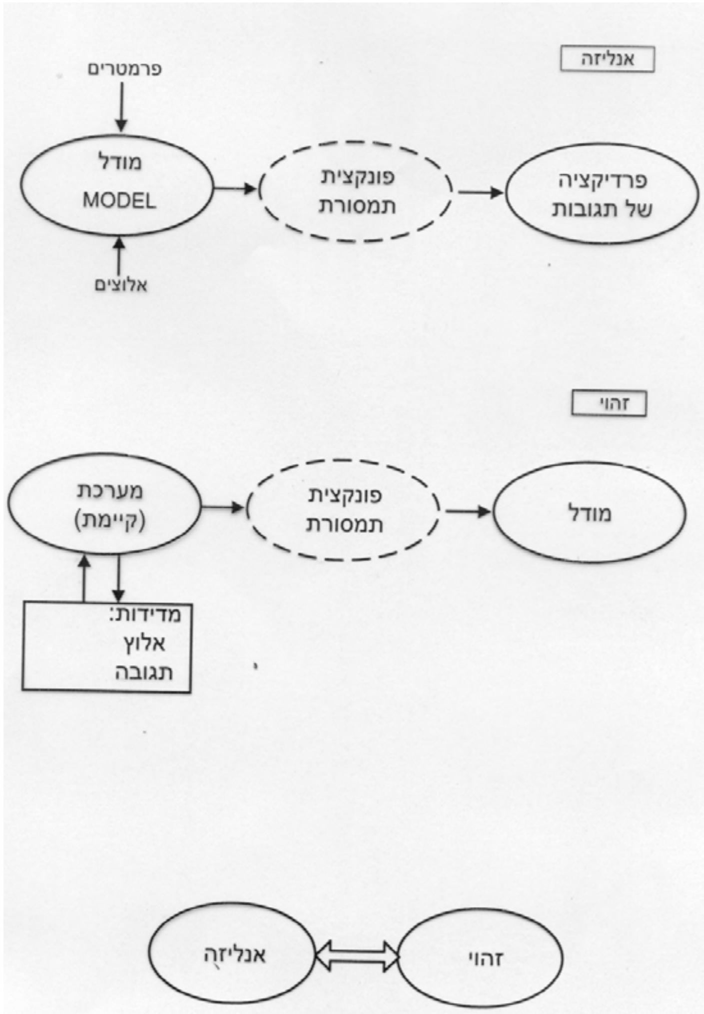
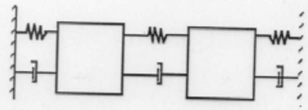


Analytical and Experimental Modal Analysis

Modal description of vibrating system

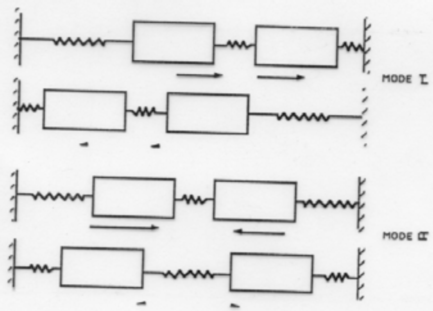


Two deflection shapes exist for each eigenfrequency.

They are described by the modal matrix

Deflection shapes are eigenvectors of modal matrix

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} = \left[(\Phi)_1 \; (\Phi)_2 \right]$$

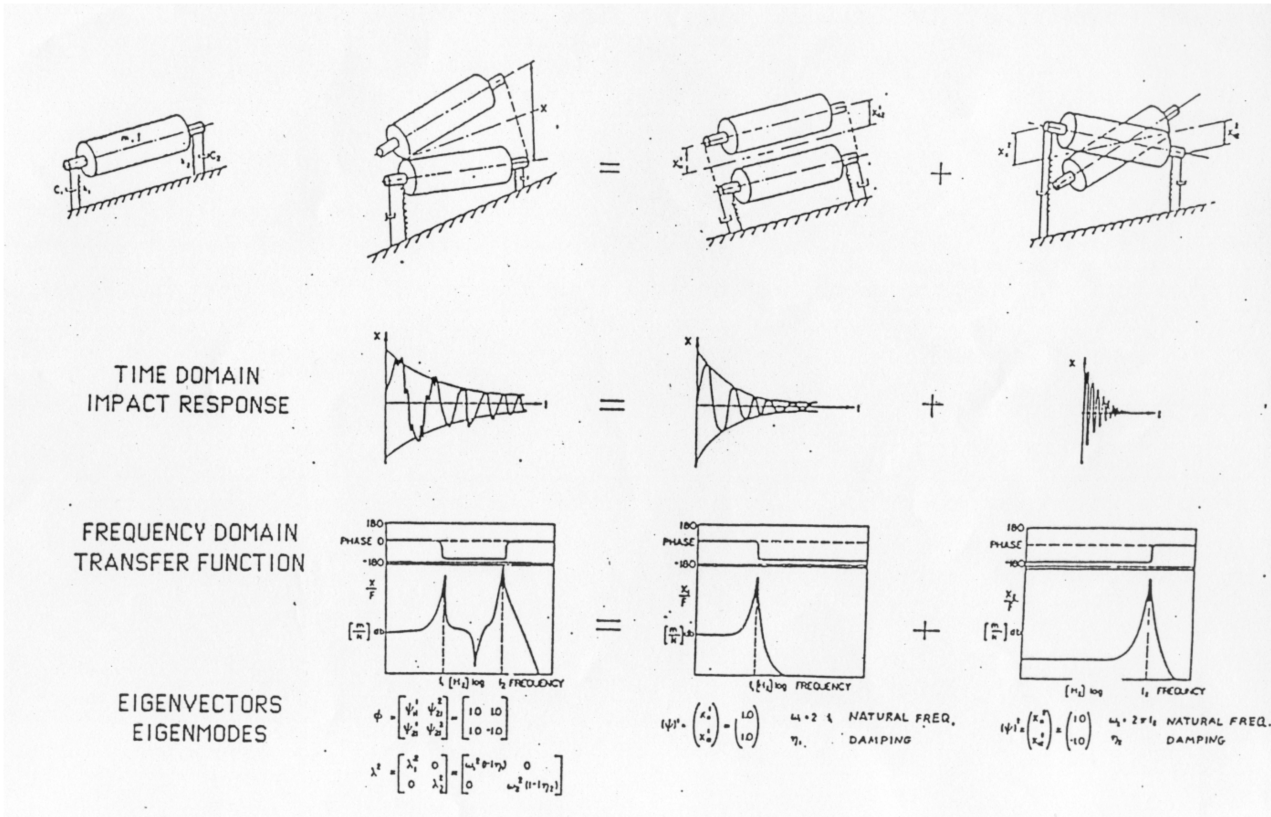


$$(\Phi)_r = \begin{bmatrix} \Phi_{1r} \\ \Phi_{2r} \end{bmatrix}$$

The eigenvalue matrix describes natural frequencies

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\lambda_r = \omega_r^2$$



Signal Processing: Modal Analysis

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + c_2(\dot{x}_1 - \dot{x}_2) + k_1 x_1 + k_2(x_1 - x_2) = f_1$$

$$m_2\ddot{x}_2 + (c_2 + c_3)\dot{x}_1 + (k_2 + k_3)x_2 - c_2\dot{x}_1 - k_2 x_1 = f_2$$

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 + (k_1 + k_2)x_1 - c_2\dot{x}_2 - k_2 x_2 = f_1$$

$$m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_3 x_2 + k_2(x_2 - x_1) = f_2$$

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad [c] = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \quad [k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

$$(x) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (\dot{x}) = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} \quad (\ddot{x}) = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} \quad (f) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

General: $[m](\ddot{x}) + [c](\dot{x}) + [k](x) = (f)$

$$m(N \times N) \quad c(N \times N) \quad x(N \times 1) \quad f(N \times 1)$$

Undamped system:

$$[c] = 0$$

$$(\ddot{x}) = -\omega^2 (\psi) \, exp(j\omega t)$$

$$-\omega^2 [m](\psi) + [k](\psi) = 0$$

$$-\lambda [m](\psi) + [k](\psi) = 0$$

$$\left\{ [m]^{-1} [k] - \lambda [I] \right\} (\psi) = 0$$

λ_i *eigenvalue*

$(\psi)_i$ *eigenvector*

$m_1 = 5\ kg\ m_2 = 10\ kg \qquad k_1 = k_2 = 2N/m \qquad k_3 = 4N/m$

$$\left| \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0 \qquad (4/5-\lambda)(3/5-\lambda) - (-1/5)(-2/5) = 0$$

$$\lambda_1 = 2/5 \qquad \omega_1 = (2/5)^{1/2} \qquad [rad/sec]$$

$$\lambda_2 = 1 \qquad \omega_2 = 1 \qquad [rad/sec]$$

$$\lambda_1 = \omega_1^2 = 2/5 \qquad \begin{bmatrix} 4/5-2/5 & -2/5 \\ -1/5 & 3/5-2.5 \end{bmatrix} \begin{pmatrix} \psi_{11} \\ \psi_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2/5\ \psi_{11} - 2/5\psi_{21} = 0$$

$$\psi_{11} = \psi_{21}$$

$$(\psi)_1 = \begin{pmatrix} \psi_{11} \\ \psi_{21} \end{pmatrix} = \begin{pmatrix} \psi_{11} \\ \psi_{11} \end{pmatrix}$$

$$\lambda_2 = \omega_2^2 = 1 \qquad (\psi)_2 = \begin{pmatrix} \psi_{12} \\ -\frac{1}{2}\psi_{12} \end{pmatrix}$$

Multi Degree of Freedom Models

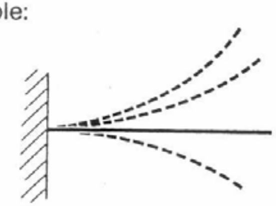
- **Mode Shape**

$$[B(p_r)]\,\{X\}_r = \{0\} \qquad (1)$$

$\{X\}_r = \{\psi\}_r$ mode shape for mode # r

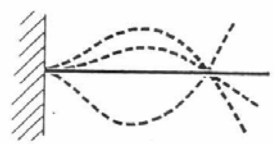
Example:

$\{\psi\}_1$



$$\begin{Bmatrix} 0{,}00 \\ ,07 \\ ,17 \\ ,33 \\ ,50 \\ ,67 \\ 1{,}00 \end{Bmatrix}$$

$\{\psi\}_2$



$$\begin{Bmatrix} 0{,}00 \\ ,28 \\ ,67 \\ 1{,}00 \\ ,67 \\ 0{,}00 \\ -,50 \end{Bmatrix}$$

$\{\psi\}_r$ is solution for homogenous equation (1), i.e. only the relative deflections are found

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$$\lambda_i \neq \lambda_j \quad (\psi)_i^T \; [m](\psi)_j = 0$$

$$(\psi)_i^T [k](\psi)_j = 0$$

Weighted Orthogonality relations

$$\lambda_i = \lambda_j \quad (\psi)_i^r [m](\psi)_i = M_i$$

$$(\psi)_i^T [k](\psi)_i = K_i$$

$$[\psi]^T [m][\psi] = \begin{bmatrix} \ddots & & \\ & M_i & \\ & & \ddots \end{bmatrix}$$

Generalized mass and stiffness

$$[\psi]^T [k][\psi] = \begin{bmatrix} \ddots & & \\ & K_i & \\ & & \ddots \end{bmatrix}$$

$$K_i/M_i = \omega_i^2$$

$$\omega_1 = (2/5)^{1/2} \quad set \;\; \psi_{11} = 1 \quad (\psi)_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = M_1 = 15$$

$$K_1 = \omega_1^2 M_1 = 6 \quad \omega_2 = 1 \quad Set \;\; \psi_{12} = 1 \quad (\psi)_2 = \begin{pmatrix} 1 \\ -\dfrac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -\dfrac{1}{2} \end{pmatrix}^T \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{pmatrix} 1 \\ -\dfrac{1}{2} \end{pmatrix} = M_2 = 15/2$$

$$K_2 = \omega_2^2 M_2 = 15/2$$

$$[M] = \begin{bmatrix} 15 & 0 \\ 0 & 15/2 \end{bmatrix} \quad [K] = \begin{bmatrix} 6 & 0 \\ 0 & 15/2 \end{bmatrix}$$

Mass normalized eigenvectors

$$M_i = 1$$

$$K_i = \omega_i^2 M_i = \omega_i^2$$

$$\omega_1 = (2/5)^{1/2} \qquad \begin{pmatrix} \phi_{11} \\ \phi_{21} \end{pmatrix}^T \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{pmatrix} \phi_{11} \\ \phi_{21} \end{pmatrix} = M_1 = 1$$

$$\phi_{11} = \phi_{21}$$

$$(\phi)_1 = \begin{pmatrix} \sqrt{1/15} \\ \hline \sqrt{1/15} \end{pmatrix}$$

$$\omega_2 = 1 \qquad \begin{pmatrix} \phi_{12} \\ \phi_{22} \end{pmatrix}^T \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{pmatrix} \Phi_{12} \\ \Phi_{22} \end{pmatrix} = M_2 = 1$$

$$(\phi)_2 = \begin{pmatrix} \sqrt{2/15} \\ -0.5\sqrt{2/15} \end{pmatrix}$$

$$(\phi)_i = \frac{1}{\sqrt{M_i}} (\psi)_i$$

Generalized coordinates

$$[\psi] = [(\psi)_1 \ldots (\psi_2) \ldots (\psi)_n]$$

$$(x) = [\psi]q \quad q = [\psi]^{-1}(x)$$

$$[m](\ddot{x}) + [k](x) = (f)$$

$$[\psi]^T[m][\psi](\ddot{q}) + [\psi]^T[k][\psi](q) = [\psi]^T(f)$$

$$[M](\ddot{q}) + [K](q) = [\psi]^T(f)$$

$$M_i\ddot{q}_i + K_i q_i = (\psi)_i(f) = f_i$$

$$\ddot{q}_i + \omega_i^2 q_i = f_i / M_i = \frac{(\psi)_i^T(f)}{(\psi)_i^T[m](\psi)_i}$$

$$\ddot{q}_i + \omega_i^2 q_i = f_i = (\phi)_i^T(f)$$

$$\psi] = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}$$

$$(x) = [\psi]q = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

$$[\psi]^T [m][\psi] = \begin{bmatrix} 15 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

$$[\psi]^T [k]\ [\psi] = \begin{bmatrix} 6 & 0 \\ 0 & \frac{15}{0} \end{bmatrix}$$

$$\begin{bmatrix} 15 & 0 \\ 0 & 15/2 \end{bmatrix} + \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & \frac{15}{2} \end{bmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$15\ddot{q}_1 + 6q_1 = f_1 + f_2$$

$$\frac{15}{2}\ddot{q}_1 + \frac{15}{2}q_2 = f_1 - \frac{f_2}{2}$$

Uncoupling of equations

Transfer Function approach

$$m\ddot{x} + kx = f\,exp(\,j\omega t\,)$$

$$(\,k - \omega^2 m\,)x = f$$

$$\left\{[\,k\,] - \omega^2[\,m\,]\right\}(\,x\,) = (\,f\,)$$

$$[\,H\,]^{-1}(\,x\,) = (\,f\,) \qquad (\,x\,) = [\,H\,](\,f\,)$$

$$[\,H\,]^{-1} = [\,k\,] - \omega^2[\,m\,]$$

$$[\,\phi\,]^T[\,H\,]^{-1}[\,\phi\,] = [\,\phi\,]^T[\,k\,][\,\phi\,] = \omega^2[\,\phi\,]^T[\,m\,][\,\phi\,] = \begin{bmatrix} \ddots & & \\ & \omega_r^2 - \omega^2 & \\ & & \ddots \end{bmatrix}$$

$$[\,H\,] = [\,\phi\,]\begin{bmatrix} \ddots & & \\ & \omega_r^2 - \omega^2 & \\ & & \ddots \end{bmatrix}^{-1}[\,\phi\,]^T$$

$$f = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f_k \\ 0 \end{pmatrix} \qquad (x) = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \phi_{21} & & & \\ \vdots & & & \\ \phi_{N1} & & & \phi_{NN} \end{bmatrix} \begin{bmatrix} \diagdown & & \\ & (\omega_r^2 - \omega^2) & \\ & & \diagdown \end{bmatrix}^{-1} \begin{bmatrix} \phi_{11} & \phi_{21} & \cdots & \phi_{N1} \\ \phi_{12} & & & \\ \vdots & & & \\ \phi_{1N} & & & \phi_{NN} \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f_2 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \vdots & & & \\ \vdots & & & \\ \phi_{N1} & & & \phi_{NN} \end{bmatrix} \begin{bmatrix} \diagdown & & \\ & (\omega_r^2 - \omega^2) & \\ & & \diagdown \end{bmatrix}^{-1} \begin{pmatrix} \phi_{k1} \\ \phi_{k2} \\ \vdots \\ \phi_{kN} \end{pmatrix} f_k$$

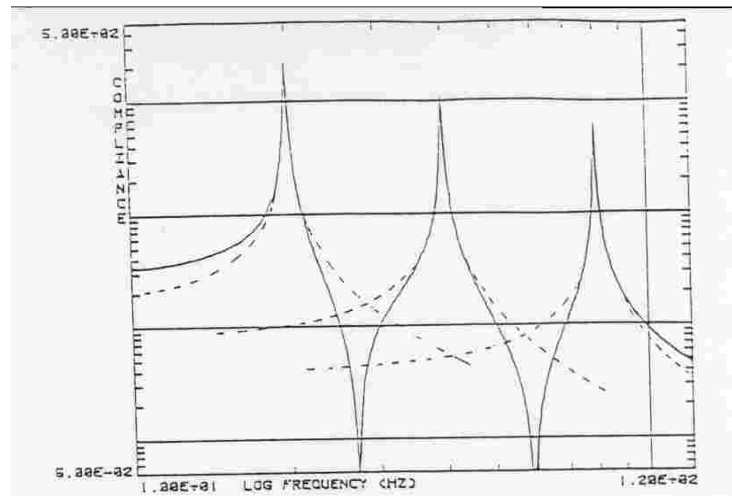
$$x_i = (\phi_{i1}\phi_{i2}\ldots\phi_{iN} \begin{bmatrix} \diagdown & & \\ & (\omega_r^2 - \omega^2) & \\ & & \diagdown \end{bmatrix}^{-1} \begin{pmatrix} \phi_{k1} \\ \vdots \\ \phi_{kN} \end{pmatrix} f_k$$

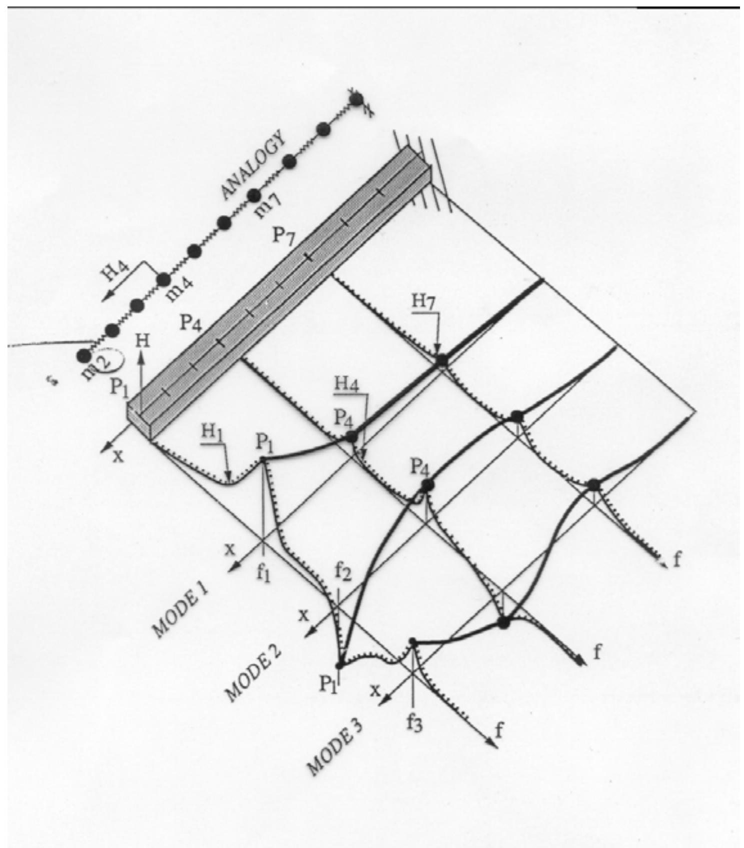
Summary:

$$H_{ik} = \sum_{r=1}^{N} \frac{\phi_{ir}\phi_{kr}}{\omega_r^2 - \omega^2}$$

$$H_{ik} = \sum_{r=1}^{N} \frac{\phi_{ir}\phi_{kr}}{\omega_r^2 - \omega^2 + j2\varepsilon_r\omega_r\omega} = \sum_{r} \frac{{}_{v}A_{ik}}{\omega_r^2 - \omega^2 + j^2\varepsilon_r\omega_r\omega}$$

$${}_{r}A_{ik} = \phi_{ir}\phi_{kr}$$

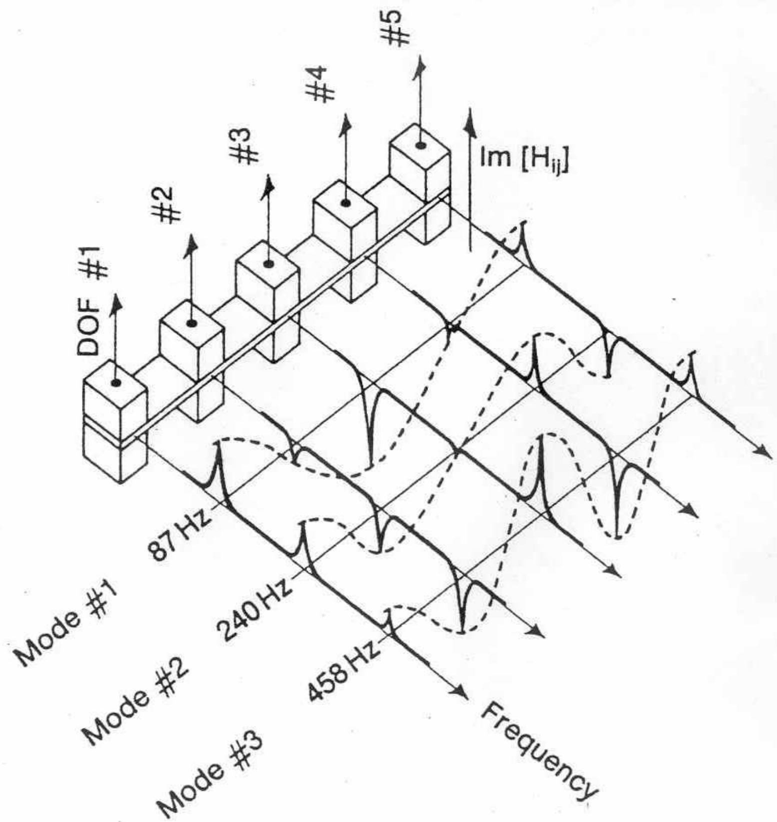




Signal Processing: Modal Analysis

Frequency Response Function and Modal Parameters

- **Mode Shapes from Quadrature Picking**

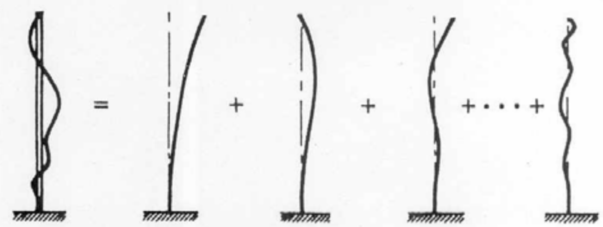


Signal Processing: Modal Analysis

Modal Analysis Introduction

Modal Behaviour

- The dynamic response of a structure can be decomposed into a discrete set of independent particular motions
- These motions are called Modes of Vibration
- A Mode is described by the Modal Parameters:
  * Natural Frequency & Damping
  * Mode Shape



$$x(y,t) = q_1(t)\{\phi\}_1 + q_2(t)\{\phi\}_2 + q_3(t)\{\phi\}_3 + \cdots + q_n(t)\{\phi\}$$

- Modal Analysis is the process of determining the modal parameters

  and ultimately to

- Create a mathematical Model

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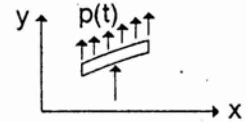
Signal Processing: Modal Analysis

Modal Analysis Introduction

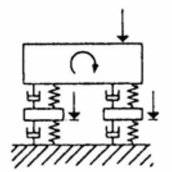
• Mathematical Models

- Wave Equation



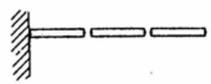
$$\frac{\partial^2}{\partial x^2}\left[IE(x)\,\frac{\partial^2 y}{\partial x^2}\right] + m(x)\,\frac{\partial^2 y}{\partial t^2} =$$

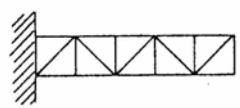
- Lumped Parameters

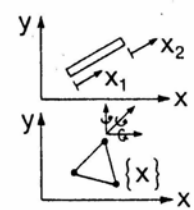


$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} =$$

- Finite Elements





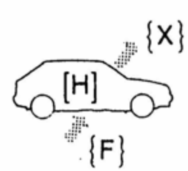


$$[k]_{2\times 2}$$

$$\{x\} \quad [k]_{18\times 1}$$

$$[m]\{\ddot{x}\} + [k]\{x\} = \{f\}$$

- Experimental Modal Model



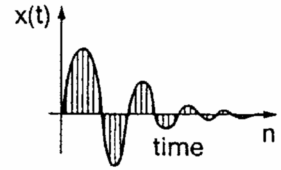
$$\{X\} = [H]\{F\}$$

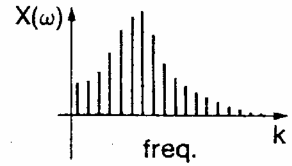
$$[H] = \sum_{r}^{2m} \frac{\{\psi\}_r\ \{\psi\}_r^T}{j\omega + \sigma_r - j\omega_{dr}}$$

The Modal Transformation

• Fourier – vs. Modal Transformation

Fourier

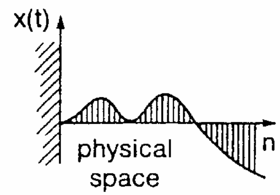


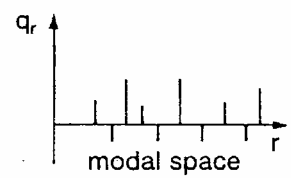


$$X(n) = \sum_{k}^{N/2} X(k)\, e^{j\frac{nk}{N}}$$

or $$x(t) = [e^{j\omega t}]\{X(\omega)\} = \sum_{n=1}^{N/2} \sin(\omega_n t) \times X_n$$

Modal





$$X(n) = \sum_{r=1}^{m} \phi_{nr} \cdot q_r$$

or $$\{x\} = [\Phi]\{q\} = \sum_{r=1}^{m} \{\phi\}_r \cdot q_r$$

Imagine:

$[e^{j\omega t}] = [\ \cdots\ ]$ and $\Phi = [\ \cdots\ ]$

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Signal Processing: Modal Analysis