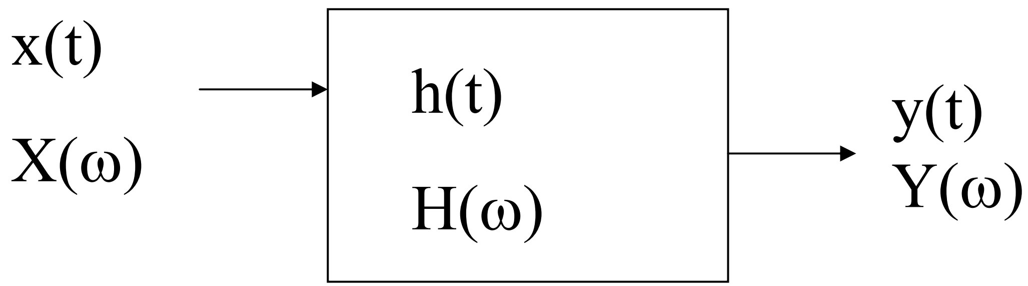
Linear systems - summary



Transients

$$y(t) = h(t) \otimes x(t) \qquad Y(\omega) = H(\omega) X(\omega)$$

Random

$$S_{yy}(\omega) = \left| H(\omega) \right|^2 S_{xx}(\omega)$$

$$S_{xy}(\omega) = H S_{xx}(\omega)$$

P_{yy}, P_{xx} are PSD's

Correlation functions

Autocorrelation $R_{xx}(\tau) = E[x(t)x(t+\tau)]$

Crosscorrelation $R_{xy}(\tau) = E[x(t)y(t+\tau)]$

Fourier Transform Relation

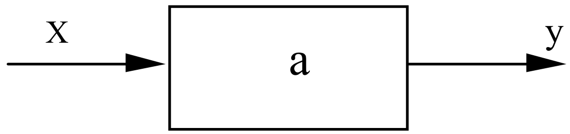
$R(\tau) \longleftrightarrow S(\omega)$ $S(\omega) = F[R(\tau)]$

$$R_{xy} = h(\tau) \otimes R_{xx}$$

White noise

$$R(\tau) = E[x(t)x(t)] = P_x\delta(\tau) \qquad (Dirac)$$

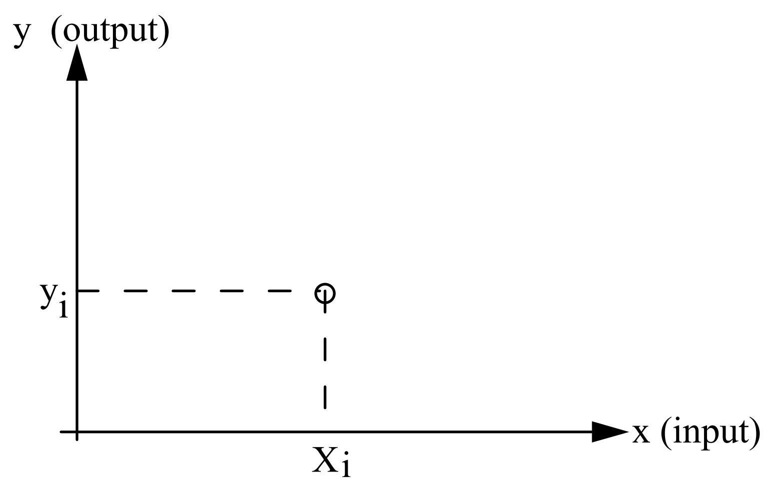
$$S(\omega) = const \qquad (all\ frequencies$$

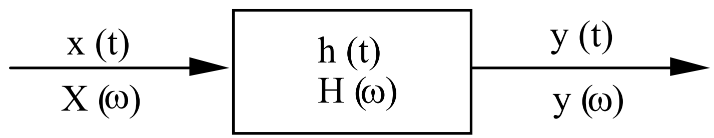


$$y = ax \qquad \hat{a} = \frac{y_i}{x_i}$$

Measurement:

$$y_i = ax_i$$





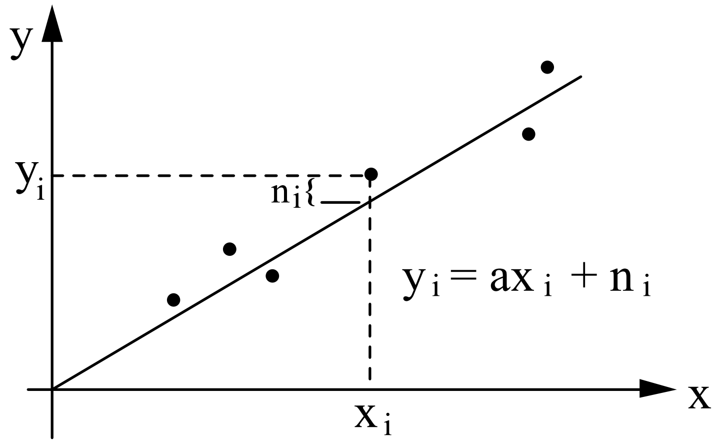
$$y(t) = h(t) \otimes x(t)$$

$$Y(\omega) = H(\omega) X(\omega)$$

$$\hat{H} = \frac{Y(\omega)}{X(\omega)}$$

$$H = |H(\omega)| \angle \Phi(\omega)$$

Signal Processing: Input/Output
Identification



$$\hat{a} = \frac{\sum_{M} x_i y_i}{\sum_{M} x_i x_i}$$

$$\hat{a} = \frac{\sum_{M} x_i^* y_i}{\sum_{M} x_i^* x_i}$$

$$\frac{de}{d\hat{a}} = \sum 2(yi - \hat{a}x_i)(-x_i) = 0$$

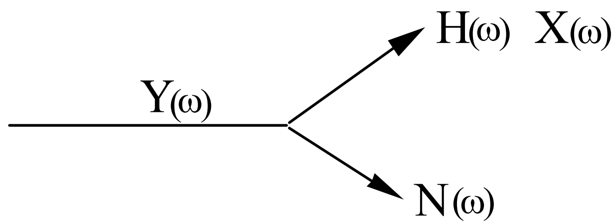
$$= X - 2\, n_i x_i = 0$$

$$\because \quad \sum n_1 k_i = 0$$

$$x_i y_i = x_i(\hat{a}\, x_i + n_i)$$

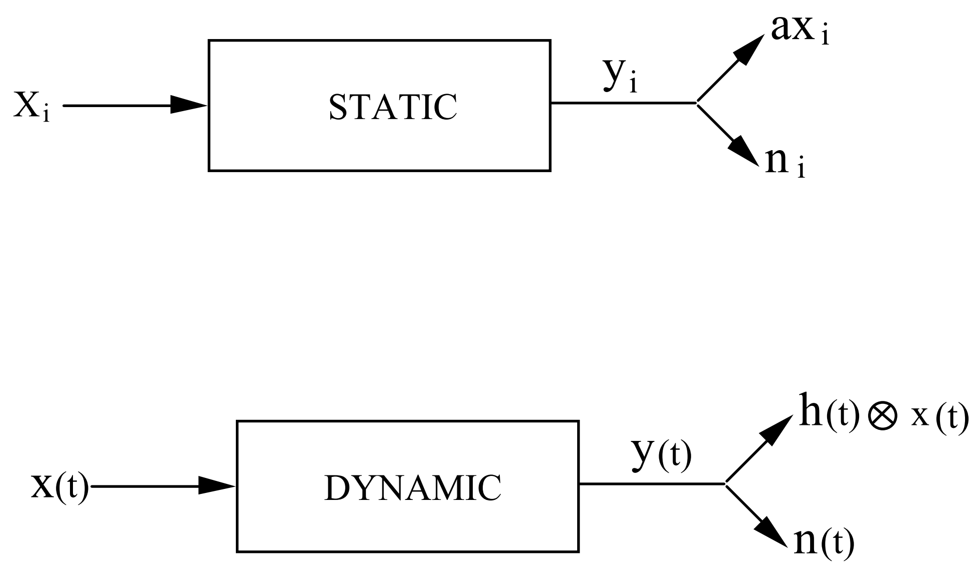
$$\sum x_i y_i = \sum x_i(\hat{a}x_i + n_i) = \hat{a}\sum x_i x_i + \sum x_i n_i = \hat{a}\sum x_i x_i$$

$$\hat{a} = \frac{\sum x_i y_i}{\sum x_i x_i}$$



$$\hat{H}(\omega) = \frac{\sum_{M} X^{*}(\omega)Y(\omega)}{\sum_{M} X^{*}(\omega)X(\omega)}$$

Signal Processing: Input/Output
Identification



Signal Processing: Input/Output
Identification

$$\frac{1}{M}\sum_{M} X^*(\omega)X(\omega) = S_{xx}(\omega)$$ **Auto Spectrum**

$$\frac{1}{M}\sum_{M} X^*(\omega)X(\omega)Y(\omega) = S_{xy}(\omega)$$ **Cross Spectrum**

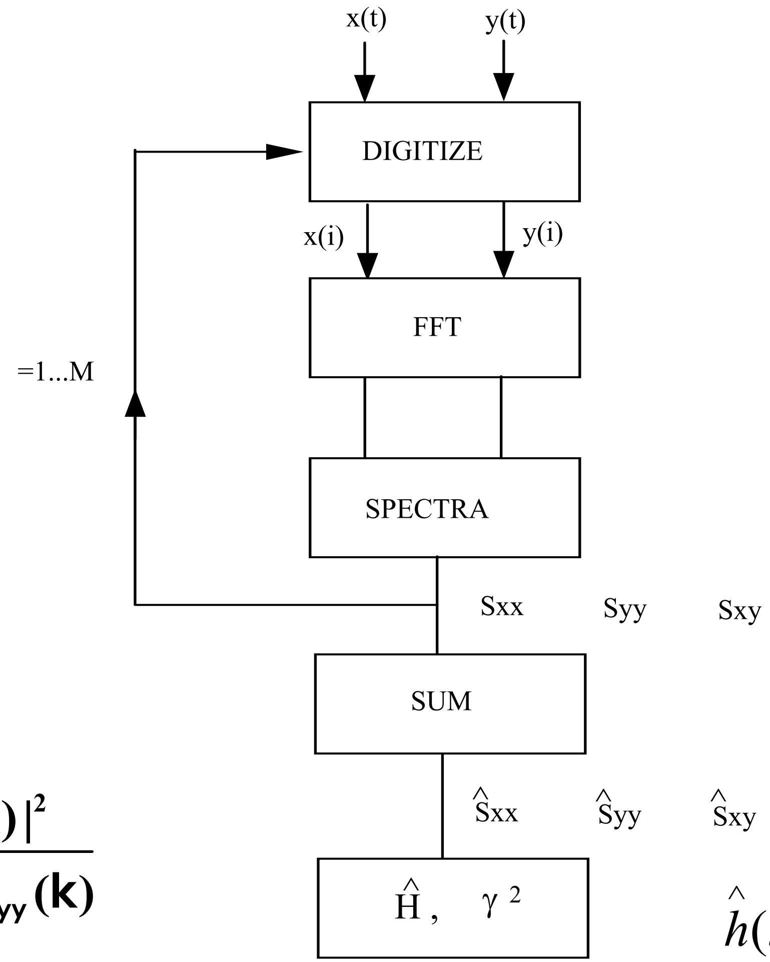
$$\hat{H}(\omega) = \frac{S_{xy}(\omega)}{S_{xx}(\omega)}$$

$$\hat{H} = \frac{S_{xy}}{S_{xx}}$$

$$\gamma^2 = \frac{\left[\frac{S_{xy}}{S_{xx}} X\right]^* \left[\frac{S_{xy}}{S_{xx}} X\right]}{Y^* Y} = \frac{\left(\frac{S_{xy}}{S_{xx}}\right)^* \left(\frac{S_{xx}}{S_{xx}}\right) S_{xx}}{S_{yy}} = \frac{\left|S_{xy}\right|^2}{S_{xx} S_{yy}}$$

$$\hat{H}(k) = \frac{\hat{S}_{xy}(k)}{\hat{S}_{xx}(k)}$$

$$\gamma^2(k) = \frac{|\hat{S}_{xy}(k)|^2}{\hat{S}_{xx}(k)\, S_{yy}(k)}$$



$$\hat{h}(i) = FFT^{-1}[\hat{H}(k)]$$

Signal Processing: Input/Output Identification

$$S_{xx}^{\ell}(k) = X_{\ell}^{*}(k) X_{\ell}(k)$$

$$S_{xx}^{\ell}(k) = Y_{\ell}^{*}(k)\ Y_{\ell}(k)$$

$$S_{xx}^{\ell}(k) = X_{\ell}^{*}(k) Y_{\ell}(h)$$

$$\hat{S}_{xx}(k) = \frac{1}{M}\sum_{\ell=1}^{M} S_{xx}^{\ell}(k)$$

$$\hat{S}_{yy}(k) = \frac{1}{M}\sum_{\ell=1}^{M} S_{yy}^{\ell}(k)$$

$$\hat{S}_{xy}(k) = \frac{1}{M}\sum_{\ell=1}^{M} S_{xy}^{\ell}(k)$$

Use of the coherence function

$\gamma^2 = 1$ implies all y caused by x

$\gamma^2 = 0$ Implies none of y is caused by x

γ^2 Is thus a measure of the identification

$\gamma^2 \rangle 0.9$ Is usually the minimum required

The coherence function may be used as a measure *independently for separate frequency ranges*

Errors

Aliasing (Sampling)

Random (averaging)

Leakage (Windows)

Bias

Insufficient resolution (N)

Input-output additive noise

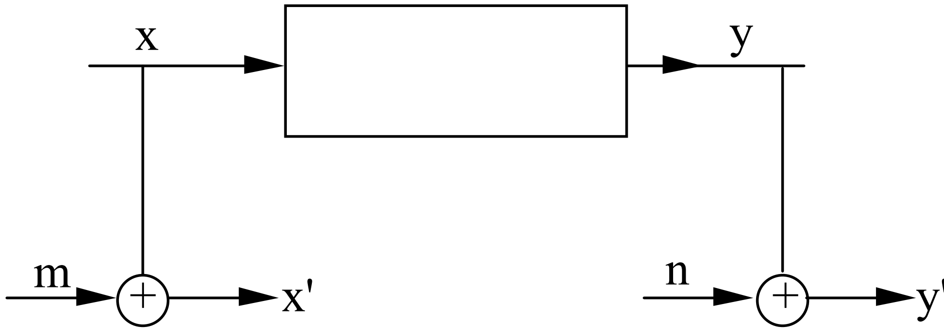
Delays

Unmeasured excitations

$\hat{S}_{xx} \quad \hat{S}_{yy}$	$\dfrac{1}{\sqrt{M}}$
$\lvert \hat{S}_{xy} \rvert$	$\dfrac{1}{\gamma\sqrt{M}}$
$\lvert \hat{H} \rvert$	$\dfrac{[1-\gamma^2]^{1/2}}{\gamma\sqrt{2M}}$
$\hat{\gamma}^2$	$\dfrac{\sqrt{2}[1-\gamma^2]}{\sqrt{M}}$

Random Errors – theoretical values (Bendat)

Note: $\hat{\gamma}^2$ Is used for error estimation



$$\hat{H} = \frac{Sx'y'}{Sx'x'} = \frac{\sum (X+M)^*(Y+N)}{\sum (X+M)^*(X+M)}$$

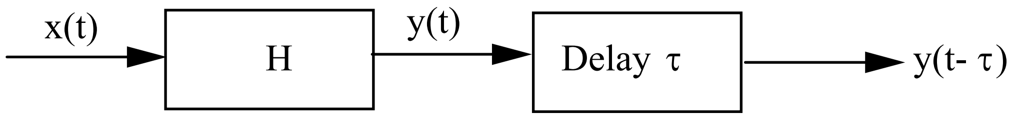
$$\hat{H} = \frac{\sum X*Y}{\sum X^*X + \sum M^*M} = \frac{S_{xy}}{S_{xx}+S_{mm}} = \frac{S_{xy}}{S_{xx}\left(1+\frac{S_{mm}}{S_{xx}}\right)} = H_o \frac{1}{1+\frac{S_{mm}}{S_{xx}}}$$

$$H_1 = \frac{S_{xy}}{S_{xx}}$$

$$H_2 = \frac{S_{yy}}{S_{xy}}$$

$$\hat{H}_2 = \frac{Sy'y'}{S_{x'y'}} = \frac{S_{yy} + S_{nn}}{S_{xy}} = H_o(1 + \frac{S_{nn}}{S_{yy}})$$

$$\hat{H}_1 = \frac{S_{xy}}{S_{xx}} = \frac{\sum H(X_1 + X_2)X_1^*}{X_1 X_1^*} = H_o \frac{Sx_1x_1 + Sx_2x_1}{Sx_1x_1}$$



Signal Processing: Input/Output
Identification

$$\gamma^2 = \frac{|S^1_{xy}|^2}{S^1_{xx} S^1_{yy}} = \frac{|X^* Y|^2}{X^* X \ \ Y^* Y} \equiv 1$$

$$\hat{H}_1 = \frac{S_{xy}}{S_{xx}} = \frac{\sum H(X_1 + X_2)X_1^*}{X_1 X_1^*} = H_o \frac{Sx_1x_1 + Sx_2x_1}{Sx_1x_1}$$

$$\gamma^2 = \frac{|S^1_{xy}|^2}{S^1_{xx} S^1_{yy}} = \frac{|X^* Y|^2}{X^* X \;\; Y^* Y} \equiv 1$$

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$$Y(\omega) = H(\omega) X(\omega)$$

$$\hat{H} = \frac{Y(\omega)}{X(\omega)}$$

$$H = \left| H(\omega) \right| \angle \Phi(\omega)$$

Signal Processing: Input/Output Identification

Example:SDOF