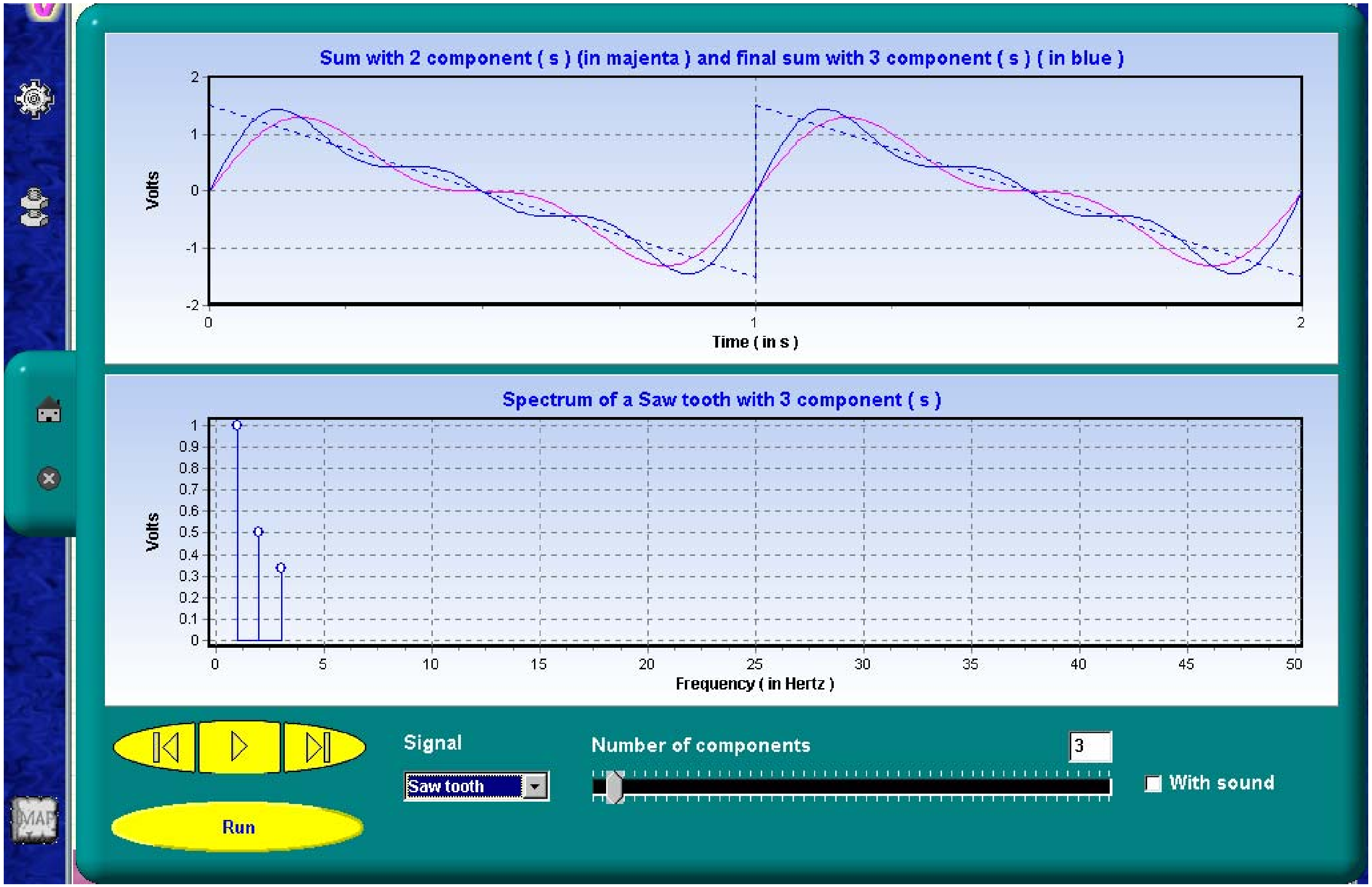

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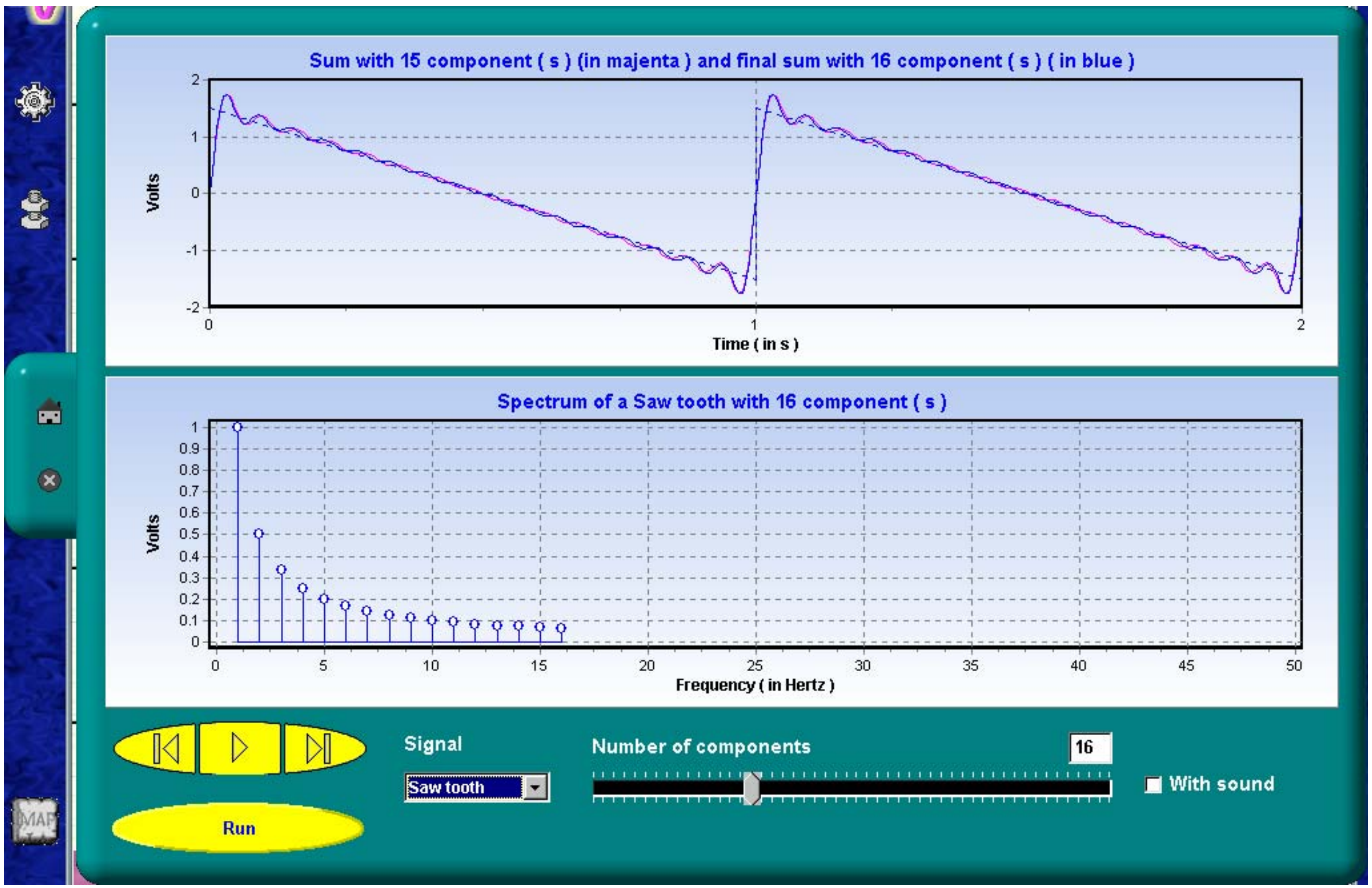

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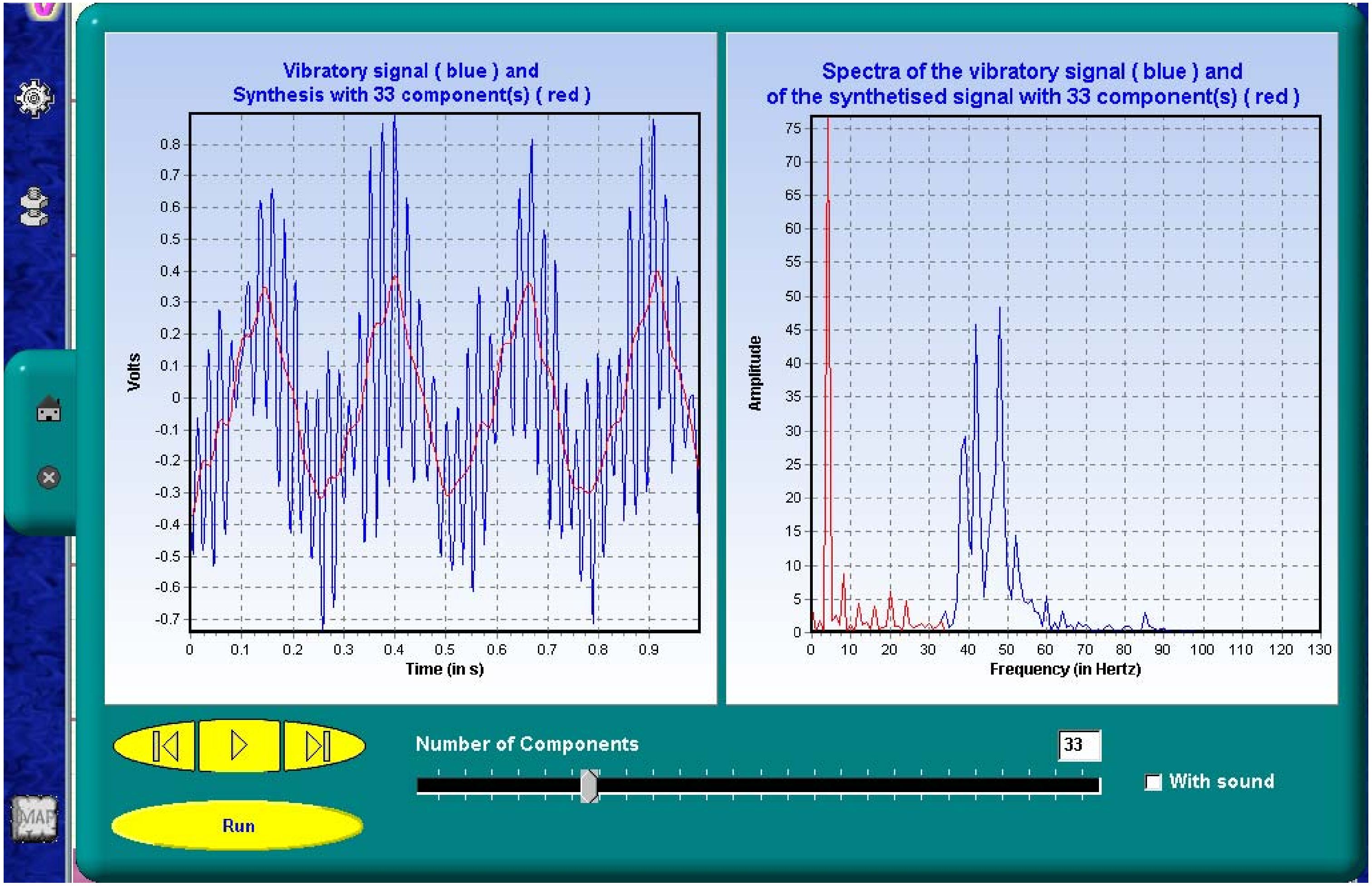

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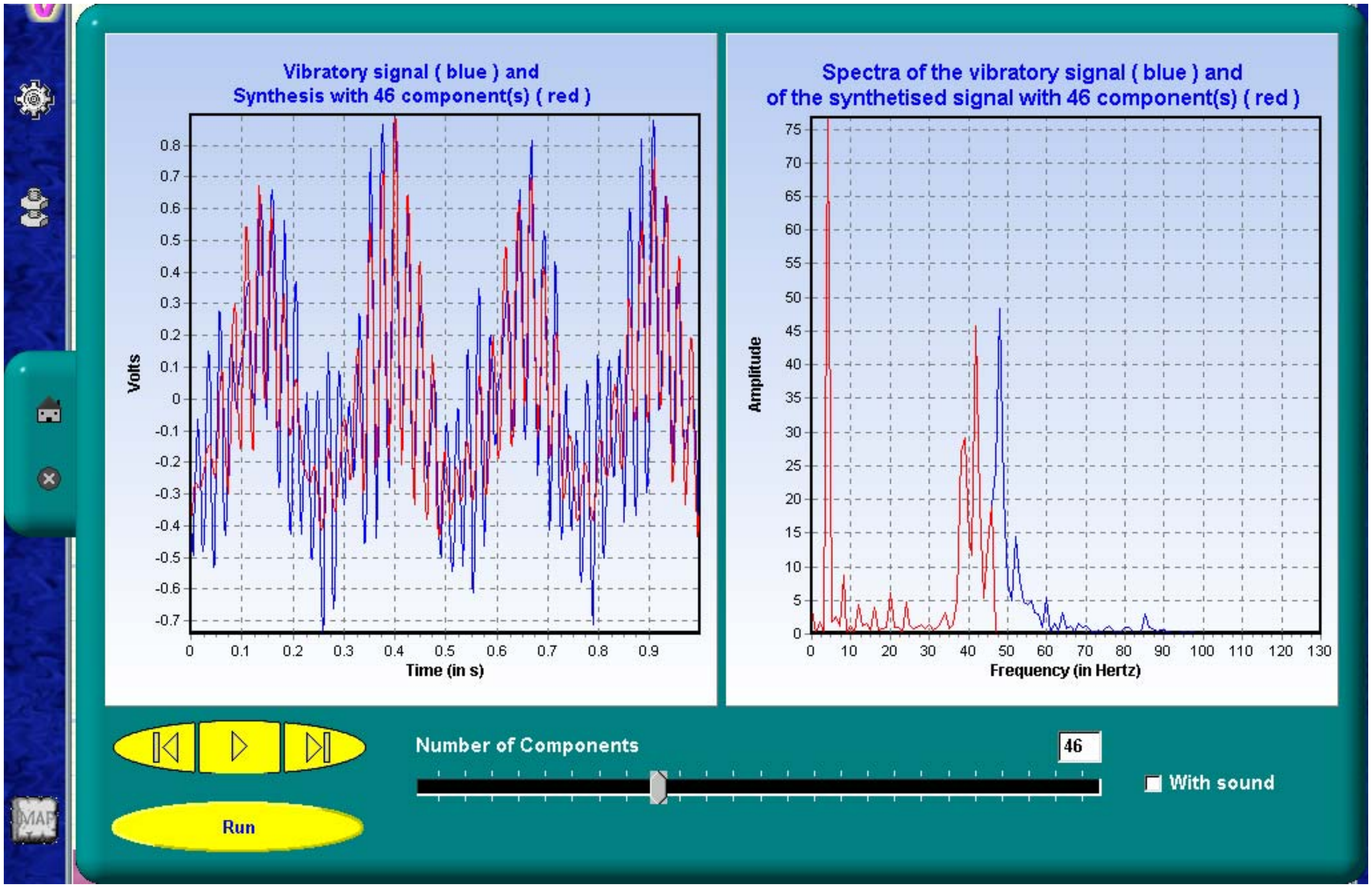

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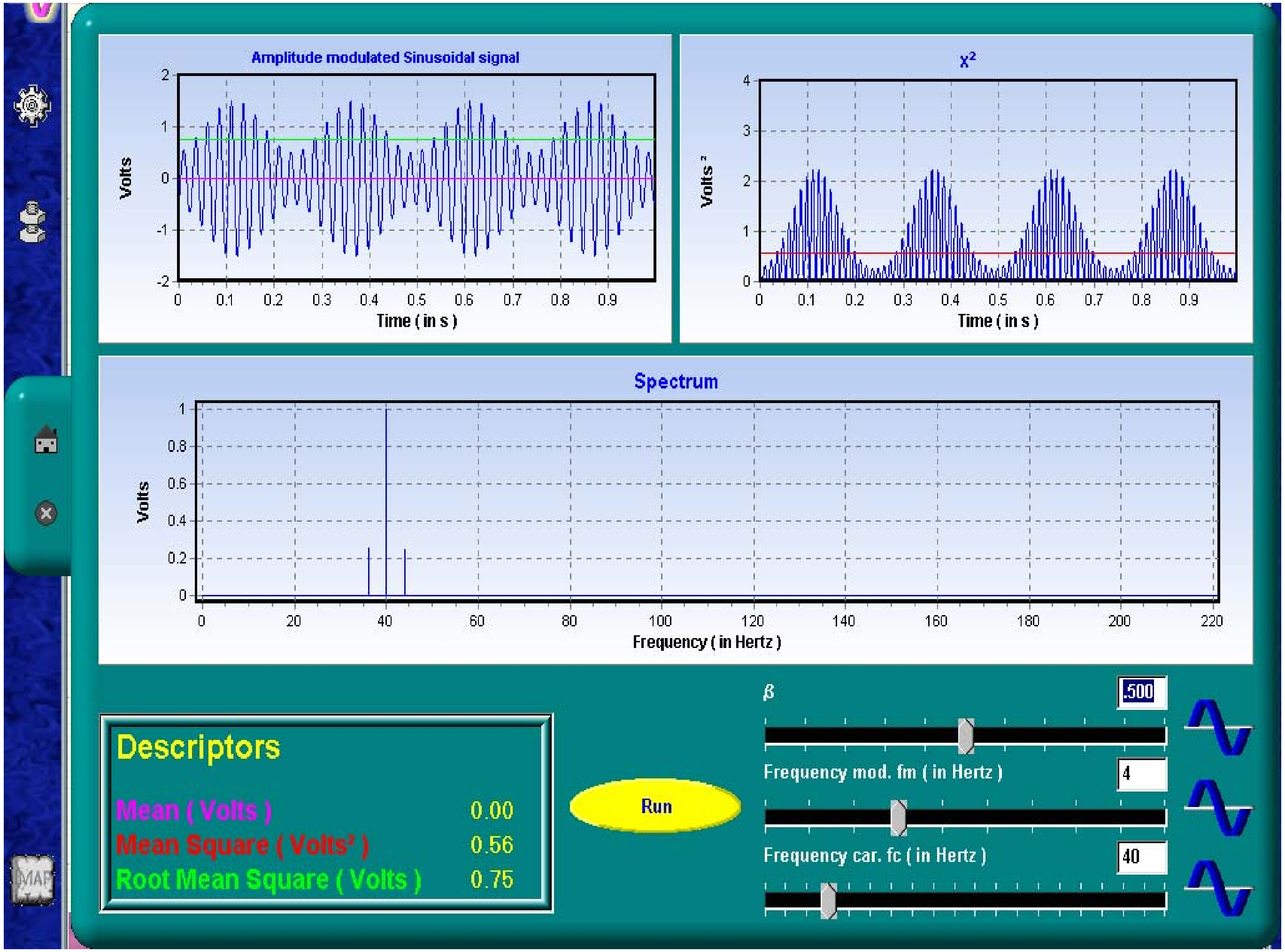

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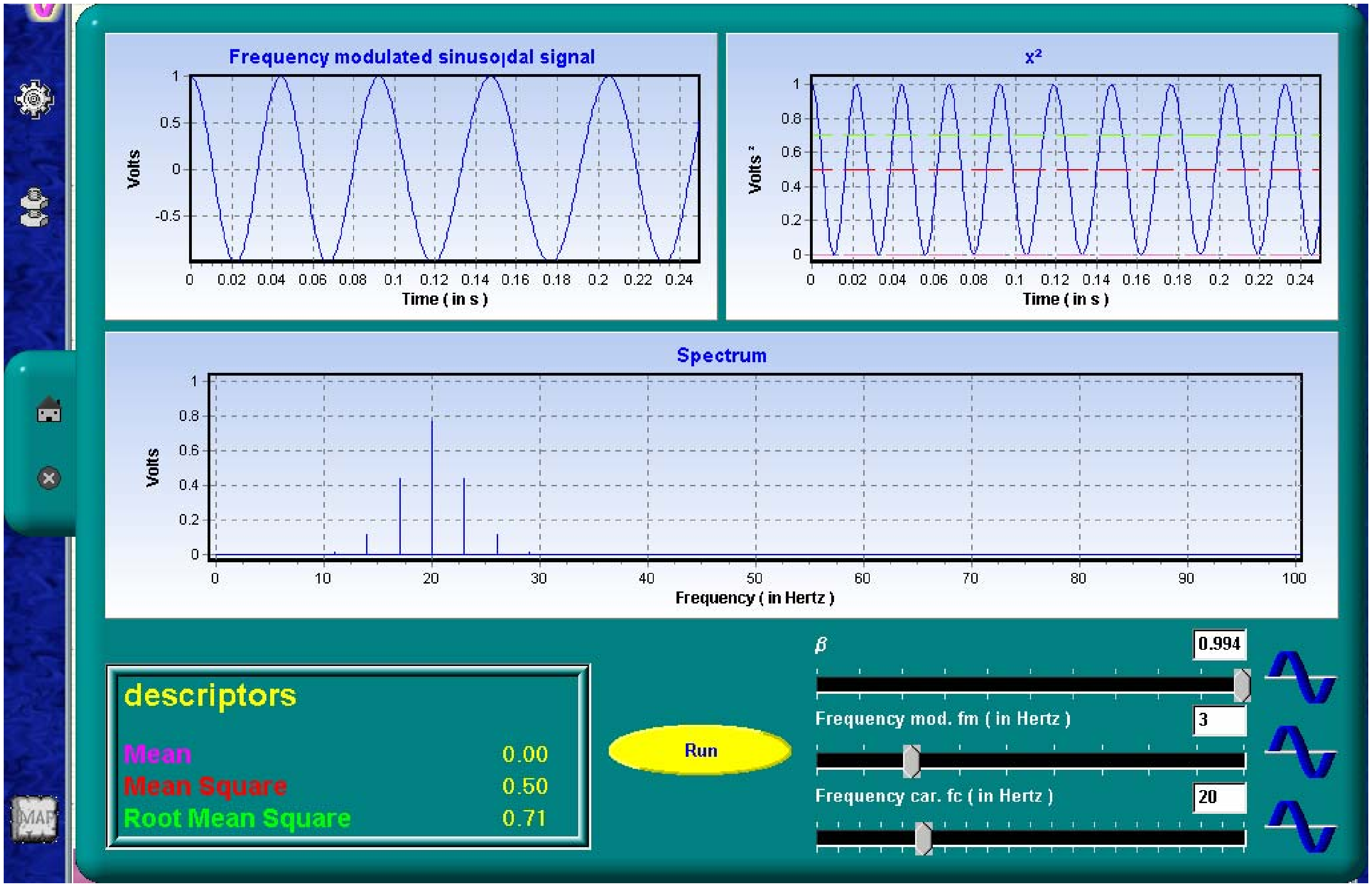
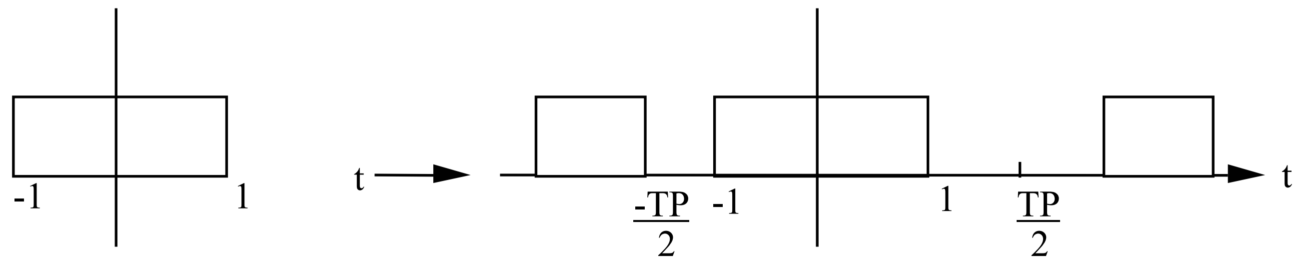


Signal Processing:Fourier

The Fourier (Integral) Transform

- **Tool for aperiodic (transient) signals.**
- **We start from a periodic signal, where we extend periodically the transient signal to be analysed. We next let the period increase to infinity, and in the limit. only the transient remains. This is shown in figure , for the specific case of a square pulse in the interval**



As Tp grows, we are left in the limit, with a single pulse at the origin.

We analyse the periodic function via a Fourier Series,
with a fundamental frequency of $f_1 = 1/T_p$.
As T_p grows larger, frequencies $n f_1$ are more densily packed with a separation of $\Delta f = f_1 = 1/T_p$

In the limit

$$T_p \to \infty \qquad \Delta f \to 0$$

Denote $n\,\Delta f = f_n$

The coefficients of the complex Fourier series

$$C_n = \lim \Delta f \int_{-T_p/2}^{T_p/2} x(t) \exp(-j2\pi\, f_n\, t)dt$$

$$T_p \to \infty$$

$$\Delta f \to 0$$

as $\Delta f \to 0, n \times f_1$ **becomes continuous, and we will write** ***f*** **instead of** ***f_n*****. We may define an amplitude density**

$$X(f) = \lim_{\Delta f \to 0}\left[\frac{C_n}{\Delta f}\right] = \lim_{T_p \to \infty} \int_{-T_p/2}^{T_p/2} x(t) \exp(-j2\pi\, ft)dt$$

It can be shown that x(t) can be derived inversely via X(f). Thus we have the Fourier Integral pair

$$x(t) = \int_{-\infty}^{\infty} X(f)\exp(j2\pi\, ft)\, df$$

$$X(f) = \int_{-\infty}^{\infty} x(t)\exp(j2\pi\, ft)\, dt$$

Often denoted symbolically by $x(t) \Rightarrow X(f)$

The scaling theorem

If x(t) $\Leftrightarrow$ X(*f*)

then

$$x(at) \leftrightarrow \frac{1}{a} X\left(\frac{f}{a}\right)$$

For a > 1 the time scale is compressed

The theorem states that, except for a normalizing factor

$$\frac{1}{a}$$

the transform X is expanded in the frequency domain.

Fourier Transforms of periodic signals.

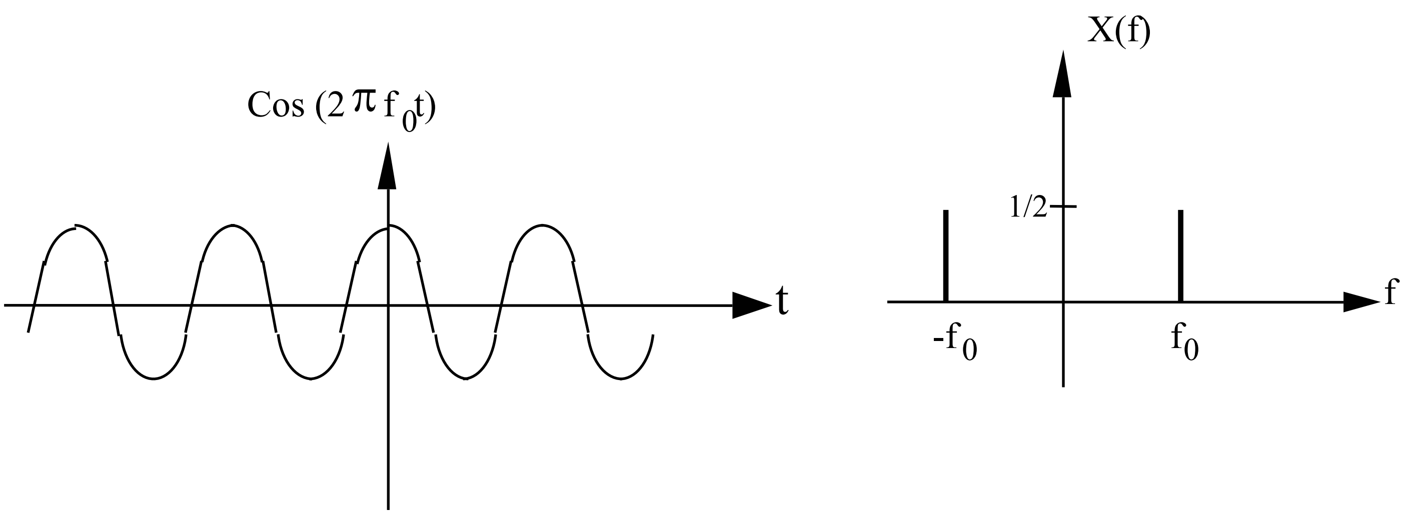
Fourier Transforms of periodic signals.

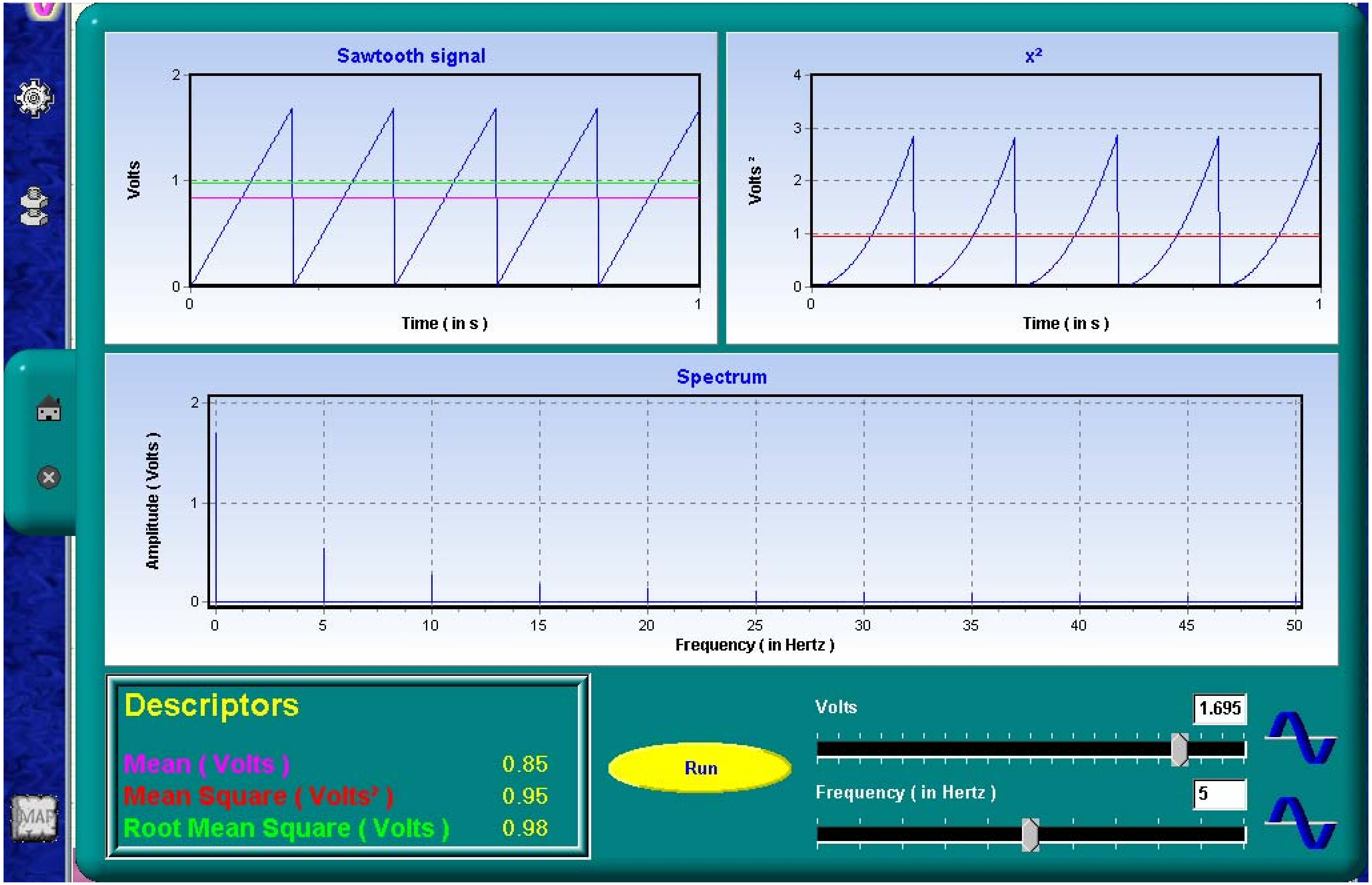
We may extend the concept of FT to power signals, like periodic functions.

It can be shown that the FT of a cosine signal

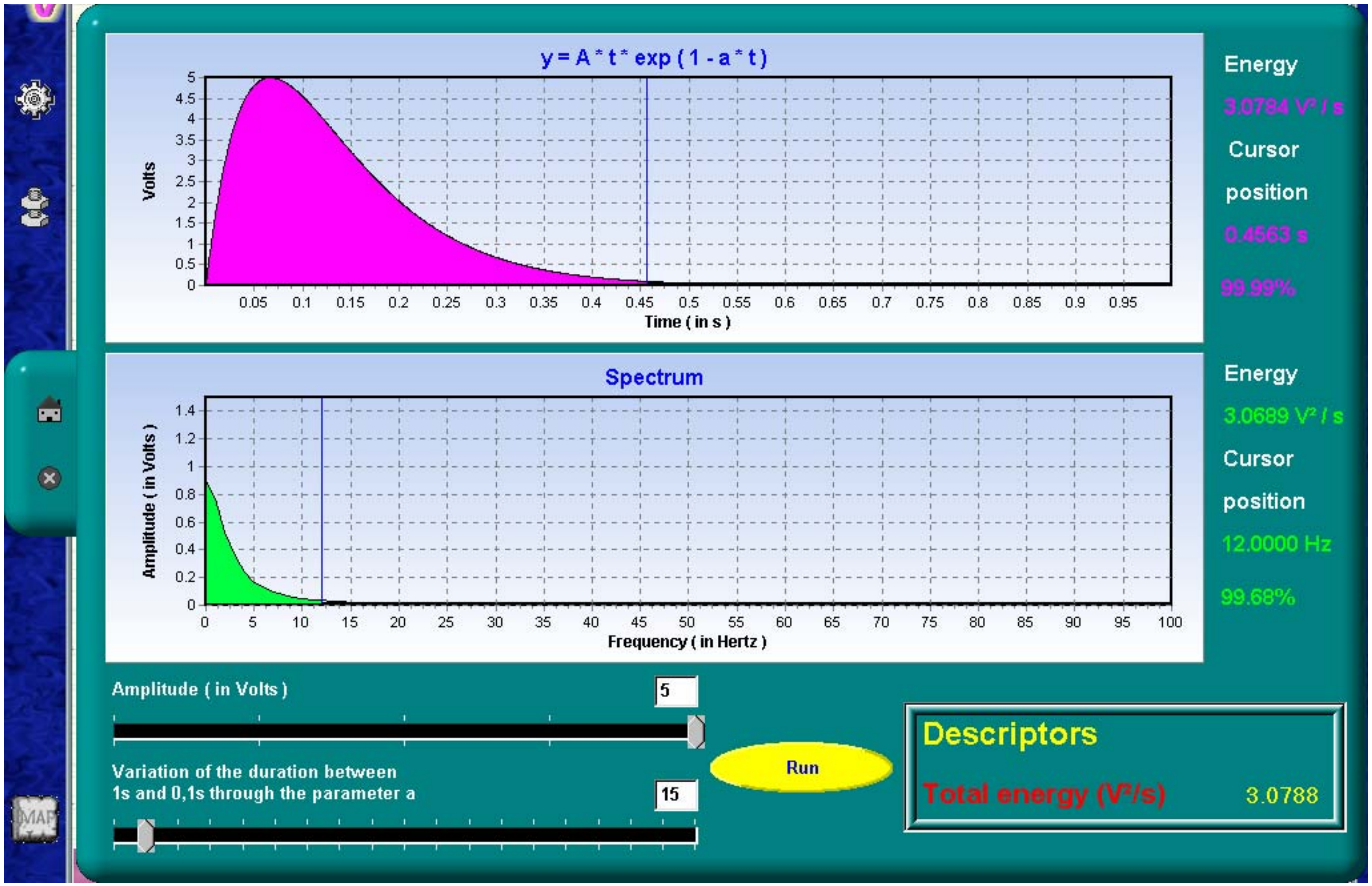
$$\cos(2\pi f_o t) \leftrightarrow \frac{1}{2}[\delta(f - f_o) + \delta(f + f_o)]$$

The transform exists only at the discrete frequencies $\pm$ *fo*.

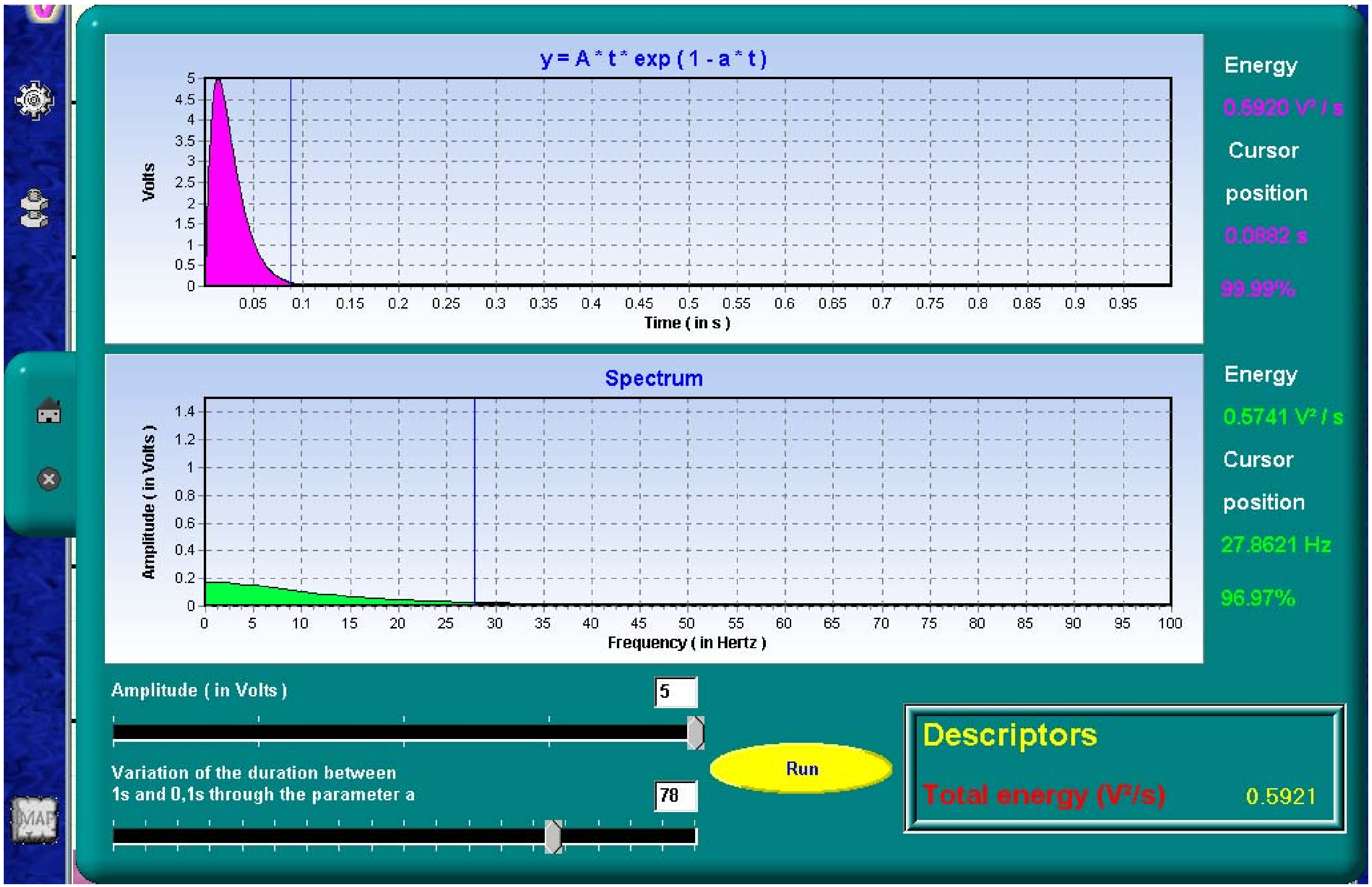




Signal Processing:Fourier



Signal Processing:Fourier



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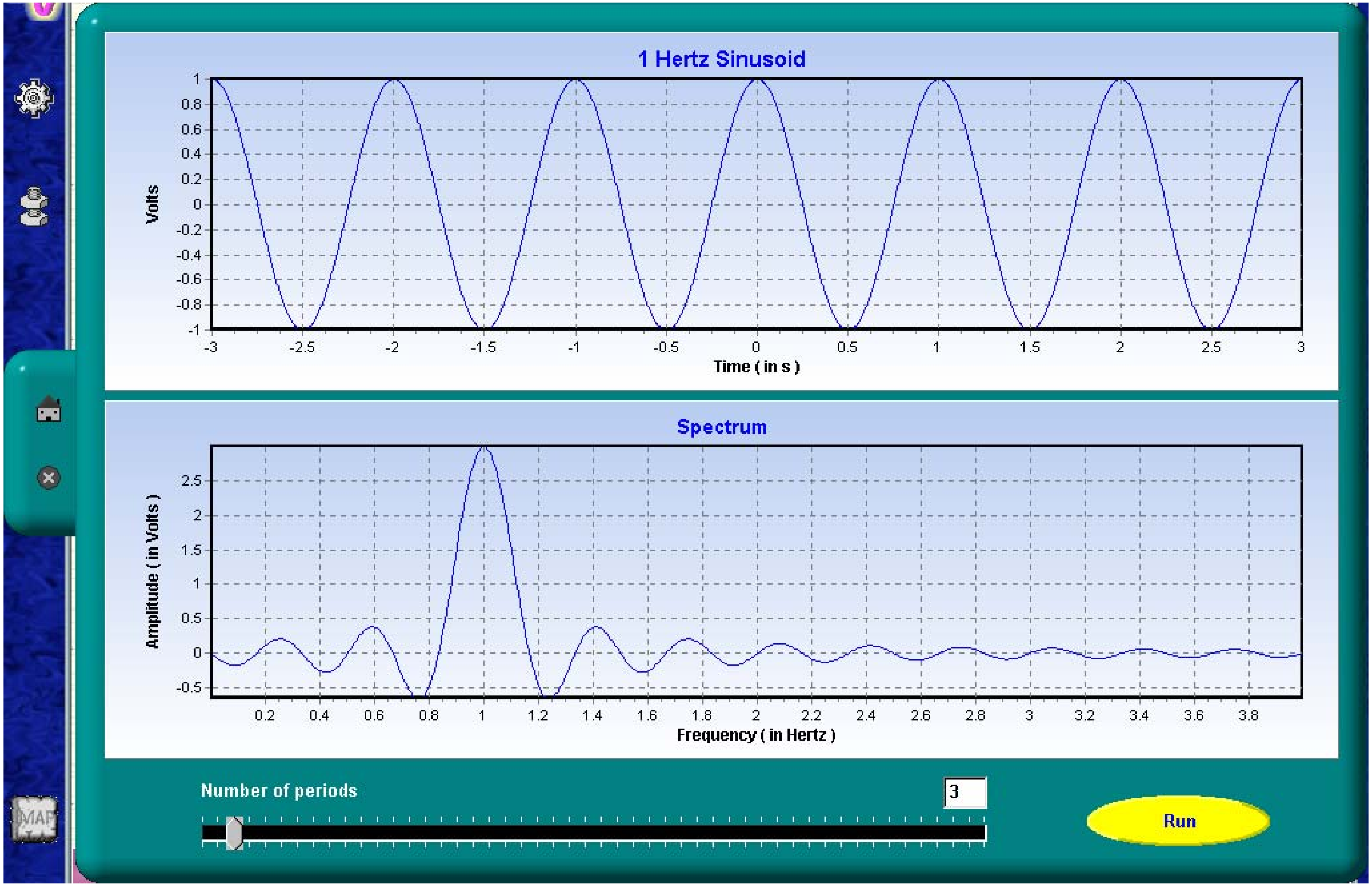
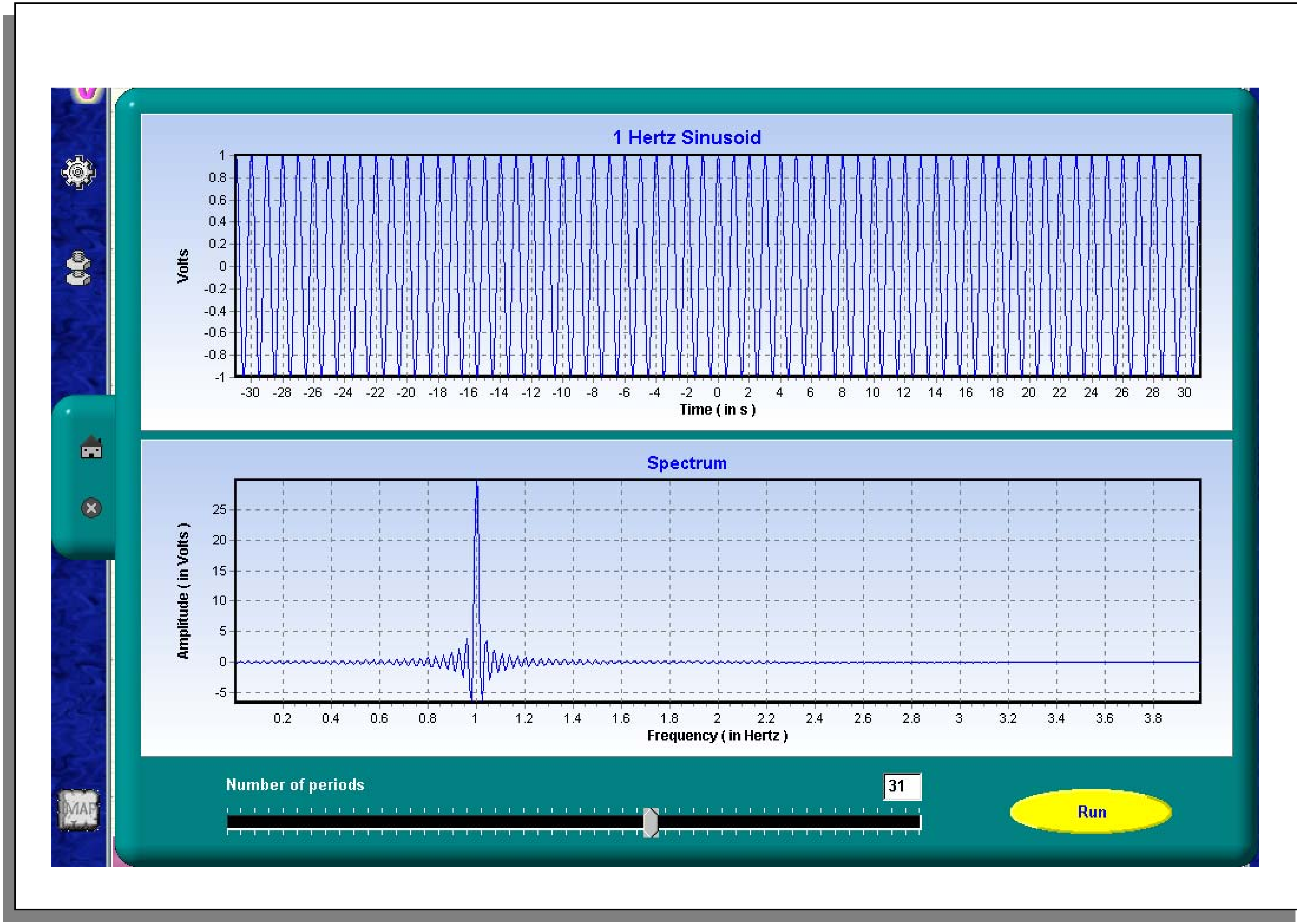
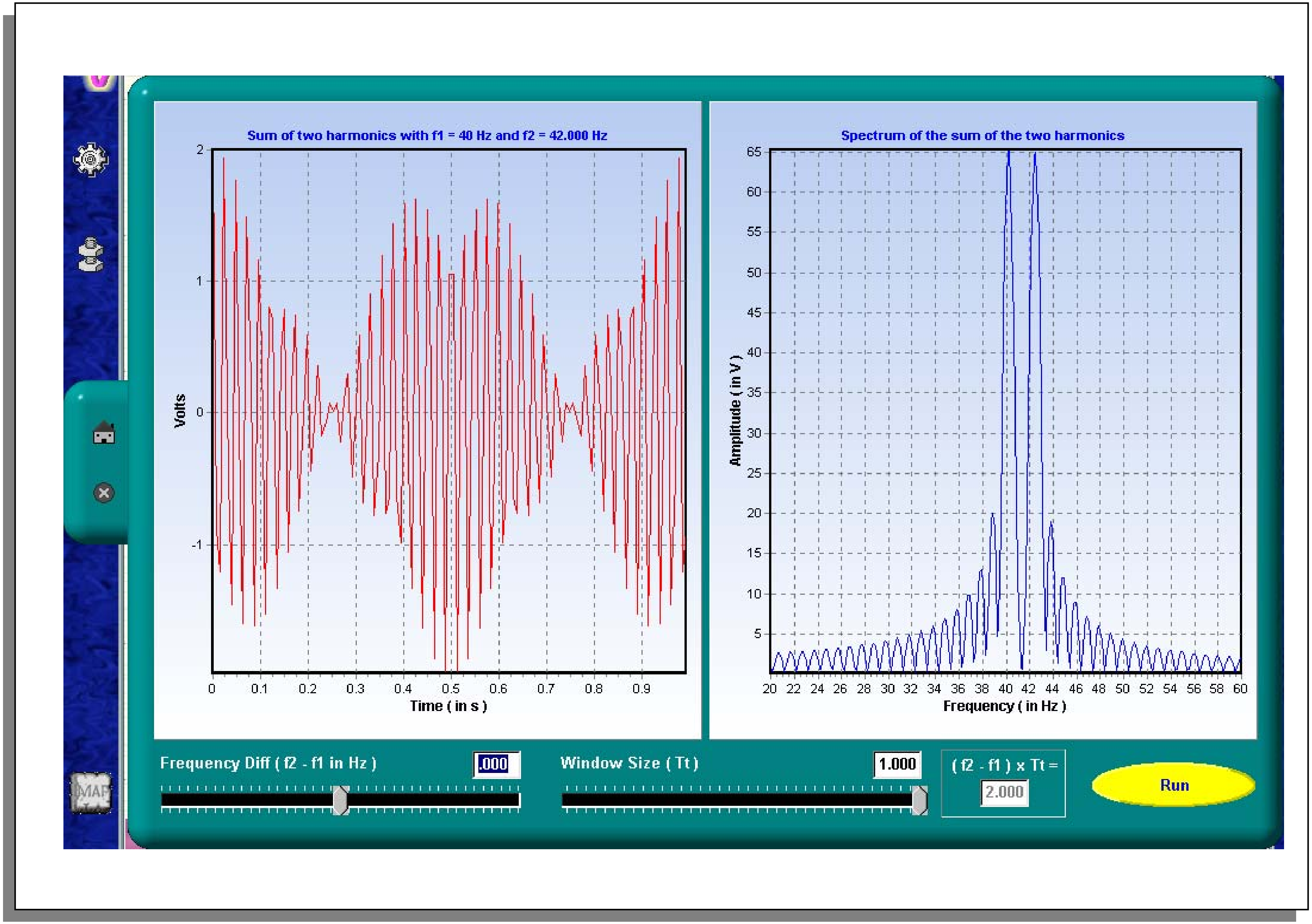


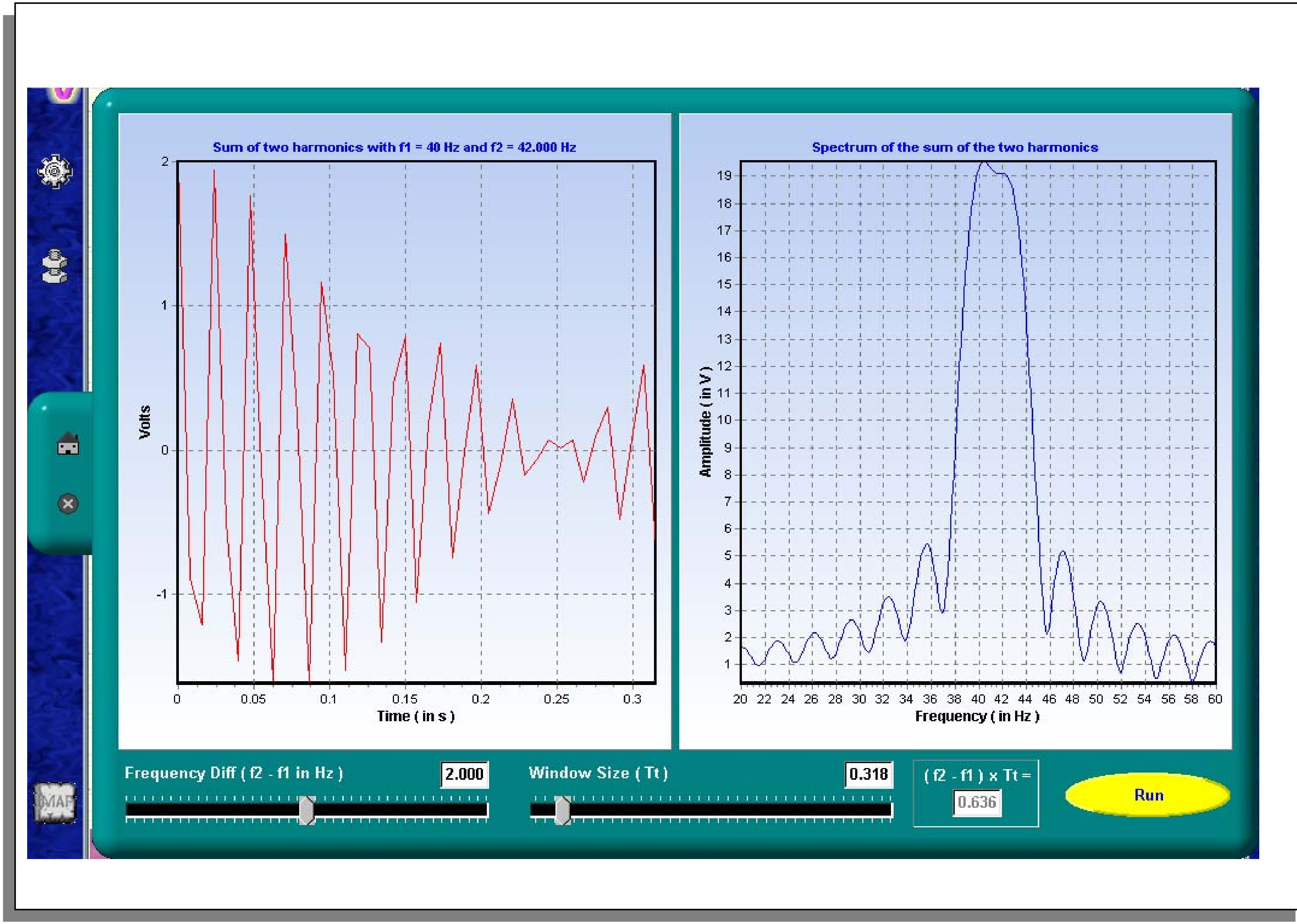
Signal Processing:Fourier



Signal Processing:Fourier



Signal Processing:Fourier



Signal Processing:Fourier

Discrete Fourier Transform (DFT)

- **We start by discretizing the FT pair**

- **The line interval t** $\rightarrow \Delta \mathrm{T}$
- **The frequency interval f** $\rightarrow \kappa\, \Delta \mathrm{T}$
- **The integral** $\int \rightarrow \sum$

Let us analyze a signal of duration T_{total}

$$T_{total} = N\,\Delta T$$

The time interval ΔT is chosen according to the sampling theorem.

The frequency interval Δf is chosen as the reciprocal of the analyzed signal length

$$\Delta f = \frac{1}{N\,\Delta T}$$

Then

$$X(f) \rightarrow X(k\Delta f) = \Delta T \sum x(i\Delta T)\exp(-j\frac{2\pi}{N})^{ik}$$

$$x(t) \rightarrow x(i\Delta T) = \frac{1}{N\Delta T}\sum X(k\Delta f)\exp(j\frac{2\pi}{N})^{ik}$$

The expression is periodic for *ik* = *N*, hence the summation for $X(k\,\Delta f)$ and $x(i\,\Delta T)$ are periodic within N.

We thus limit the summation to N samples, resulting in the DFT pair

$$X(k\Delta f) = \Delta T \sum_{o}^{N-1} x(i\Delta T)\exp(-j\frac{2\pi}{N})^{ik} \quad E.U.\ X(kf)[V-\sec]$$

$$x(i\Delta T) = \frac{1}{N\Delta T}\sum_{o}^{N-1} X(d\Delta f)\exp(j\frac{2\pi}{N})^{ik} \quad E.U.\ X(k)[V]$$

A normalized DFT pair, for $\Delta T = 1$, is usually computed by most procedures

$$X(k) = \sum_{o}^{N-1} x(i) W_N^{ik}$$

$$x(i) = \frac{1}{N} \sum_{o}^{N-1} X(k) W_N^{-ik}$$

with the compact notation

$$W_N = \exp(-j\frac{2\pi}{N})$$

The DFT is a transform between two sequences of N samples

$$X(k) \leftrightarrow x(i)$$

In general it is a transform between two complex sequences

The frequency scale

a) $k = 0\,,\;\; f = k\,\Delta f = 0$

This is the zero (DC) frequency. Here

$$X(0) = \sum_{o}^{N-1} x(i)$$

and equals N times the average of x(i)

b) $k = \frac{N}{2}$, $f = \frac{N}{2}\Delta f = \frac{N}{2}\frac{1}{N\Delta T} = \frac{1}{2\Delta T} = f_{NYQUIST}$

c) From the periodicity of $\exp(j\,2\pi / N)$, and hence the periodicity of *x(i)* and *X(k)*, we may define negative indices, and

$$X(-k) = X(N-k)$$

The upper part of X(k), with *k>N/2,* may thus be interpreted as transforms for negative frequencies.

The DFT of real signals.

- **For most engineering measurements, x(t) and hence x(i) is real. From**

$$x(i) = \frac{1}{N}\sum_{o}^{N-1} X(k)\exp(j\frac{2\pi}{N})^{ik}$$

it follows that X(k) must satisfy specific conditions in order to have the summation result in real samples. These conditions are one of a specific symmetry, with

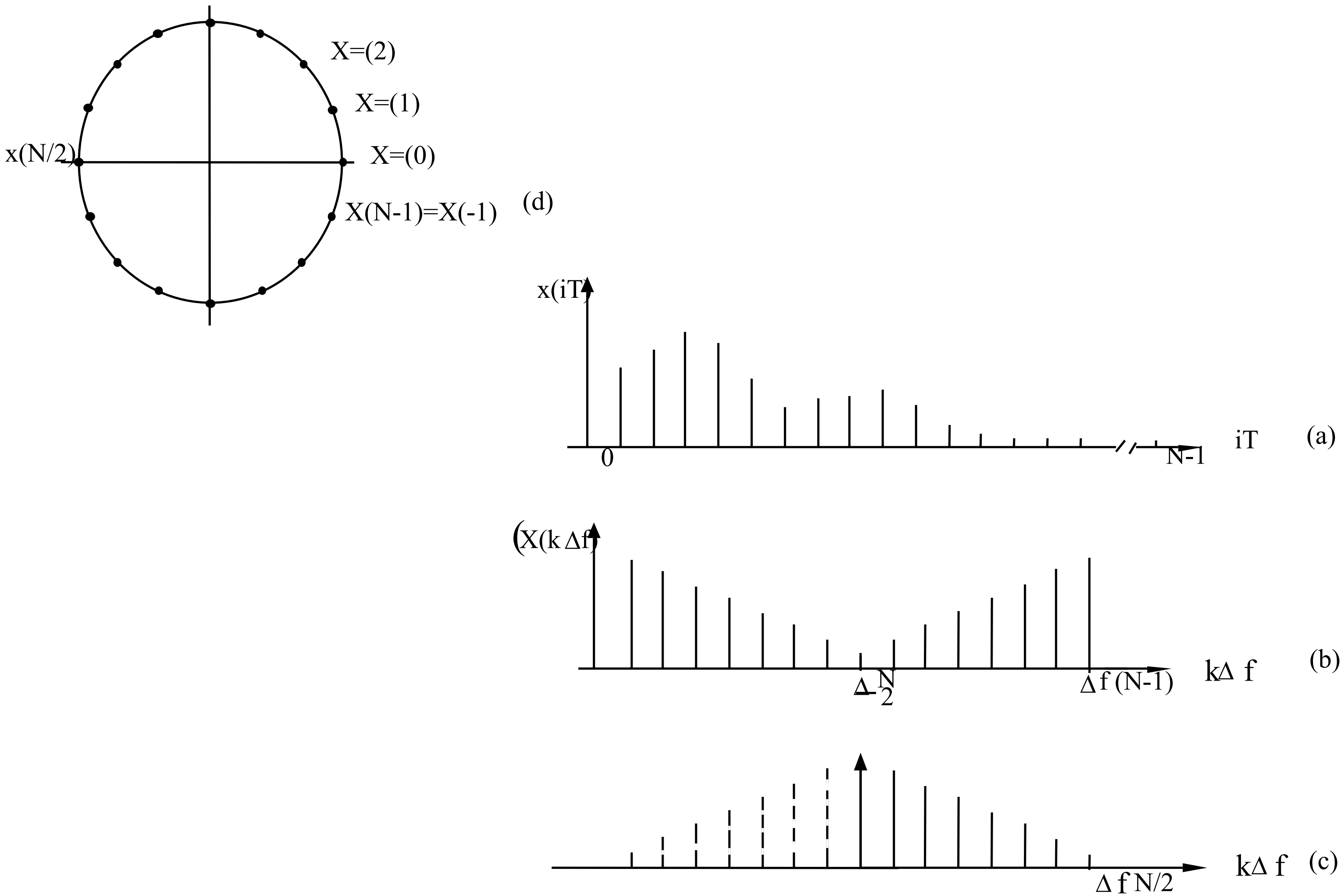
$$X(-k) = X(k)^*$$

thus

$$|X(-k)| = |X(k)|$$

and

$$\arg[X(-k)] = -\arg[X(k)]$$


X=(2)
X=(1)
x(N/2)
X=(0)
X(N-1)=X(-1)
(d)
x(iT)
0
N-1
iT
(a)
X(k Δf)
Δ N/2
Δ f (N-1)
kΔ f
(b)
Δ f N/2
kΔ f
(c)

The Fast Fourier Transform (FFT)

This is a procedure for an extremely efficient computation of the DFT pair.

The computation for a specific X(k) necessitates N operations.

Each operation consists of complex multiplication and summation. The complete sequence of X, k = 0.... N - 1, necessitates N^2 such computations.

The FFT utilizes symmetries in the computation steps, to achieve huge savings. The necessary number of operations is reduced roughly to N $\log_2 N$. The savings for large N are enormous.

Many FFT procedures, especially for small machines, work best with a number of samples N equal to some power of 2:

$N = 2^P$

N = 32, 64, 128.... 4096....

Basic procedure for analyzing a signal via the FFT

1) Choose ΔT, the sampling interval

2) Choose N a power of 2

The resulting resolution in the frequency domain is

$$\Delta f = \frac{1}{N\,\Delta T}$$

The frequency scale is $$f = \frac{k}{N\,\Delta T}$$

Example:

Analyzing range : 0 - 500 Hz

Sampling interval

$$\Delta T = \frac{1}{2.5\,f_{\max}} = \frac{1}{2.5 \times 500} = 0.8\ [m\sec]$$

a) The necessary $\Delta f = 5$

$$N = \frac{1}{\Delta f \times \Delta T} = \frac{1.25 \times 10^3}{5} = 250$$

and we would choose N = 256

b) The necessary $\Delta f = 2 \quad [Hz]$

$$N = \frac{1}{\Delta f \times \Delta T} = 750$$

and we may choose

N = 512 resulting in $\Delta f = 2.44 \quad [Hz]$

or

N = 1024 resulting in $\Delta f = 1.22 \quad [Hz]$