ON THE $H^\infty$ FIXED-LAG SMOOTHING: HOW TO EXPLOIT THE INFORMATION PREVIEW

by

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On the $H^\infty$ Fixed-Lag Smoothing: How to Exploit the Information Preview*

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Abstract

The problem of the continuous-time $H^\infty$ fixed-lag smoothing over the infinite horizon is studied. The first solution to the problem is derived in terms of one algebraic Riccati equation of the same dimension as in the filtering case and the mechanism by which the performance improvements with respect to the $H^\infty$ filtering occur is clarified. It is shown that the $H^\infty$ smoother exploits the information preview in an “$H^2$ manner.”

1 Introduction

In this paper the so-called fixed-lag smoothing problem is studied. The problem is to estimate a linear combination of the system states at time $t$, $t \in \mathbb{R}^+$, on the basis of the measurements available up to time $t+h$ for a given $h > 0$, which is referred to as the smoothing lag. This problem corresponds to the case, where some delay in supplying the estimate can be tolerated, like in many signal processing and communication applications. It is clear that potentially smoothers can achieve a better performance than corresponding filters ($h=0$) and the larger is the smoothing lag, the better estimate can be obtained. The questions are how to exploit this potential and what is the price (e.g., in terms of complexity of the resulting estimator) one has to pay to achieve this improvement?

The interest to the fixed-lag smoothing can be traced back to (Wiener, 1949) and in the context of the $H^2$ (Kalman) theory the problem is currently well understood, see (Anderson and Moore, 1979) and the references therein. On the other hand, the $H^\infty$ version of the fixed-lag smoothing problem is less studied and only few results are available in the literature, all in the discrete time. Grimble (1991, 1996) solved the problem using the polynomial $H^2$ embedding approach, Theodor and Shaked (1994) addressed time-varying finite horizon smoothing by game theoretic methods, while Zhang et al. (2000) used a Krein space polynomial approach. Arguably, the underlying idea in all these solutions is to treat the delay on equal footing with the rest of the system dynamics by incorporating the delay into the transfer function (or the state-space matrices) of the process. Since in the discrete time the delay dynamics are finite dimensional, this trick enables one to reduce the smoothing problem to a standard filtering

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one that, in turn, can be solved using known methods. The conceptual simplicity, however, is somewhat misleading, since the computational burden grows rapidly as the smoothing lag gets larger and the clear physically meaningful structure of the delay element does not show up in the final solution. The latter complicates any further analysis, e.g., the effect of the smoothing lag on the solution properties cannot be traced back. To cope with the complexity, Colaneri, Maroni, and Shaked (1998) proposed a special modification of the J-spectral factorization procedure aimed to handle the associated discrete delay. The essence of the idea is first to solve the problem for zero smoothing lag (filtering) and then to apply an iterative adjustment of the resulting solution to get rid of zeros at infinity caused by the delay. The latter adjustment is rather simple, so the large lag does not give rise to numerical difficulties. Yet the dependence of the resulting solution on the problem data is not readily recoverable, so both the structure of the smoother and the dependence of the achievable performance on \( h \) are not clearly seen. Moreover, the whole procedure becomes more complicated when the associated filtering problem is unsolvable and the solution does not appear to be extendable to the continuous-time setting.

Summarizing, the currently available solutions to the \( H^\infty \) fixed-lag smoothing problem are all limited to the discrete-time case, are considerably more complicated than the corresponding \( H^2 \) solutions, and fall short in accounting for the effect of the smoothing lag on the achievable estimation performance.

In this paper a novel approach to the solution of the infinite-horizon \( H^\infty \) fixed-lag smoothing problem is proposed and applied to solve the continuous-time version of the problem. As in (Colaneri et al., 1998), the solution is based on the J-spectral factorization machinery, but the delay is handled in a completely different fashion. Following the idea of Meinsma and Zwart (2000), on the first stage the transfer matrix to be factorized is modified so that the parts involving delay are excluded from the factorization procedure completely. This reduces the problem to the J-spectral factorization of a finite-dimensional transfer matrix and also suggests the structure of the resulting smoother. The latter consists of an FIR (finite impulse response) part in parallel connection with a finite-dimensional system having the structure of the \( H^\infty \) filter. Furthermore, the Hamiltonian matrix associated with the modified J-spectral factorization problem turns out to be similar to the Hamiltonian matrix associated with the filtering J-spectral factorization. Using this fact, the effect of the smoothing lag \( h \) on the achievable estimation performance is quantified.

The main technical contributions of this paper are as follows:

- the first solution to the continuous-time \( H^\infty \) fixed-lag smoothing problem is proposed;
- it is shown that the necessary and sufficient solvability conditions for the smoothing problem are based on one Riccati equation of the same dimension as in the filtering case;
- it is shown that the sub-optimal \( H^\infty \) smoother has structure similar to that of the \( H^2 \) optimal smoother and is compatible with the latter by the complexity;
- the first analysis of the effect of the smoothing lag \( h \) on the achievable smoothing performance is carried out;
- it is demonstrated that the use towards this end the inverse Riccati solution enables one to account for the effect of \( h \) on the achievable performance in an elegant manner.

Moreover, this paper shows clearly that the mechanism by which the previewed information is exploited by the \( H^\infty \) smoother (from both the smoother structure and the achievable estimation performance points of view) is similar to that in the \( H^2 \) case. This leads one to the qualitative conclusion that the \( H^\infty \) smoother exploits the information preview in an \( H^2 \) manner.
The paper is organized as follows. In Section 2 some preliminary results on the J-canonical factorization of rational transfer matrices are collected. In Section 3 the smoothing problem is formulated and solved. Section 4 is devoted to the analysis of the effect of the smoothing lag on the achievable performance. A numerical example illustrating the proposed solution is discussed in Section 5. Some concluding remarks are provided in Section 6.

Notation  Given a matrix $M$, $\|M\|$ and $\|M\|_F$ denote its spectral and Frobenius matrix norms, respectively, and $M^T$ denotes the transpose of $M$. Given a transfer matrix $G(s)$, its conjugate is defined as $G(s)^* = G^*(-s)$ and, when $G(s)$ is stable, $\|G(s)\|_2$ and $\|G(s)\|_\infty$ denote its $H^2$ and $H^\infty$ norms, respectively. The notation $\mathcal{C}_r(G, Q)$ stands for the chain scattering transformation $\mathcal{C}_r(G, Q) \triangleq (G_{21} + G_{22}Q)(G_{11} + G_{12}Q)^{-1}$. The completion operator $\pi_h(\cdot)$ is defined as follows:

$$\pi_h \left\{ e^{-sh} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \right\} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - e^{-sh} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}.$$ 

It can be seen that $\omega = \pi_h(\cdot) \zeta$ defines the FIR mapping $\omega(t) = C \int_{-h}^t e^{A(t-h)s} B \zeta(s) \, ds$.

Finally, a $2n \times 2n$ Hamiltonian matrix $H$ is said to belong to $\text{dom}(\text{Ric})$ if it has no imaginary axis eigenvalues and its stable (corresponding to the open left half plane eigenvalues) co-spectral subspace is complementary to $\ker \begin{bmatrix} 0 & I \end{bmatrix}$. In other words, $H \in \text{dom}(\text{Ric})$ if there exist an $n \times n$ matrix $Y = Y'$ such that $\begin{bmatrix} I & Y \end{bmatrix} H = A_s \begin{bmatrix} I & Y \end{bmatrix}$ for some Hurwitz $A_s$. The matrix $Y$ above is unique and thus the function $Y = \text{Ric}(H)$ is well defined. The definitions above are actually dual to the conventionally used (Zhou et al., 1995), yet they are better suited for the estimation problems studied in the paper.

2 Preliminaries

In this section some preliminary results related to the J-canonical factorization of rational functions are collected. More details can be found in (Hassibi et al., 1999).

Consider the $m \times m$ transfer matrix

$$\Phi_a(s) = \begin{bmatrix} A_a & -Q_a & B_{a1} \\ 0 & -A'_a & B_{a2} \\ C_{a1} & C_{a2} & D_a \end{bmatrix}$$

such that $\det(D_a) \neq 0$. For given $p$ and $q$ satisfying $p + q = m$ define the sign-indefinite matrix $J_{p,q} = \text{diag}\{1_p, -1_q\}$. When the dimensions are irrelevant or clear from the context, the simpler notation $J$ will be used. The following definition plays an important role in the sequel:

Definition 1. $\Phi_a$ is said to admit the $J_{p,q}$-canonical factorization (or simply the J-canonical factorization) if there exists a proper transfer matrix

$$\Omega_a(s) = \begin{bmatrix} \Omega_{a11}(s) & \Omega_{a12}(s) \\ \Omega_{a21}(s) & \Omega_{a22}(s) \end{bmatrix},$$

where the partitioning is compatible with that of $J_{p,q}$, such that

i) $\Phi_a(s) = \Omega_a(s) J_{p,q} \Omega_a^{-1}(s)$;

ii) $\Omega_a^{-1}(s) \in H^\infty$;

iii) $\Omega_{a11}(s) \in H^\infty$. 

Note that when the requirement $\Omega_a \in H^\infty$ is added, the J-canonical factorization becomes the J-spectral factorization. Yet such a requirement is not necessary for the solution procedure below.

For finite-dimensional systems the J-canonical factorization can be efficiently performed by state-space methods. The following result, which is essentially from (Hassibi et al., 1999) (see §12.9 and Theorem 16.1.14 there), can be formulated:

**Lemma 1.** Let

$$H_a = \begin{bmatrix} A_a & -Q_a \\ 0 & -A_a' \end{bmatrix} - \begin{bmatrix} B_{a1} \\ B_{a2} \end{bmatrix} D_a^{-1} \begin{bmatrix} C_{a1} & C_{a2} \end{bmatrix}.$$  

Then $\Phi_a$ admits the $J_{p,q}$-canonical factorization iff

(a) there exists a matrix $M_a$ such that $M_a J_{p,q} M_a' = D_a$ and

(b) $H_a \in \text{dom}(\text{Ric})$ and $Y_a = \text{Ric}(H_a) \succeq 0$.

Furthermore, if these conditions hold, then one possible choice for $\Omega_a$ is

$$\Omega_a(s) = \begin{bmatrix} A_a & (B_{a1} + Y_a B_{a2}) D_a^{-1} \\ C_{a1} & I \end{bmatrix} M_a.$$

## 3 Problem formulation and solution

Let

$$\begin{bmatrix} G_1(s) \\ G_2(s) \end{bmatrix} = \begin{bmatrix} A & B \\ C_1 & 0 \\ C_2 & D_2 \end{bmatrix}$$  \hspace{1cm} (1)

and suppose that the following assumptions hold:

(A1): $(C_2, A)$ is detectable;

(A2): $\begin{bmatrix} A - j\omega I & B \\ C_2 & D_2 \end{bmatrix}$ has full row rank $\forall \omega \in \mathbb{R}$;

(A3): $D_2 D_2' = I$.

Assumption (A3) is made just to simplify the exposition and can easily be relaxed to $D_2 D_2' > 0$.

The $H^\infty$ fixed-lag smoothing problem is then posed as follows:

**SP_h:** Given $G_1(s)$ and $G_2(s)$ and the smoothing lag $h \geq 0$, determine whether there exists a proper transfer matrix $K(s)$, which guarantees

$$\|e^{-sh} G_1(s) - K(s) G_2(s)\|_\infty < \gamma$$

for a given $\gamma > 0$ and then characterize all such $K$ when one exists.

Note that when $h = 0$ the problem **SP_0** becomes the standard $H^\infty$ filtering problem extensively studied in the literature, see (Nagpal and Khargonekar, 1991; Shaked and Theodor, 1992; Hassibi et al., 1999). This case can actually be thought of as the particular case of the general $H^\infty$ problem with the difference that the internal stability is not required. Another special case of **SP_h** solved in the literature (Nagpal and Khargonekar, 1991; Hassibi et al., 1999) is **SP_\infty** which corresponds to the infinite-horizon version of the fixed-interval smoothing problem.

We start with the following lemma, which expresses the solution of **SP_h** in terms of a J-canonical factorization:
Lemma 2. Let
\[
\Phi_\alpha(s) = \begin{bmatrix}
G_2(s) & 0 \\
e^{-sh}G_1(s) & I \\
\end{bmatrix} \begin{bmatrix}
I & 0 \\
0 & -\gamma^2 I \\
\end{bmatrix} \begin{bmatrix}
G_2(s) & 0 \\
e^{-sh}G_1(s) & I \\
\end{bmatrix}.
\] (2)

Then \(SP_h\) is solvable iff \(\Phi_\alpha(s)\) admits the \(J\)-canonical factorization with the factor \(\Omega_\alpha(s)\) and then the set of all solutions to \(SP_h\) is parametrized as
\[
K(s) = C_r(\Omega_\alpha(s), Q(s))
\]
for any \(Q \in H^\infty\) so that \(\|Q\|_{\infty} < 1\) and \(C_r(\Omega_\alpha(\infty), Q(\infty))\) exists. Moreover, if in addition \(\Omega_{\alpha 12}(s)\) is strictly proper, then the “central solution” \(C_r(\Omega_\alpha, 0)\) maximizes the entropy of the smoothing error.

Proof. The result is just the continuous-time version of (Hassibi et al., 1999, Theorem 10.5.1) modulo the fact that the strictly properness of \(\Omega_{\alpha 12}\) is only required to show the maximum entropy property of the central solution.

Unlike the finite-dimensional case, the result of Lemma 2 is not readily useful. The difficulty is to find the required \(J\)-canonical factorization of \(\Phi_\alpha\), whose \((1,2)\) and \((2,1)\) blocks contain infinite-dimensional elements \(e^{sh}\) and \(e^{-sh}\), respectively. Yet it turns out that this infinite-dimensional \(J\)-canonical factorization problem can be converted to an equivalent finite-dimensional one. To this end, define the Hamiltonian matrix
\[
H = \begin{bmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22}
\end{bmatrix} = \begin{bmatrix}
A & -BB' \\
0 & -A'
\end{bmatrix} - \begin{bmatrix}
BD_2' \\
C_2
dref{3}
\end{bmatrix} \begin{bmatrix}
C_2 & -D_2 B'
\end{bmatrix}
\]
and the symplectic matrix function
\[
\Sigma(\tau) = \begin{bmatrix}
\Sigma_{11}(\tau) & \Sigma_{12}(\tau) \\
\Sigma_{21}(\tau) & \Sigma_{22}(\tau)
\end{bmatrix} = e^{-H \tau}.
\]
The following result can be formulated:

Lemma 3. Let
\[
\Phi(s) = \begin{bmatrix}
A & -BB' \\
0 & -A'
\end{bmatrix} \begin{bmatrix}
BD_2' \\
C_2
dref{3}
\end{bmatrix} - \begin{bmatrix}
\Sigma_{12}(h)C_1' \\
\Sigma_{11}(h)C_1'
\end{bmatrix}
\]
and
\[
\Delta(s) = -\pi_h \left\{ e^{-sh} \begin{bmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22}
\end{bmatrix} \begin{bmatrix}
BD_2' \\
C_2
dref{3}
\end{bmatrix} \right\}.
\] (4)

Then \(\Phi_\alpha\) given by (2) admits the \(J\)-canonical factorization iff so does \(\Phi\) and, moreover, \(\Omega_\alpha = \left[ \begin{bmatrix} 1 & 0 \end{bmatrix} \right] \Omega\), where \(\Omega_\alpha\) and \(\Omega\) are the \(J\)-canonical factors of \(\Phi_\alpha\) and \(\Phi\), respectively.

Proof. First, note that for any \(S \in H^\infty\)
\[
\Phi_\alpha = \left[ \begin{bmatrix} 1 & 0 \\ S & 1 \end{bmatrix} \Phi_\beta \left[ \begin{bmatrix} 1 & 0 \\ S & 1 \end{bmatrix} \right] \right] \sim.
\]
where
\[
\Phi_\beta = \begin{bmatrix}
G_2G_2^\sim & G_2G_2^R^- \\
RG_2G_2^\sim & G_1G_1^\sim - G_1G_2^\sim(G_2G_2^\sim)^{-1}G_2G_1^\sim + RG_2G_2^R^- - \gamma^2 I
\end{bmatrix}
\]
and
\[
R = e^{-sh}G_1G_2^\sim(G_2G_2^\sim)^{-1} - S
\]
(the inverse above exists whenever (A3) holds). Obviously, \(\Phi_\alpha = \Omega_\alpha J \Omega_\alpha^\sim\) iff \(\Omega_\alpha = \begin{bmatrix} 1 & 0 \end{bmatrix} \Omega_\beta\) for any \(\Omega_\beta\) such that \(\Phi_\beta = \Omega_\beta J \Omega_\beta^\sim\). Since the transfer matrix \(\begin{bmatrix} 1 & 0 \end{bmatrix}\) is invertible in \(H^\infty\), conditions (ii) of Definition 1 for \(\Omega_\alpha\) and \(\Omega_\beta\) are equivalent. Finally, the \((1,1)\) sub-blocks of \(\Omega_\alpha\) and \(\Omega_\beta\) coincide that implies the equivalence of condition (iii) for \(\Omega_\alpha\) and \(\Omega_\beta\).

Now, to prove the Lemma one needs to show that if \(S = \Delta\) (which is stable), then \(\Phi_\beta = \Phi\). To show this note, that
\[
G_1G_2^\sim(G_2G_2^\sim)^{-1} = \begin{bmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22}
\end{bmatrix}
\]
\[
C_2^2 = \begin{bmatrix}
BD_2' \\
0
\end{bmatrix}
\]

It is seen now that the choice \(S = \Delta\) leads to
\[
R = \begin{bmatrix}
H_{11} & H_{12} & BD_2' \\
H_{21} & H_{22} & C_2^2 \\
C_1\Sigma_11(h) & C_1\Sigma_12(h) & 0
\end{bmatrix}
\]
which is finite-dimensional. Then the result follows by substituting this expression into the definition of \(\Phi_\beta\) and performing some tedious yet straightforward simplifications.

**Remark 3.1.** The idea of extracting the infinite-dimensional bi-stable part from the J-canonical factorization was proposed by Meinsma and Zwart (2000) to solve a two-block problem with dead time. That problem corresponds in our case to \(h < 0\) and requires an upper triangular extracted factor. The fact that the infinite-dimensional factor in the proof of Lemma 3 is lower triangular simplifies the analysis considerably, since the \((1,1)\) sub-block of \(\Phi\) is left unchanged and consequently the complete reduction to a finite-dimensional problem is possible. This is unlike (Meinsma and Zwart, 2000), where the extracted factor alters the \((1,1)\) sub-block that, in turn, gives rise to an additional “infinite-dimensional” solvability condition.

Now, using Lemmas 1 and 3 the solution to \(SP_h\) can be formulating. To this end define
\[
H_\gamma(\tau) = \Sigma^{-1}(\tau)(H + \gamma^{-2}\begin{bmatrix} 0 & 0 \\
C_1^h & 0
\end{bmatrix})\Sigma(\tau)
\]
(note that \(H_\gamma(0)\) is the Hamiltonian matrix associated with the filtering problem for \(G_1\) and \(G_2\)). Then:

**Theorem 1.** \(SP_h\) is solvable iff \(H_\gamma(h) \in \text{dom}(\text{Ric})\) and \(Y(h) = \text{Ric}(H_\gamma(h)) \geq 0\). Furthermore, if these conditions hold, then the set of all smoothers solving \(SP_h\) is parametrized as \(K = \Delta + C_2^2(\Omega, Q)\), where \(\Delta\) is given by (4),
\[
\Omega(s) = \begin{bmatrix}
A & BD_2' + Y(h)C_2^2 & \frac{1}{\gamma}(\Sigma_2'(h) - Y(h)\Sigma_2'(h))C_2^2 \\
C_2 & I & 0 \\
C_1\Sigma_1(h) & 0 & I
\end{bmatrix},
\]
and \(Q \in H^\infty\) satisfies \(\|Q\|_\infty < \gamma\) but otherwise arbitrary.
Proof. First, apply Lemma 1 to $\Phi$. The Hamiltonian matrix $H_a$ becomes

$$H_a = \begin{bmatrix} A & -BB' \\ 0 & -A' \end{bmatrix} - \begin{bmatrix} BD_2' & -\Sigma_1'(h)C_1' \\ C_2' & \Sigma_{11}'(h)C_1' \end{bmatrix}J^{-1}\begin{bmatrix} C_1 & -D_2B' \\ C_1\Sigma_{11}(h) & C_1\Sigma_{12}(h) \end{bmatrix}$$

$$= H + \gamma^{-2}\begin{bmatrix} -\Sigma_1'(h) \\ \Sigma_{11}'(h) \end{bmatrix} C_1' C_1 \begin{bmatrix} \Sigma_1(h) & \Sigma_{12}(h) \end{bmatrix}.$$

Since $\Sigma(h)$ is symplectic,

$$\Sigma^{-1}(h) = \begin{bmatrix} \Sigma_{22}'(h) & -\Sigma_{12}'(h) \\ -\Sigma_{12}'(h) & \Sigma_{11}'(h) \end{bmatrix}$$

and then, since $H$ and $\Sigma(h)$ are commute,

$$H_{\alpha} = H + \gamma^{-2}\Sigma^{-1}(h)\begin{bmatrix} 0 \\ C_1' \end{bmatrix} \begin{bmatrix} C_1 & 0 \end{bmatrix}\Sigma(h) = \Sigma^{-1}(h)H\gamma(0)\Sigma(h).$$

This yields the solvability conditions. The smoother formula follows directly from Lemma 3 by noticing that $\mathcal{C}_r\left(\left[\begin{array}{cc} I & 0 \\ \Delta & I \end{array}\right] \Omega, Q\right) = \Delta + \mathcal{C}_r(\Omega, Q)$. Finally, the realization of $\Omega$ is readily obtained from the formula in Lemma 1 with $M_a = \left[\begin{array}{cc} I & 0 \\ 0 & \gamma \end{array}\right]$ and an appropriate scaling of $Q$.

As stated by Lemma 2, the so-called central solution ($Q = 0$) possesses the maximum entropy property. The explicit formula for such a smoother is given below:

**Corollary 1.** Let the conditions of Theorem 1 hold. Then the unique smoother maximizing the entropy of the smoothing error is $K = \Delta + K_c$, where

$$K_c(s) = \frac{A - (BD_2' + Y(h)C_2')C_2}{C_1\Sigma_{11}(h)} \begin{bmatrix} BD_2' + Y(h)C_2' \\ 0 \end{bmatrix}$$

and $\Delta$ is given by (4).

Theorem 1 yields the complete solution to $\text{SP}_h$. It is worth stressing that it requires only one Riccati equation of the same dimension as in the filtering case to be solved. The smoother itself, however, is infinite-dimensional because of the FIR term $\Delta$. In fact, the $H^\infty$ smoother reminds the $H^\infty$ predictor (Mirkin, 2000). The latter also includes a finite-dimensional part and an FIR part. The difference is that in the prediction case the infinite-dimensional part is in the feedback connection with the rational part, like in the Smith predictor scheme, while in the smoothing case these components are in the feedforward connection. Another qualitative difference between the $H^\infty$ predictor and smoother is that the infinite-dimensional part of the latter does not depend on the performance level $\gamma$ and, moreover, coincides with the infinite-dimensional part of the $H^2$ smoother ($\gamma \to \infty$). Thus, one may conclude that the $H^\infty$ smoother exploits the information preview in an “$H^2$ fashion.”

The similarity between the Hamiltonian matrices for the filtering ($h = 0$) and smoothing ($h > 0$) cases is intriguing. It suggests that the dependence of the achievable $H^\infty$ performance on the size the smoothing lag can be quantified and an additional insight into the mechanism by which the previewed information is exploited in the $H^\infty$ setting can be provided. This is indeed true and the next section is devoted to such an analysis.
4 The effect of the smoothing lag on the achievable performance

The solvability conditions for $\text{SP}_h$ in Theorem 1 are expressed in terms of the Riccati solution $Y(h)$. Yet the use of the latter for the analysis of the effect of $h$ on the achievable performance is complicated by the fact that $Y(h)$ is discontinuous as a function of $h$. On the other hand, it is known (Scherer, 1990; Gahinet, 1994) that the null space of $Y(0)$ depends only upon properties of the realization of $G_2(s)$ and does not depend on $\gamma$. Moreover, as will be shown below, the null space of $Y(h)$ does not depend on the smoothing lag $h$ either. These facts suggest that the effect of $h$ can easier be accounted for in terms of the inverse of $Y(h)$.

To eliminate the singular part of $Y(h)$ from the analysis and hence simplify the exposition, assume that

(A4): $BD_2' = 0$;

(A5): $(A, B)$ has no stable uncontrollable modes.

Assumption (A4) stays that the measurement noise is independent of the system disturbance, while (A5) just rules out the problem redundancy. These assumptions thus are quite reasonable. It is worth stressing, however, that they can easily be omitted by applying a similarity transformation that extracts the redundant states, see §4.1 for the details.

Assumptions (A1)–(A5) guarantee (Zhou et al., 1995, Chapter 13) that $H \in \text{dom}(\text{Ric}(H)$ (H is defined by (3)) and, moreover, $\text{Ric}(H) > 0$. Hence, the Hamiltonian matrix

$$
\tilde{H} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} H \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -A' & -C_1' C_2 \\ -B B' & A \end{bmatrix} \in \text{dom}(\text{Ric})
$$

too and $\tilde{Y}_\kappa = \text{Ric}(\tilde{H}) > 0$ (since $\tilde{Y}_\kappa = \text{Ric}(H)^{-1}$). Note that $\tilde{Y}_\kappa$ is the stabilizing solution to the following Riccati equation:

$$
-\tilde{Y}_\kappa A - A' \tilde{Y}_\kappa + C_1' C_2 - \tilde{Y}_\kappa B B' \tilde{Y}_\kappa = 0
$$

(5)

so that the matrix $A_\kappa = -(A + B B' \tilde{Y}_\kappa)$ is Hurwitz. Furthermore, the spectrum of $A_\kappa$ coincides with that of the “closed-loop state-matrix” of the Kalman filter associated with $G_2(s)$.

Similarly, the following Hamiltonian matrix is associated with $Y(h)^{-1}$ (whenever it exists):

$$
\tilde{H}_Y(\tau) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} H_Y(\tau) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \tilde{\Sigma}^{-1}(\tau)(\tilde{H} + \gamma^{-2}[0 \ C_1' C_1]) \tilde{\Sigma}(\tau),
$$

where $\tilde{\Sigma}(\tau) = e^{-\tilde{H}_\tau \tau}$. The following result plays an important role in the sequel:

**Lemma 4.** Let $\gamma_\infty$ be the maximal $\gamma$ for which $\tilde{H}_Y(0)$ has eigenvalues on the $j\omega$-axis. Then the following statements are equivalent:

i) $\gamma > \gamma_\infty$;

ii) the (fixed-interval) smoothing problem $\text{SP}_\infty$ is solvable;

iii) $\tilde{H}_Y(h) \in \text{dom}(\text{Ric})$ for any $h \geq 0$.

**Proof.** See §A.1 in Appendix.

**Remark 4.1.** The state-space solution to the fixed-interval smoothing problem $\text{SP}_\infty$ was given by Nagpal and Khargonekar (1991). Yet the solvability condition in Lemma 4 is different from that in (Nagpal and Khargonekar, 1991) and more suitable for the further analysis of the effect of $h$ on the achievable performance.
Remark 4.2. As a matter of fact, $\gamma_\infty$ can be expressed in terms of the $H^\infty$ norm of a stable system. To this end, note that

$$\tilde{H}_r(0) = \begin{bmatrix} 1 & -\hat{Y}_k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A'_k & \gamma^{-2}C'_1C_1 \\ -BB' & -A_k \end{bmatrix} \begin{bmatrix} 1 & \hat{Y}_k \\ 0 & 1 \end{bmatrix}.$$  

Hence, $\gamma_\infty = \|C_1(sI - A_k)^{-1}B\|_\infty$ by (Zhou et al., 1995, Lemma 4.7).

We shall also need the inverse $H^\infty$ Riccati equation associated with the filtering problem:

$$-\tilde{Y}_rA - A'\tilde{Y}_r + C'_2C_2 - \frac{1}{\gamma^2}C'_1C_1 - \tilde{Y}_rBB'\tilde{Y}_r = 0.$$  

(6)

According to Lemma 4 the stabilizing solution to (6), $\tilde{Y}_r = \text{Ric}(\tilde{H}_r(0))$, is well defined whenever $\gamma > \gamma_\infty$. Moreover, as $\lim_{r \to \infty} \tilde{Y}_r = \hat{Y}_k$ and $\tilde{Y}_r$ is monotonically non-decreasing function of $\gamma$ (Gahinet, 1994), $\tilde{Y}_r \leq \hat{Y}_k$ for all $\gamma > \gamma_\infty$.

We are now in the position to state the main result of this section:

**Theorem 2.** The following statements are equivalent:

i) $\text{SP}_h$ is solvable;

ii) $\gamma > \gamma_\infty$ and $\tilde{Y}(h) = \text{Ric}(\tilde{H}_r(h)) > 0$;

iii) $\gamma > \gamma_\infty$ and $Q_r(h) > 0$, where $Q_r(t)$ is the solution to the differential Riccati equation

$$-Q_r(t) = A'Q_r(t) + Q_r(t)A - C'_2C_2 + Q_r(t)BB'Q_r(t), \quad Q_r(0) = \tilde{Y}_r,$$

which is monotonically non-decreasing and $\lim_{t \to \infty} Q_r(t) = \hat{Y}_k > 0$ for all $\gamma > \gamma_\infty$;

iv) $\gamma > \gamma_\infty$ and $\|C_r e^{A_k h} B_k\| < 1$, where $B_k$ and $C_r$ are any matrices satisfying

$$B_kB'_k = \hat{Y}_k^{-1} - W_c \geq 0 \quad \text{and} \quad C'_2C_r = (I - (\hat{Y}_k - \tilde{Y}_r)W_c)^{-1}(\hat{Y}_k - \tilde{Y}_r) \geq 0,$$

respectively, and $W_c \geq 0$ is the solution to the Lyapunov equation

$$A_kW_c + W_cA'_k + BB' = 0.$$  

(8)

Moreover, $\|C_r e^{A_k h} B_k\|$ is monotonically non-increasing function of $h$.

*Proof.* See §A.2 in Appendix.

As follows from Theorem 2, any performance $\gamma > \gamma_\infty$ is achievable provided that the smoothing lag $h$ is “large enough.” This can be clearly seen from, for example, statement iv) of Theorem 2 since $\|C_r e^{A_k h} B_k\|$ approaches zero as $h \to \infty$. This property of the $H^\infty$ smoother is rather expectable. What is less obvious, is the fact that the “convergence” rate in the $H^\infty$ case is completely determined by the dynamics of the corresponding Kalman filter (see the comment after eqn. (5)).

The last observation can be further developed by noting that the $H^2$ solution is obtained from the $H^\infty$ one by taking $\gamma^{-2} = 0$. Then, using the estimator in Corollary 1 and calculating the $H^2$ norm of the resulting error transfer matrix $e^{-sh}G_1 - KG_2$, one can end up with the following formula:

$$\tilde{J}_h = \min_{K \in H^\infty} \|e^{-sh}G_1 - KG_2\|_2^2$$

$$= \tilde{J}_\infty + \|C_1 e^{A_k h} B_k\|_F^2,$$

(9)
Consider \( \delta_{\infty} = \text{tr}(C_1W_cC_1') \) is the achievable performance of the fixed-interval \( H^2 \) smoothing problem. Thus, the \( H^2 \) fixed-lag smoothing performance approaches the fixed-interval one exponentially and the speed of convergence is determined by the matrix \( A_{\infty} \), exactly as in the \( H^\infty \) case. This observation supports the claim that the \( H^\infty \) smoother exploits the information preview in an "\( H^2 \) fashion" (cf. the discussion at the end of Section 3).

There exists, however, a remarkable difference between the \( H^2 \) and \( H^\infty \) solutions. As seen from (9), the \( H^2 \) performance is improved all the time as the smoothing lag \( h \) grows (though the improvement becomes negligible as \( h \) exceeds several times the dominant time constant of \( A_{\infty} \), see (Anderson and Moore, 1979)). On the other hand, in the \( H^\infty \) case there always exists a finite \( h_\gamma \) such that \( \|C_\gamma e^{A_{\infty}h_{\gamma}B_{\infty}}\| < 1 \) for all \( h > h_\gamma \) and every \( \gamma > \gamma_\infty \). In other words, any performance level \( \gamma > \gamma_\infty \) is achievable with a finite smoothing lag. Moreover, the quantity \( h_\infty \), which corresponds to the limit \( \gamma \downarrow \gamma_\infty \) and is the function of the problem data, can be regarded as the maximal smoothing lag making sense from the \( H^\infty \) performance point of view. Indeed, any further increase of \( h \) does not improve the achievable \( H^\infty \) performance at all! Note, however, that the increase of the smoothing lag beyond \( h_\infty \) might affect some other properties of the solution, e.g., its \( H^2 \) performance (see Section 5).

### 4.1 Relaxation of (A4) and (A5)

As discussed at the beginning of this section, assumptions (A4) and (A5) are made to guarantee the invertibility of \( Y(h) \). Yet it turns out that \( \ker Y(h) \) is independent of both \( \gamma \) and \( h \). This fact enables one to eliminate the singular part of the Riccati solution from the further analysis. To this end note, that there exists a unitary matrix \( U = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \) such that

\[
U'(A - BD_2'C_2)U = \begin{bmatrix} \bar{A} & ? \\ 0 & \bar{A}_s \end{bmatrix}, \quad U'B(I - D_2'D_2) = \begin{bmatrix} \bar{B} \\ 0 \end{bmatrix},
\]

\( \bar{A}_s \) is Hurwitz, and the pair \( (\bar{A}, \bar{B}) \) has no stable uncontrollable modes (here "?" stands for an irrelevant block). Denote

\[
\bar{C}_1 = C_1U_1 \quad \text{and} \quad \bar{C}_2 = C_2U_1.
\]

Then it is readily verified that

\[
Y(h) = U_1\bar{Y}(h)U_1',
\]

where \( \bar{Y}(h) \) is the Riccati solution associated with Hamiltonian matrix built upon \( \bar{A}, \bar{B}, \bar{C}_1, \) and \( \bar{C}_2 \) instead of \( A - BD_2'C_2, B(I - D_2'D_2), C_1, \) and \( C_2 \) respectively. Moreover, the absence of stable uncontrollable modes in \( (\bar{A}, \bar{B}) \) implies that \( \bar{Y}(h) \) is nonsingular. Thus all results of this section remain valid in the general case with the replacements \( A - BD_2'C_2 \rightarrow \bar{A}, B(I - D_2'D_2) \rightarrow \bar{B}, \) and \( C_1 \rightarrow \bar{C}_1 (i = 1, 2) \).

### 5 Numerical example

Consider \( SP_h \) for the following simple first-order system:

\[
G_1(s) = \begin{bmatrix} \frac{a\omega}{s - \alpha} & 0 \\ \frac{a\omega}{s - \alpha} & 1 \end{bmatrix}, \quad \text{and} \quad G_2(s) = \begin{bmatrix} \frac{a\omega}{s - \alpha} & 1 \end{bmatrix},
\]

with \( \alpha \neq 0 \) and \( q > 0 \). It can be verified that in this case \( \gamma_\infty = q/\sqrt{q^2 + 1} \) and for any \( \gamma > \gamma_\infty \)

\[
\bar{Y}_\gamma = \frac{1}{aq}(\sqrt{1 + q^2(1 - \gamma^{-2})} - 1).
\]
When $G(s)$ is stable ($a < 0$) $\tilde{Y}_γ > 0$ for all $γ > γ_∞$ and therefore smoothing offers no benefits over filtering. On the other hand, if $a > 0$, then $\tilde{Y}_γ > 0$ iff $γ > 1 > γ_∞$. Thus, in this case smoothing leads to superior estimation performance and, moreover, the lower is the system noise intensity (i.e., the smaller is $q$), the larger is the potential advantage of smoothing over filtering, see Fig. 1(a).

Assume throughafter that $a > 0$ and denote $η = 1/\sqrt{q^2 + 1} < 1$. Using condition iv) of Theorem 2 it can be shown that the minimal achievable $H_∞$ performance $γ_{min}$ for $SP_h$ is

$$γ_{min} = \begin{cases} \frac{1}{2}((1-η)e^{ah/η} + (1+η)e^{-ah/η}) & \text{if } h ≤ h_∞ = \frac{1}{2a}\ln\frac{1+η}{1-η} \\ γ_∞ = \sqrt{1-η^2} & \text{otherwise.} \end{cases}$$

The plot $γ_{min}$ vs. $h$ is depicted in Fig. 1(b) for the case of $a = q = 1$. It is seen that $γ_∞$ is achievable with $h = h_∞ ≈ 0.623$ and any further increase of $h$ does not improve the $H_∞$ estimation performance. The increase of $h$, however, does affect the Riccati solution $Y(h)$ and, consequently, the central estimator in Corollary 1. This is clearly seen from Fig. 1(c), which shows the solutions $Q_γ(h) = 1/Y(h)$ to the differential Riccati equation (7) for two different...
values of \( \gamma \): \( \gamma = \gamma_\infty \) (solid line) and \( \gamma = 1.1 \gamma_\infty \) (dashed line). According to condition iii) of Theorem 2, \( h_\infty \) corresponds to the zero crossing of \( Q_\gamma(h) \) and the latter then continues to change until it practically converges to \( \tilde{Y}_\kappa \) at \( h \approx 2.5 \) (here \( \gamma = \gamma_\infty \) is assumed).

It is then interesting to see how the variation of \( Q_\gamma(h) \) in \( h > h_\infty \) affects the estimation error. To this end, consider Fig. 2(a), which depicts the magnitudes plots of \( G_1 - e^{jh}KG_2 \) under the central smoother \( K \) achieving \( \gamma_{\min} \) for several values of \( h \in [0, 2.5] \). It is seen that in the interval \([0, h_\infty]\) the error transfer function is virtually all-pass and its \( H_\infty \) norm decreases as \( h \) increases. The further increase of \( h \), which keeps the peak of the error unchanged, has completely different effect on the estimation error and shows up in the decrease of the error bandwidth. In other words, the increase of \( h \) beyond \( h_\infty \) reduces the \( H^2 \) norm of \( G_1 - e^{jh}KG_2 \) until the latter reaches its minimal value as \( Q_\gamma \) converges to \( \tilde{Y}_\kappa \). This agree well with the known fact (Hassibi et al., 1999, Ch. 10) that as the smoothing lag \( h \to \infty \) both \( H^2 \) and \( H_\infty \) approaches result in the same estimator, which actually minimizes the estimation error energy for every input. The reader is referred to Figs. 2(a) and 2(b) to compare the \( H_\infty \) and \( H^2 \) solutions and see how the both approach the same error transfer function as \( h \) increases, yet from different directions.

### 6 Concluding remarks

In this paper the first solution to the continuous-time \( H_\infty \) fixed-lag smoothing problem has been derived. The derivation has been based on the reduction of the associated infinite-dimensional \( J \)-spectral factorization problem to an equivalent finite-dimensional one. The resulting smoother consists of two components connected in parallel: a finite-dimensional estimator reminiscent of the \( H_\infty \) filter and an FIR block of the length of the smoothing lag. The solvability conditions are based on one algebraic Riccati equation of the same dimension as the Riccati equation associated with the filtering problem. It has also been shown that any performance level \( \gamma > \gamma_\infty \) can be achieved if the smoothing lag is “large enough” (\( \gamma_\infty \) stands for the \( H_\infty \) performance level of the infinite-horizon version of the fixed-interval smoothing problem). Moreover, the effect of the smoothing lag on the achievable performance has been quantified.
Note that the implementation issues for the $H^\infty$ smoother have not been addressed in the paper. The potential problem might be in implementing the FIR block, which is built upon a Hamiltonian matrix and requires the matrix exponentials to be computed. Yet the computation of the matrix exponents for matrices having positive eigenvalues, especially for a large smoothing lag, might be a difficult numerical problem. Hence, alternative realizations for the $H^\infty$ fixed-lag smoother that involve only matrix exponentials of Hurwitz matrices is required. This is the subject of future research.

Another question raised by the current research is the $H^\infty$ performance of the $H^2$ (Kalman) smoothers. The results of the numerical simulation in Section 5 show that when the smoothing lag $h$ increases the $H^2$ and $H^\infty$ smoothers approach each other. This phenomenon is yet to be quantified.

Finally note, that the technique proposed in the paper can be applied to the finite-horizon and discrete-time systems mutatis mutandis. The only difference seems to be in more cumbersome derivation, although the results are expected to be of the same form.

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References


Appendix

A.1 Proof of Lemma 4

\(i\) \(\iff\) \(ii\)

Standard completing to square arguments (Hassibi et al., 1999, §10.4.2) yield that the optimal performance level for \(SP_{\infty,r}\) can be characterized as follows:

\[
\gamma_{\infty}^2 = \sup_{\omega \in [0,\infty)} \sigma(\Psi(j\omega)),
\]

where

\[
\Psi = G_1(1 - G_2^{-1}G_2^{-1}G_2) G_1^{-1} = \begin{bmatrix} \hat{H}_{11} & \hat{H}_{12} \\ \hat{H}_{21} & \hat{H}_{22} \end{bmatrix} \begin{bmatrix} C_1' \\ 0 \end{bmatrix}.
\]

Clearly, \(\gamma > \gamma_{\infty}\) iff \(\gamma^2 I - \Psi(j\omega) > 0\) for all \(\omega\). Since \(\Psi(s)\) is strictly proper, the latter condition is equivalent to the requirement that \((\gamma^2 I - \Psi)^{-1}\) has no poles on the imaginary axis. The equality \(\gamma_{\infty} = \gamma_{\infty}\) then follows by standard arguments associated with the computation of the \(H_{\infty}\) norm (see, e.g., Zhou et al., 1995, Lemma 4.7)) and the fact that \(H\) has no imaginary axis eigenvalues.

\(i\) \(\iff\) \(iii\)

It follows from the reasoning above that \(H_{\infty}(0)\) has imaginary axis eigenvalues \(\forall \gamma \in [0,\gamma_{\infty})\). Therefore, \(H_{\infty}(0) \notin \text{dom}(\text{Ric})\) for any \(\gamma \leq \gamma_{\infty}\) and one only needs to prove that \(H_{\infty}(h) \in \text{dom}(\text{Ric})\) whenever \(\gamma > \gamma_{\infty}\). The first step toward this end is to prove the latter for \(h = 0\):

**Claim 1.** \(H_{\infty}(0) \in \text{dom}(\text{Ric})\) for all \(\gamma > \gamma_{\infty}\).

**Proof.** Since \(H_{\infty}(0)\) has no \(j\omega\)-axis eigenvalues, there exists a full row rank matrix \([\bar{\gamma}_\alpha \ \bar{\gamma}_\beta]\) such that

\[
[\bar{\gamma}_\alpha \ \bar{\gamma}_\beta] H_{\infty}(0) = A_L [\bar{\gamma}_\alpha \ \bar{\gamma}_\beta]
\]

for some Hurwitz \(A_L\). Now, to prove the Claim it is only left to show that \(\bar{\gamma}_\alpha\) is nonsingular. But the latter is guaranteed by the fact that \((A, B)\) has no stable uncontrollable modes, see (Zhou et al., 1995, Theorem 13.7).

Now, note that \(H_{\infty}(h) \in \text{dom}(\text{Ric})\) iff there exists a matrix \(\bar{Y}(h)\) satisfying

\[
[1 \ \bar{Y}(h)] H_{\infty}(h) = A_L(h) [1 \ \bar{Y}(h)]
\]

for some Hurwitz \(A_L(h)\). In particular, when \(h = 0\)

\[
[1 \ \bar{Y}(0)] H_{\infty}(0) = [1 \ \bar{Y}(0)] \bar{\Xi}(h) H_{\infty}(h) \bar{\Xi}(h)^{-1} = A_L(0) [1 \ \bar{Y}(0)],
\]

where \(\bar{\Xi}(0) = \text{Ric}(H_{\infty}(0))\). Hence, if \(H_{\infty}(0) \in \text{dom}(\text{Ric})\), then \(H_{\infty}(h) \in \text{dom}(\text{Ric})\) too iff

\[
\det(\bar{\Xi}_{11}(h) + \bar{Y}_y \bar{\Xi}_{21}(h)) \neq 0
\]

and in that case

\[
\bar{Y}(h) = (\bar{\Xi}_{11}(h) + \bar{Y}_y \bar{\Xi}_{21}(h))^{-1}(\bar{\Xi}_{12}(h) + \bar{Y}_y \bar{\Xi}_{22}(h)).
\]

\(10\)
Consider the following first-order system:

\[
\begin{bmatrix}
Y_{\alpha} & Y_{\beta}
\end{bmatrix} = -\begin{bmatrix}
Y_{\alpha} & Y_{\beta}
\end{bmatrix} \dot{\Phi}
\]

(11)

with the initial condition \(\begin{bmatrix} Y_{\alpha}(0) & Y_{\beta}(0) \end{bmatrix} = \begin{bmatrix} 1 & \bar{\Phi} \end{bmatrix}\). Clearly, (10) is equivalent to the non-singularity of \(Y_{\alpha}(t)\) and \(\bar{Y}(h) = Y_{\alpha}^{-1}(h)Y_{\beta}(h)\). On the other hand, it is the standard result from the Riccati theory that \(\bar{Y}(h)\) as in the last formula satisfies \(\bar{Y}(h) = Q_{Y}(h)\), where \(Q_{Y}(t)\) is the solution to to the differential Riccati equation (7). Thus, in order to show the existence of \(\bar{Y}(h)\) it is sufficient to show that (7) has no escape points on the interval \([0, h]\). To this end the following result is required:

**Claim 2.** \(Q_{Y}(t)\) is monotonically non-decreasing function of \(t\) in the sense that \(Q_{Y}(t_{1}) \geq Q_{Y}(t_{2})\) whenever \(t_{1} \geq t_{2}\). Moreover, \(Q_{Y}(t)\) is bounded and \(\lim_{t \to \infty} Q_{Y}(t) = \bar{Y}_{k} > 0\).

**Proof.** Differentiating (7) by \(t\) one gets:

\[
\dot{Q}_{Y} = -(A + BB'Q_{Y})'Q_{Y} - \dot{Q}_{Y}(A + BB'Q_{Y}).
\]

Using the fact that \(Q_{Y}(0) = \text{Ric}(\bar{H}_{Y}(0))\) it is readily seen that \(\dot{Q}_{Y}(t)|_{t=0} = \frac{1}{\gamma}C_{1}'C_{1} \geq 0\). Then, denoting by \(\Phi_{Q}(t, 0)\) the state transition matrix associated with \(-(A + BB'Q_{Y})\), one gets:

\[
\dot{Q}_{Y}(t) = \frac{1}{\gamma} \Phi_{Q}'(t, 0)C_{1}'C_{1}\Phi_{Q}(t, 0) \geq 0,
\]

from which the monotonicity statement of the Claim follows immediately.

Now, since \(\bar{Y}_{Y}\) is monotonically non-decreasing function of \(\gamma\) (Gahinet, 1994), \(\bar{Y}_{Y} \leq \bar{Y}_{k}\) for every \(\gamma > \gamma_{\infty}\). Define the function

\[
Q_{\Delta}(t) = \bar{Y}_{k} - Q_{Y}(t),
\]

(12)

which is monotonically non-increasing. It can then be verified that \(Q_{\Delta}\) satisfies

\[
\dot{Q}_{\Delta} = A_{\kappa}'Q_{\Delta} + Q_{\Delta}A_{\kappa} + Q_{\Delta}BB'Q_{\Delta},
\]

\(Q_{\Delta}(0) = \bar{Y}_{k} - \bar{Y}_{Y} \geq 0\),

where \(A_{\kappa} = -(A + BB'\bar{Y}_{k})\) is Hurwitz and \(\dot{Q}_{\Delta}(t) \leq 0\) for all \(t\). It follows from this equation that if there exists a vector \(\eta \neq 0\) such that \(Q_{\Delta}(t)|_{\eta} = 0\), then \(\dot{Q}_{\Delta}(t)|_{\eta} = 0\) too. Then, the continuity of \(Q_{\Delta}(t)\) means that no eigenvalues of \(Q_{\Delta}\) can cross 0. Since \(Q_{\Delta}\) is symmetric, it has only real eigenvalues and thus its inertia is unchanged. This, in turn, implies that \(Q_{\Delta}(t) \geq 0\) for all \(t\). Therefore, as \(Q_{\Delta}\) is monotonically non-increasing, its limit as \(t \to \infty\) exists. Finally, since \(A_{\kappa}\) is Hurwitz, the only equilibrium of \(Q_{\Delta}\) in the positive semi-definite region is \(Q_{\Delta} = 0\). \(\square\)

As follows from Claim 2, \(\bar{Y}(h)\) exists for all \(h \geq 0\) that, in turn, implies that \(\bar{H}_{Y}(h) \in \text{dom}(\text{Ric})\). This completes the proof of Lemma 4.

### A.2 Proof of Theorem 2

It follows from Lemma 4 that \(\gamma_{\infty}\) is the lower bound for the achievable smoothing performance for every \(h\). It therefore will be assumed throughout the rest of the proof that \(\gamma > \gamma_{\infty}\).

\(i) \iff ii)\)

By Theorem 1 \(\text{SP}_{h}\) is solvable iff \(H_{Y}(h) \in \text{dom}(\text{Ric})\) and \(Y(h) = \bar{Y}^{-1}(h) > 0\) (the latter follows from assumptions (A4) and (A5)). If \(H_{Y}(h) \in \text{dom}(\text{Ric})\), then the solvability is clearly equivalent to the positive definiteness of \(\bar{Y}(h)\). On the other hand, for every \(\gamma > \gamma_{\infty}\) \(H_{Y}(h) \notin \text{dom}(\text{Ric})\) iff \(\bar{Y}(h)\) is singular and hence condition \(ii)\) does not hold. This completes the proof.
\(ii) \iff iii)\)

As follows from the proof of Lemma 4, \(Q_\gamma(h) = \tilde{Y}(h)\) for every \(h \geq 0\). Then the equivalence of \(ii)\) and \(iii)\) follows by Claim 2 on p. 16.

\(iii) \iff iv)\)

Consider equation (11) and denote
\[
\begin{bmatrix}
Z_\alpha & Z_\beta
\end{bmatrix} = \begin{bmatrix}
Y_\alpha & Y_\beta
\end{bmatrix} \begin{bmatrix}
I & \tilde{Y}_k
0 & -I
\end{bmatrix},
\]
so that \(Q_\Delta = Z_\alpha^{-1}Z_\beta\), where \(Q_\Delta(t)\) is defined by (12). Since
\[
\begin{bmatrix}
I & \tilde{Y}_k
0 & -I
\end{bmatrix} \begin{bmatrix}
I & \tilde{Y}_k
0 & -I
\end{bmatrix}^{-1} = \begin{bmatrix}
A'_k & 0
BB' & -A_k
\end{bmatrix},
\]
(11) can be rewritten as
\[
\begin{bmatrix}
Z_\alpha & Z_\beta
\end{bmatrix} = \begin{bmatrix}
Z_\alpha & Z_\beta
\end{bmatrix} \begin{bmatrix}
-A'_k & 0
-BB' & A_k
\end{bmatrix}
\]
with the initial conditions \(\begin{bmatrix}
Z_\alpha(0) & Z_\beta(0)
\end{bmatrix} = \begin{bmatrix}
I & Q_\Delta(0)
\end{bmatrix}\). Solving this equation one gets:
\[
Z_\beta(t) = Q_\Delta(0)e^{A_\kappa t} \quad \text{and} \quad Z_\alpha(t) = (I - Q_\Delta(0)W(t))e^{-A_\kappa t},
\]
where \(W(t) = \int_0^t e^{A_\kappa s}BB'e^{A_\kappa s}ds \geq 0\) (note, that \(W_c = \lim_{t \to \infty} W(t)\)), and then:
\[
Q_\Delta(t) = e^{A_\kappa t}(I - Q_\Delta(0)W(t))^{-1}Q_\Delta(0)e^{A_\kappa t} = e^{A_\kappa t}I - \Theta W(t)\Theta = I - \Theta e^{A_\kappa t}.
\]
where \(\Theta = Q_\Delta(0)^{1/2}\). Because \(Q_\Delta\) is bounded (by Claim 2), \(\Theta W(t)\Theta < I\) for all \(t\). Using the explicit formula for \(Q_\Delta\), the condition \(Q(t) > 0\) can be rewritten as
\[
\tilde{Y}_k - e^{A_\kappa t}\Theta(I - \Theta W(t)\Theta)^{-1}\Theta e^{A_\kappa t} > 0
\]
or, equivalently,
\[
\begin{bmatrix}
\tilde{Y}_k
\Theta e^{A_\kappa t}
\end{bmatrix} \begin{bmatrix}
e^{A_\kappa t}\Theta
I - \Theta W(t)\Theta
\end{bmatrix} > 0.
\]
Then, as \(\tilde{Y}_k > 0\), the latter holds iff the Schur complement of \(\tilde{Y}_k\) is positive definite of, i.e., iff
\[
\Theta(e^{A_\kappa t}\tilde{Y}_k^{-1}e^{A_\kappa t} + W(t))\Theta < I.
\]
Yet since \(W(t) = W_c - e^{A_\kappa t}W_c e^{A_\kappa t}\), the inequality above can be reformulated as
\[
\Theta e^{A_\kappa t}B_kB'_ke^{A_\kappa t}\Theta < I - \Theta W_c\Theta.
\]
Now, noticing that \(C'_c = \Theta(I - \Theta W_c\Theta)^{-1}\Theta\) the latter inequality can be equivalently rewritten as \(C_y Y(t)C'_c < I\), where \(Y(t) = e^{A_\kappa t}B_kB'_ke^{A_\kappa t}\). The reasoning above yields that
\[
Q(h) > 0 \iff \|C_y e^{A_\kappa h}B_k\| < 1.
\]
Finally, the monotonicity of the norm in the last inequality follows from the fact that
\[
\tilde{Y} = e^{A_\kappa t}(A_\kappa(\tilde{Y}_k^{-1} - W_c) + (\tilde{Y}_k^{-1} - W_c)A'_k)e^{A_\kappa t}
\]
\[
= -e^{A_\kappa t}\tilde{Y}_k^{-1}C_2^T C_2 \tilde{Y}_k^{-1} e^{A_\kappa t} \leq 0. \quad \text{(by (5) and (8))}
\]
This completes the proof of Theorem 2.